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SUBORDINATION AND SUPERORDINATION FOR MEROMORPHIC FUNCTIONS ASSOCIATED WITH THE MULTIPLIER TRANSFORMATION

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Abstract. The purpose of the present paper is to obtain some subordination and superordination preserving properties involving a certain family of multiplier transformations for meromorphic functions in the open unit disk. The sandwich-type theorems for these linear operators are also considered.

1. Introduction

Let $\mathcal{H} = \mathcal{H}(\mathbb{U})$ denote the class of analytic functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N} = \{1, 2, \cdots\}$, let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H} : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\}.$$

Let f and F be members of H. The function f is said to be subordinate to F, or F is said to be superordinate to f, if there exists a function w analytic in $\mathbb{U}$, with $w(0) = 0$ and $|w(z)| < 1$, and such that $f(z) = F(w(z))$. In such a case, we write $f \prec F$ or $f(z) \prec F(z)$. If the function F is univalent in $\mathbb{U}$, then $f \prec F$ if and only if $f(0) = F(0)$ and $f(\mathbb{U}) \subset F(\mathbb{U})$ (cf. [8], [13]).

Definition 1. ([7]) Let $\phi : \mathbb{C}^2 \to \mathbb{C}$ and let h be univalent in $\mathbb{U}$. If p is analytic in $\mathbb{U}$ and satisfies the differential subordination

$$\phi(p(z), zp'(z)) \prec h(z), \tag{1.1}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant if $p \prec q$ for all p satisfying (1.1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1.1) is said to be the best dominant.

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Definition 2. ([8]) Let $\varphi : \mathbb{C}^2 \to \mathbb{C}$ and let h be analytic in $\mathbb{U}$. If p and $\varphi(p(z), zp'(z))$ are univalent in $\mathbb{U}$ and satisfy the differential superordination

$$h(z) \prec \varphi(p(z), zp'(z)), \tag{1.2}$$

then p is called a solution of the differential superordination. An analytic function q is called a subordinant of the solutions of the differential superordination, or more simply a subordinant if $q \prec p$ for all p satisfying (1.2). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants q of (1.2) is said to be the best subordinant.

Definition 3. ([8]) We denote by $\mathcal{Q}$ the class of functions f that are analytic and injective on $\overline{\mathbb{U}} \backslash E(f)$, where

$$E(f) = \left\{ \zeta \in \partial\mathbb{U} : \lim_{z \to \zeta} f(z) = \infty \right\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U} \backslash E(f)$.

Let Σ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k$$

which are analytic in the punctured open unit disk $\mathbb{D} = \{z \in \mathbb{C} : 0 < |z| < 1\}$ with $a_0 \neq 0$. For any $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, we denote the multiplier transformations D_l^n of functions $f \in \Sigma$ by

$$D_\lambda^n f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} \left(\frac{k+1+\lambda}{\lambda} \right)^n a_k z^k \ \ (\lambda > 0;\ z \in \mathbb{U}). \tag{1.3}$$

Obviously, we have

$$D_\lambda^s (D_\lambda^t f(z)) = D_\lambda^{s+t} f(z)$$

for all nonnegative integers s and t. The operators D_λ^n and D_1^n are the multiplier transformations introduced and studied by Sarangi and Uralegaddi [12] and Uralegaddi and Somanatha [14], [15], respectively. It is easily verified from (1.3) that

$$z(D_\lambda^n f(z))' = \lambda D_\lambda^{n+1} f(z) - (\lambda+1) D_\lambda^n f(z). \tag{1.4}$$

By using of the principle of subordination, Miller et al. [9] obtained some subordination theorems involving certain integral operators for analytic functions in $\mathbb{U}$ (see also [1], [10]). Moreover, Miller and Mocanu [8] considered differential superordinations, as the dual problem of differential subordinations (see also [2]). In the present paper, we investigate the subordination and superordination preserving properties of the multiplier transformation D_λ^n defined by (1.3) with the sandwich-type theorems. It should be remarked that the results presented here are new ones which are not studied by another researchers yet.

The following lemmas will be required in our present investigation.

