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DECOMPOSITION OF CONTINUITY AND COMPLETE CONTINUITY IN SMOOTH FUZZY TOPOLOGICAL SPACES

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Abstract. In this paper, fuzzy α^*-set, fuzzy C-set, fuzzy AB-set, fuzzy t-set, fuzzy B-set, etc., are introduced in the sense of Sostak [12] and Ramadan [9]. By using these sets, a decomposition of fuzzy continuity and complete fuzzy continuity are provided. Characterization of smooth fuzzy extremally disconnected spaces is also obtained in this connection.

1. Introduction and preliminaries

The concept of fuzzy set was introduced by Zadeh [17] in his classical paper. Fuzzy sets have applications in many fields such as information [11] and control [15]. Chang [1] introduced the notion of a fuzzy topology. Later Lowen [6] redefined what is now known as stratified fuzzy topology. Sostak [12] introduced the notion of fuzzy topology as an extension of Chang and Lowen's fuzzy topology. Later on he has developed the theory of fuzzy topological spaces in [13] and [14]. After that several authors [2],[3],[4],[5],[6],[7] and [8] have reintroduced the same definition and studied fuzzy topological spaces being unaware of Sostak's work. They refered to the fuzzy topology in the sense of Chang and Lowen as the topology on fuzzy subsets. Fuzzy β-regular sets, the class of weak fuzzy AB-sets and a new characterization of fuzzy extremally disconnected spaces are introduced and studied by Uma, Roja and Balasubramanian [16]. In this paper r-fuzzy β-regular sets, the class of r-weak fuzzy AB-sets are introduced and studied. Decomposition of fuzzy continuous functions are also proved.

Throughout this paper, let X be a nonempty set, $I = [0, 1]$ and $I_0 = (0, 1]$. For $\langle \in I, T(x) = \langle$ for all $x \in X$. For $\alpha \in I, \overline{\alpha}(x) = \alpha$ for all $x \in X$.

Definition 1.1. [12] *A function $T : I^X \to I$ is called a smooth fuzzy topology on X if it satisfies the following conditions :*

(1) $T(\overline{0}) = T(\overline{1}) = 1$,

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(2) $T(\mu_1 \wedge \mu_2) \geq T(\mu_1) \wedge T(\mu_2)$ *for any* $\mu_1, \mu_2 \in I^X$,
(3) $T(\vee_{j\in\Gamma}\mu_j) \geq \wedge_{j\in\Gamma}T(\mu_j)$ *for any* $\{\mu_j\}_{j\in\Gamma} \in I^X$.

The pair (X, T) *is called a smooth fuzzy topological space.*

Remark 1.1. Let (X, T) be a smooth fuzzy topological space. Then, for each $r \in I_0, T_r = \{\mu \in I^X : T(\mu) \geq r\}$ is Chang's fuzzy topology on X.

Definition 1.2. [10] *Let* (X, T) *be a smooth fuzzy topological space. For each* $\lambda \in I^X$, $r \in I_0$, *an operator* $C_T : I^X \times I_0 \to I^X$ *is defined as follows:* $C_T(\lambda, r) = \wedge\{\mu : \mu \geq \lambda, T(\overline{1}-\mu) \geq r\}$. *For* $\lambda, \mu \in I^X$ *and* $r, s \in I_0$, *it satisfies the following conditions:*

(1) $C_T(\overline{0}, r) = \overline{0}$.
(2) $\lambda \leq C_T(\lambda, r)$.
(3) $C_T(\lambda, r) \vee C_T(\mu, r) = C_T(\lambda \vee \mu, r)$.
(4) $C_T(\lambda, r) \leq C_T(\lambda, s)$, *if* $r \leq s$.
(5) $C_T(C_T(\lambda, r), r) = C_T(\lambda, r)$.

Proposition 1.1. [10] *Let* (X, T) *be a smooth fuzzy topological space. For each* $\lambda \in I^X, r \in I_0$, *an operator* $I_T : I^X \times I_0 \to I^X$ *is defined as follows:* $I_T(\lambda, r) = \vee\{\mu : \mu \leq \lambda, T(\mu) \geq r\}$. *For* $\lambda, \mu \in I^X$ *and* $r, s \in I_0$, *it satisfies the following conditions :*

(1) $I_T(\overline{1} - \lambda, r) = \overline{1} - C_T(\lambda, r)$.
(2) $I_T(\overline{1}, r) = \overline{1}$.
(3) $\lambda \geq I_T(\lambda, r)$.
(4) $I_T(\lambda, r) \wedge I_T(\mu, r) = I_T(\lambda \wedge \mu, r)$.
(5) $I_T(\lambda, r) \geq I_T(\lambda, s)$, *if* $r \leq s$.
(6) $I_T(I_T(\lambda, r), r) = I_T(\lambda, r)$.