Lemma 1.1. ([5]) *Suppose that the function* $H : \mathbb{C}^2 \to \mathbb{C}$ *satisfies the condition*:

$$\mathrm{Re}\{H(is, t)\} \leq 0,$$

for all real s *and* $t \leq -n(1+s^2)/2$*, where* n *is a positive integer. If the function* $p(z) = 1 + p_n z^n + \cdots$ *is analytic in* $\mathbb{U}$ *and*

$$\mathrm{Re}\{H(p(z), zp'(z))\} > 0 \quad (z \in \mathbb{U}),$$

then $\mathrm{Re}\{p(z)\} > 0$ *in* $\mathbb{U}$.

Lemma 1.2. ([6]) *Let* $\beta, \gamma \in \mathbb{C}$ *with* $\beta \neq 0$ *and let* $h \in \mathcal{H}(\mathbb{U})$ *with* $h(0) = c$*. If* $\mathrm{Re}\{\beta h(z) + \gamma\} > 0$ *for* $z \in \mathbb{U}$*, then the solution of the differential equation*:

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z) \quad (z \in \mathbb{U})$$

with $q(0) = c$ *is analytic in* $\mathbb{U}$ *and satisfies* $\mathrm{Re}\{\beta q(z) + \gamma\} > 0$ *for* $z \in \mathbb{U}$.

Lemma 1.3. ([7]) *Let* $p \in \mathcal{Q}$ *with* $p(0) = a$ *and let* $q(z) = a + a_n z^n + \cdots$ *be analytic in* $\mathbb{U}$ *with* $q(z) \not\equiv a$ *and* $n \in \mathbb{N}$*. If* q *is not subordinate to* p*, then there exist points* $z_0 = r_0 e^{i\theta} \in \mathbb{U}$ *and* $\zeta_0 \in \partial\mathbb{U} \setminus E(f)$*, for which* $q(\mathbb{U}_{r_0}) \subset p(\mathbb{U})$,

$$q(z_0) = p(\zeta_0) \quad \text{and} \quad z_0 q'(z_0) = m\zeta_0 p'(\zeta_0) \quad (m \geq n).$$

A function $L(z, t)$ defined on $\mathbb{U} \times [0, \infty)$ is the subordination chain (or Löwner chain) if $L(\cdot, t)$ is analytic and univalent in $\mathbb{U}$ for all $t \in [0, \infty)$, $L(z, \cdot)$ is continuously differentiable on $[0, \infty)$ for all $z \in \mathbb{U}$ and $L(z, s) \prec L(z, t)$ for $0 \leq s < t$.

Lemma 1.4. ([8]) *Let* $q \in \mathcal{H}[a, 1]$*, let* $\varphi : \mathbb{C}^2 \to \mathbb{C}$ *and set* $\varphi(q(z), zq'(z)) \equiv h(z)$*. If* $L(z, t) = \varphi(q(z), tzq'(z))$ *is a subordination chain and* $p \in \mathcal{H}[a, 1] \cap \mathcal{Q}$*, then*

$$h(z) \prec \varphi(p(z), zp'(z))$$

implies that

$$q(z) \prec p(z).$$

Furthermore, if $\varphi(q(z), zp'(z)) = h(z)$ *has a univalent solution* $q \in \mathcal{Q}$*, then* q *is the best subordinant.*

Lemma 1.5. ([11]) *The function* $L(z, t) = a_1(t)z + \cdots$*, with* $a_1(t) \neq 0$ *and* $\lim_{t\to\infty} |a_1(t)| = \infty$*, is a subordination chain if and only if*

$$\mathrm{Re}\left\{\frac{z\partial L(z, t)/\partial z}{\partial L(z, t)/\partial t}\right\} > 0 \quad (z \in \mathbb{U};\ 0 \leq t < \infty).$$

2. Main results

Firstly, we begin by proving the following subordination theorem involving the multiplier transformation D_λ^n defined by (1.3).

Theorem 2.1. *Let $f, g \in \Sigma$. Suppose that*

$$\mathrm{Re}\left\{1+\frac{z\phi''(z)}{\phi'(z)}\right\} > -\delta \tag{2.1}$$

$$\left(\phi(z) := (1-\alpha)zD_\lambda^{n+1}g(z)+\alpha zD_\lambda^n g(z);\ \lambda>0;\ 0\le\alpha<1;\ z\in\mathbb{U}\right),$$

where

$$\delta = \frac{(1-\alpha)^2+\lambda^2-|(1-\alpha)^2-\lambda^2|}{4\lambda(1-\alpha)} \tag{2.2}$$

If f and g satisfy the following subordination condition :

$$(1-\alpha)zD_\lambda^{n+1}f(z)+\alpha zD_\lambda^n f(z) \prec (1-\alpha)zD_\lambda^{n+1}g(z)+\alpha zD_\lambda^n g(z), \tag{2.3}$$

then

$$zD_\lambda^n f(z) \prec zD_\lambda^n g(z). \tag{2.4}$$

Moreover, the function $zD_\lambda^n g(z)$ is the best dominant.

Proof. Let us define the functions F and G, respectively, by

$$F(z) := zD_\lambda^n f(z) \quad \text{and} \quad G(z) := zD_\lambda^n g(z), \tag{2.5}$$

We first show that, if the function q is defined by

$$q(z) := 1+\frac{zG''(z)}{G'(z)} \quad (z\in\mathbb{U}), \tag{2.6}$$

then

$$\mathrm{Re}\{q(z)\}>0 \quad (z\in\mathbb{U}).$$

Taking the logarithmic differentiation on both sides of the second equation in (2.5) and using (1.4) for $g\in\Sigma$, we obtain

$$\lambda\phi(z) = \lambda G(z)+(1-\alpha)zG'(z). \tag{2.7}$$

Now, by differentiating both sides of (2.7), we obtain the relationship:

$$\begin{aligned} 1+\frac{z\phi''(z)}{\phi'(z)} &= 1+\frac{zG''(z)}{G'(z)}+\frac{zq'(z)}{q(z)+\lambda/(1-\alpha)} \\ &= q(z)+\frac{zq'(z)}{q(z)+\lambda/(1-\alpha)} \equiv h(z). \end{aligned} \tag{2.8}$$

We see from (2.1) that

$$\mathrm{Re}\left\{h(z)+\frac{\lambda}{1-\alpha}\right\}>0 \quad (z\in\mathbb{U}),$$

holds true and by using Lemma 1.2, we conclude that the differential equation (2.8) has a solution $q\in\mathcal{H}(\mathbb{U})$ with $q(0)=h(0)=1$. Let us put

$$H(u,v) = u+\frac{v}{u+\lambda/(1-\alpha)}+\delta, \tag{2.9}$$

where δ is given by (2.2). From (2.1), (2.8) and (2.9), we obtain

$$\mathrm{Re}\{H(q(z), zq'(z))\} > 0 \quad (z \in \mathbb{U}).$$

Now we proceed to show that $\mathrm{Re}\{H(is,t)\} \leq 0$ for all real s and $t \leq -(1+s^2)/2$. From (2.9), we have

$$\begin{aligned}\mathrm{Re}\{H(is,t)\} &= \mathrm{Re}\left\{is + \frac{t}{is+\lambda/(1-\alpha)} + \delta\right\} \\ &= \frac{t(\lambda/(1-\alpha))}{|\lambda/(1-\alpha)+is|^2} + \delta \\ &\leq -\frac{E_\delta(s)}{2|\lambda/(1-\alpha)+is|^2},\end{aligned} \tag{2.10}$$

where

$$E_\delta(s) := \left(\frac{\lambda}{1-\alpha} - 2\delta\right)s^2 - \frac{\lambda}{1-\alpha}\left(2\delta\frac{\lambda}{1-\alpha} - 1\right). \tag{2.11}$$

For δ given by (2.2), we can prove easily that the expression $E_\delta(s)$ given by (2.11) is positive or equal to zero. Hence from (2.9), we see that $\mathrm{Re}\{H(is,t)\} \leq 0$ for all real s and $t \leq -(1+s^2)/2$. Thus, by using Lemma 1.1, we conclude that $\mathrm{Re}\{q(z)\} > 0$ for all $z \in \mathbb{U}$. That is, q is convex in $\mathbb{U}$.