Definition 1.3. [9] Let (X, T) be a smooth fuzzy topological space. For $\lambda \in I^X$ and $r \in I_0$, λ is called

(1) r-fuzzy semiopen (briefly, r-fso) if $\lambda \leq C_T(I_T(\lambda, r), r)$.
(2) r-fuzzy semiclosed (briefly, r-fsc) if $\lambda \geq I_T(C_T(\lambda, r), r)$.

Definition 1.4. [9] *Let* (X, T) *be a smooth fuzzy topological space. For* $\lambda \in I^X$ *and* $r \in I_0$,

(1) $SI_T(\lambda, r) = \vee\{\mu \in I^X : \mu \leq \lambda, \mu$ *is r-fso* $\}$ *is called r-fuzzy semi interior of* λ.
(2) $SC_T(\lambda, r) = \wedge\{\mu \in I^X : \mu \geq \lambda, \mu$ *is r-fsc* $\}$ *is called r-fuzzy semi closure of* λ.

2. Decomposition of r-fuzzy regular sets

In this section, r-fuzzy regular open set, r-fuzzy β-open set, r-fuzzy α^*-set, r-fuzzy C-set, r-fuzzy AB-set, r-fuzzy t-set, r-fuzzy B-set, etc. are introduced. Further interrelations among them are discussed providing counter examples wherever necessary.

Definition 2.1. Let (X, T) be a smooth fuzzy topological space. For $\lambda \in I^X$ and $r \in I_0$,λ is called

(1) r-fuzzy preopen if $\lambda \leq I_T(C_T(\lambda, r), r)$.
(2) r-fuzzy regular open if $\lambda = I_T(C_T(\lambda, r), r)$.
(3) r-fuzzy -open if $\lambda \leq C_T(I_T(C_T(\lambda, r), r), r)$.
(4) r-fuzzy -open if $\lambda \leq I_T(C_T(I_T(\lambda, r), r), r)$.

Note 2.1. The complements of r-fuzzy regular open and r-fuzzy β-open sets are r-fuzzy regular closed and r-fuzzy β-closed sets respectively.

Definition 2.2. Let (X, T) be a smooth fuzzy topological space. For $\lambda \in I^X$ and $r \in I_0$,

(1) β-$C_T(\lambda, r) = \wedge\{\mu \in I^X : \mu \geq \lambda$ and μ is r-fuzzy β-closed $\}$ is called r-fuzzy β-closure of λ.
(2) β-$I_T(\lambda, r) = \vee\{\mu \in I^X : \mu \leq \lambda$ and μ is r-fuzzy β-open$\}$ is called r-fuzzy β-interior of λ.

Definition 2.3. A smooth fuzzy topological space (X, T) is said to be smooth fuzzy extremally disconnected if $\lambda \in T$ implies that $C_T(\lambda, r) \in T$.

Definition 2.4. Let (X, T) be a smooth fuzzy topological space. For $\lambda, \mu, \gamma \in I^X$ and $r \in I_0$, λ is called a

(1) r-fuzzy α^*-set if $I_T(\lambda, r) = I_T(C_T(I_T(\lambda, r), r), r)$.
(2) r-fuzzy C-set if $\lambda = \mu \wedge \gamma$ where $T(\mu) \geq r$and γ is r-fuzzy α^*-set.
(3) r-fuzzy semiregular set if it is r-fuzzy semiopen and r-fuzzy semiclosed.
(4) r-fuzzy AB-set if $\lambda = \mu \wedge \gamma$ where $T(\mu) \geq r$ and γ is r-fuzzy semi regular.
(5) r-fuzzy t-set if $I_T(\lambda, r) = I_T(C_T(\lambda, r), r)$.
(6) r-fuzzy B-set if $\lambda = \mu \wedge \gamma$ where $T(\mu) \geq r$ and γ is r-fuzzy t-set.

Definition 2.5. Let (X, T) be a smooth fuzzy topological space. For $\lambda \in I^X$ and $r \in I_0$, λ is called r-fuzzy β-regular if it is both r-fuzzy β-open and r-fuzzy β-closed.

Remark 2.1. Every r-fuzzy semiregular set is r-fuzzy β-regular but not conversely.