Next, we prove that the subordination condition (2.3) implies that

$$F(z) \prec G(z) \tag{2.12}$$

for the functions F and G defined by (2.5). Without loss of generality, we can assume that G is analytic and univalent on $\overline{\mathbb{U}}$ and $G'(\zeta) \neq 0$ for $|\zeta| = 1$. For this purpose, we consider the function $L(z,t)$ given by

$$L(z,t) := G(z) + \frac{(1-\alpha)(1+t)}{\lambda} zG'(z) \quad (z \in \mathbb{U};\ 0 \leq t < \infty).$$

We note that

$$\left.\frac{\partial L(z,t)}{\partial z}\right|_{z=0} = G'(0)\left(\frac{\lambda + (1-\alpha)(1+t)}{\lambda}\right) \neq 0 \quad (0 \leq t < \infty;\ \lambda > 0).$$

This shows that the function

$$L(z,t) = a_1(t)z + \cdots$$

satisfies the condition $a_1(t) \neq 0$ for all $t \in [0,\infty)$. Furthermore, we have

$$\mathrm{Re}\left\{\frac{z\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\right\} = \mathrm{Re}\left\{\frac{\lambda}{1-\alpha} + (1+t)\left(1 + \frac{zG''(z)}{G'(z)}\right)\right\} > 0.$$

Therefore, by virtue of Lemma 1.5, $L(z,t)$ is a subordination chain. We observe from the definition of a subordination chain that

$$L(\zeta,t) \notin L(\mathbb{U},0) = \phi(\mathbb{U}) \quad (\zeta \in \partial\mathbb{U};\ 0 \leq t < \infty)$$

Now suppose that F is not subordinate to G, then by Lemma 1.3, there exists points $z_0 \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U}$ such that

$$F(z_0) = G(\zeta_0) \quad \text{and} \quad z_0F(z_0) = (1+t)\zeta_0 G'(\zeta_0) \quad (0 \leq t < \infty).$$

Hence we have

$$\begin{aligned} L(\zeta_0, t) &= G(\zeta_0) + \frac{(1-\alpha)(1+t)}{\lambda}\zeta_0 G'(\zeta_0) \\ &= F(z_0) + \frac{1-\alpha}{\lambda} z_0 F'(z_0) \\ &= (1-\alpha) z_0 D_\lambda^{n+1} f(z_0) + \alpha z_0 D_\lambda^n f(z_0) \in \phi(\mathbb{U}), \end{aligned}$$

by virtue of the subordination condition (2.3). This contradicts the above observation that $L(\zeta_0, t) \notin \phi(\mathbb{U})$. Therefore, the subordination condition (2.3) must imply the subordination given by (2.12). Considering $F(z) = G(z)$, we see that the function G is the best dominant. This evidently completes the proof of Theorem 2.1. □

We next prove a dual problem of Theorem 2.1, in the sense that the subordinations are replaced by superordinations.

Theorem 2.2. *Let $f, g \in \Sigma$. Suppose that*

$$\mathrm{Re}\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} > -\delta$$

$$\left(\phi(z) := (1-\alpha) z D_\lambda^{n+1} g(z) + \alpha z D_\lambda^n g(z);\ \lambda > 0;\ 0 \le \alpha < 1;\ z \in \mathbb{U}\right),$$

where δ is given by (2.2). *If $(1-\alpha) z D_\lambda^{n+1} f(z) + \alpha z D_\lambda^n f(z)$ is univalent in $\mathbb{U}$ and $z D_\lambda^n f(z) \in \mathcal{H}[1,1] \cap \mathcal{Q}$, then*

$$(1-\alpha) z D_\lambda^{n+1} g(z) + \alpha z D_\lambda^n g(z) \prec (1-\alpha) z D_\lambda^{n+1} f(z) + \alpha z D_\lambda^n f(z) \tag{2.13}$$

implies that

$$z D_\lambda^n g(z) \prec z D_\lambda^n f(z).$$

Moreover, the function $z D_\lambda^n g(z)$ is the best subordinant.

Proof. Let us define the functions F and G, respectively, by (2.5). We first note that, if the function q is defined by (2.6), by using (2.7), then we obtain

$$\begin{aligned} \phi(z) &= G(z) + \frac{1-\alpha}{\lambda} z G'(z) \\ &=: \varphi(G(z), zG'(z)). \end{aligned} \tag{2.14}$$

After a simple calculation, the equation (2.13) yields the relationship:

$$1 + \frac{z\phi''(z)}{\phi'(z)} = q(z) + \frac{zq'(z)}{q(z) + \lambda/(1-\alpha)}.$$

Then by using the same method as in the proof of Theorem 2.1, we can prove that $\mathrm{Re}\{q(z)\} > 0$ for all $z \in \mathbb{U}$. That is, G defined by (2.5) is convex(univalent) in $\mathbb{U}$.

Next, we prove that the subordination condition (2.13) implies that

$$F(z) \prec G(z) \tag{2.15}$$

for the functions F and G defined by (2.5). Now consider the function $L(z,t)$ defined by

$$L(z,t) := G(z) + \frac{(1-\alpha)t}{\lambda} zG'(z) \quad (z \in \mathbb{U};\ 0 \le t < \infty).$$

Since G is convex and $\lambda/(1-\alpha) > 0$, we can prove easily that $L(z,t)$ is a subordination chain as in the proof of Theorem 2.1. Therefore according to Lemma 1.4, we conclude that the superordination condition (2.13) must imply the superordination given by (2.15). Furthermore, since the differential equation (2.14) has the univalent solution G, it is the best subordinant of the given differential superordination. Therefore we complete the proof of Theorem 2.2. □

If we combine this Theorem 2.1 and Theorem 2.2, then we obtain the following sandwich-type theorem.

Theorem 2.3. *Let* $f, g_k \in \Sigma (k=1,2)$. *Suppose that*

$$\mathrm{Re}\left\{1 + \frac{z\phi_k''(z)}{\phi_k'(z)}\right\} > -\delta$$

$$\left(\phi_k(z) := (1-\alpha)zD_\lambda^{n+1} g_k(z) + \alpha z D_\lambda^n g_k(z); k=1,2; \lambda > 0; 0 \le \alpha < 1; z \in \mathbb{U}\right), \tag{2.16}$$

where δ *is given by* (2.2). *If* $(1-\alpha)zD_\lambda^{n+1} f(z) + \alpha z D_\lambda^n f(z)$ *is univalent in* $\mathbb{U}$ *and* $zD_\lambda^n f(z) \in \mathcal{H}[1,1] \cap \mathcal{Q}$, *then*

$$\phi_1(z) \prec (1-\alpha)zD_\lambda^{n+1} f(z) + \alpha z D_\lambda^n f(z) \prec \phi_2(z)$$

implies that

$$zD_\lambda^n g_1(z) \prec D_\lambda^n f(z) \prec zD_\lambda^n g_2(z).$$

Moreover, the functions $zD_\lambda^n g_1(z)$ *and* $zD_\lambda^{n+1} g_2(z)$ *are the best subordinant and the best dominant, respectively.*

The assumption of Theorem 2.3, that the functions $(1-\alpha)zD_\lambda^{n+1} f(z) + \alpha z D_\lambda^n f(z)$ and $zD_\lambda^n f(z)$ need to be univalent in $\mathbb{U}$, may be replaced another condition in the following result.

Corollary 2.1. *Let* $f, g_k \in \Sigma (k=1,2)$. *Suppose that the condition* (2.16) *is satisfied and*

$$\mathrm{Re}\left\{1 + \frac{z\psi''(z)}{\psi'(z)}\right\} > -\delta \tag{2.17}$$

$$\left(\psi(z) := (1-\alpha)zD_\lambda^{n+1} f(z) + \alpha z D_\lambda^n f(z);\ z \in \mathbb{U}\right),$$

where δ *is given by* (2.2). *Then*

$$\phi_1(z) \prec (1-\alpha)zD_\lambda^{n+1} f(z) + \alpha z D_\lambda^n f(z) \prec \phi_2(z)$$

implies that

$$zD_\lambda^n g_1(z) \prec zD_\lambda^n f(z) \prec zD_\lambda^n g_2(z).$$

Moreover, the functions $zD_\lambda^n g_1(z)$ *and* $zD_\lambda^n g_2(z)$ *are the best subordinant and the best dominant, respectively.*

Proof. In order to prove Corollary 2.1, we have to show that the condition (2.17) implies the univalence of $\psi(z)$ and $F(z) := zD_\lambda^n f(z)$. Since δ given by (2.2) in Theorem 2.1 satisfies $0 < \delta \leq 1/2$, the condition (2.17) means that ψ is a close-to-convex function in $\mathbb{U}$ (see [4]) and hence ψ is univalent in $\mathbb{U}$. Furthermore, by using the same techniques as in the proof of Theorem 2.1, we can prove the convexity(univalence) of F and so the details may be omitted. Therefore, from Theorem 2.3, we obtain Corollary 2.1. □

Next, we consider the integral operator F_c $(c > 0)$ defined by

$$F_c(f)(z) := \frac{c}{z^{c+1}} \int_0^z t^c f(t)dt \quad (c > 0). \tag{2.18}$$

The integral operator F_c defined by (2.18) has been widely used in geometric function theory. In particular, Goel and Sohi [3] investigated some integral preserving properties for the classes of starlike, convex and close-to-convex functions, respectively. Moreover, Uralegaddi and Somanatha [14,15] obtained some inclusion relationships for classes of meromorphic functions in connection with the integral operator F_1.

Now, we obtain the following sandwich-type result involving the integral operator defined by (2.18).

Theorem 2.4. *Let* $f, g_k \in \Sigma(k = 1, 2)$. *Suppose that*

$$\operatorname{Re}\left\{1 + \frac{z\phi_k''(z)}{\phi_k'(z)}\right\} > -\delta \tag{2.19}$$

$$(\phi_k(z) := zD_\lambda^n g_k(z);\ k = 1, 2;\ \lambda > 0;\ c > 0;\ z \in \mathbb{U}),$$

where

$$\delta = \frac{1 + c^2 - |1 - c^2|}{4c} \quad (c > 0). \tag{2.20}$$

If $zD_\lambda^n f(z)$ *is univalent in* $\mathbb{U}$ *and* $zD_\lambda^n F_c(f)(z) \in \mathcal{H}[1, 1] \cap \mathcal{Q}$, *then*

$$zD_\lambda^n g_1(z) \prec zD_\lambda^n f(z) \prec zD_\lambda^n g_2(z)$$

implies that

$$zD_\lambda^n F_c(g_1)(z) \prec zD_\lambda^n F_c(f)(z) \prec zD_\lambda^n F_c(g_2)(z).$$

Moreover, the functions $zD_\lambda^n F_c(g_1)(z)$ *and* $zD_\lambda^n F_c(g_2)(z)$ *are the best subordinant and the best dominant, respectively.*

Proof. Let us define the functions F and G_k $(k = 1, 2)$ by

$$F(z) := zD_\lambda^n F_c(f)(z) \ \text{ and } \ G_k(z) := zD_\lambda^n F_c(g_k)(z),$$

respectively. From the definition of the integral operator F_c defined by (2.18), we obtain

$$z(D_\lambda^n F_c(f)(z))' = czD_\lambda^n f(z) - (c + 1)zD_\lambda^n F_c(f)(z) \tag{2.21}$$

Then from (2.19) and (2.21), we have

$$c\phi_k(z) = cG_k(z) + zG_k'(z). \tag{2.22}$$

Setting

$$q_k(z) = 1 + \frac{zG_k''(z)}{G_k'(z)} \quad (k = 1, 2;\ z \in \mathbb{U}),$$

and differentiating both sides of (2.22), we obtain

$$1 + \frac{z\phi_k''(z)}{\phi_k'(z)} = q_k(z) + \frac{zq_k'(z)}{q_k(z) + c}.$$

The remaining part of the proof is similar to that of Theorem 2.1 and so we may omit for the proof involved. □

By using the same methods as in the proof of Corollary 2,1, we have the following result.

Corollary 2.2. *Let* $f, g_k \in \Sigma(k = 1, 2)$. *Suppose that the condition* (2.19) *is satisfied and*

$$\mathrm{Re}\left\{1 + \frac{z\psi''(z)}{\psi'(z)}\right\} > -\delta$$

$$\left(\psi(z) := zD_\lambda^n f(z) :\ \lambda > 0;\ z \in \mathbb{U}\right),$$

where δ *is given by* (2.20). *Then*

$$zD_\lambda^n g_1(z) \prec zD_\lambda^n f(z) \prec zD_\lambda^n g_2(z)$$

implies that

$$zD_\lambda^n F_c(g_1)(z) \prec zD_\lambda^n F_c(f)(z) \prec zD_\lambda^n F_c(g_2)(z).$$

Moreover, the functions $zD_\lambda^n F_c(g_1)(z)$ *and* $zD_\lambda^n F_c(g_2)(z)$ *are the best subordinant and the best dominant, respectively.*

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