Example 2.1. Let $X = \{a, b\}$. Let $\lambda_1, \lambda_2 \in I^X$, where $\lambda_1(a) = \frac{1}{2}, \lambda_1(b) = \frac{1}{2}; \lambda_2(a) = 0, \lambda_2(b) = \frac{1}{2}$. Define a smooth fuzzy topology $T : I^X \to I$ as follows :

$$T(\lambda) = \begin{cases} 1, & \lambda = \overline{0} \quad or \quad \overline{1}; \\ \frac{1}{2}, & \lambda = \lambda_1; \\ \frac{2}{3}, & \lambda = \lambda_2; \\ 0, & otherwise. \end{cases}$$

Let $\mu \in I^X$ be defined as $\mu(a) = \frac{1}{2}, \mu(b) = 0$. Let $r = \frac{1}{2}$. Then $\mu \leq C_T(I_T(C_T(\mu, \frac{1}{2}), \frac{1}{2}), \frac{1}{2})$. Hence μ is $\frac{1}{2}$-fuzzyβ-open. Similarly μ is $\frac{1}{2}$-fuzzy β-closed since $\mu \geq I_T(C_T(I_T(\mu, \frac{1}{2}), \frac{1}{2}), \frac{1}{2})$.Thus μ is $\frac{1}{2}$-fuzzy β-regular.But μ is not $\frac{1}{2}$-fuzzy semiregular since $\mu \nleq C_T(I_T(\mu, \frac{1}{2}), \frac{1}{2})$ and $\mu \ngeq I_T(C_T(\mu, \frac{1}{2}), \frac{1}{2})$.

Definition 2.6. Let (X, T) be a smooth fuzzy topological space. For $\lambda \in I^X$ and $r \in I_0$,λ is called a r-weak fuzzy AB-set if $\lambda = \mu \wedge \gamma$ where $T(\mu) \geq r$ and γ is r-fuzzy β-regular. The collection of all r-weak fuzzy AB-sets in (X, T) will be denoted by r-$wfAB(X)$.

Remark 2.2. For some r-fuzzy sets defined above, we have the following implications:

$$\begin{array}{ccccc} r-fuzzy\ \ semi\ \ regular & \rightarrow & r-fuzzy\ \ AB-set & \rightarrow & r-fuzzy\ \ B-set \\ \downarrow & & \downarrow & & \downarrow \\ r-fuzzy\ \ \beta-regular & \rightarrow & r-weakfuzzy\ \ AB-set & \rightarrow & r-fuzzy\ \ C-set \end{array}$$

The following examples show that the converse of above implications need not be true.

Example 2.2. Let $X = \{a, b\}$. Let $\lambda_1, \lambda_2 \in I^X$, where $\lambda_1(a) = \frac{1}{2}, \lambda_1(b) = 1; \lambda_2(a) = 0$, $\lambda_2(b) = 1$. Define a smooth fuzzy topology $T : I^X \to I$ as follows :

$$T(\lambda) = \begin{cases} 1, & \lambda = \overline{0} \quad or \quad \overline{1}; \\ \frac{1}{2}, & \lambda = \lambda_1; \\ \frac{2}{3}, & \lambda = \lambda_2; \\ 0, & otherwise. \end{cases}$$

Let $\gamma \in I^X$ be defined by $\gamma(a) = \frac{1}{2}, \gamma(b) = 0$. Let $r = \frac{1}{2}$. Then $I_T(\gamma, \frac{1}{2}) = I_T(C_T(\gamma, \frac{1}{2}), \frac{1}{2})$. Hence γ-is a $\frac{1}{2}$ -fuzzy t-set. Here $\gamma = \lambda_1 \wedge \gamma$, $T(\lambda_1) \geq \frac{1}{2}$ and γ is a $\frac{1}{2}$-fuzzy t-set. Thus γ is a $\frac{1}{2}$-fuzzy B-set. Also, $I_T(\gamma, \frac{1}{2}) = I_T(C_T(I_T(\gamma, \frac{1}{2}), \frac{1}{2}), \frac{1}{2})$. Thus γ is a $\frac{1}{2}$-fuzzy α^*-set and hence γ is a $\frac{1}{2}$-fuzzy C-set. But $\gamma \nleq C_T(I_T(C_T(\gamma, \frac{1}{2}), \frac{1}{2}), \frac{1}{2})$. Hence γ is not $\frac{1}{2}$-fuzzy β-regular. Thus γ is not a $\frac{1}{2}$-weak fuzzy AB-set.

Example 2.3. Let $X = \{a, b\}$. Let $\lambda_1, \lambda_2 \in I^X$, where $\lambda_1(a) = \frac{1}{2}, \lambda_1(b) = \frac{1}{2}$;$\lambda_2(a) = 0, \lambda_2(b) = \frac{1}{2}$. Define a smooth fuzzy topology $T : I^X \to I$ as follows :

$$T(\lambda) = \begin{cases} 1, & \lambda = \overline{0} \quad or \quad \overline{1}; \\ \frac{1}{2}, & \lambda = \lambda_1; \\ \frac{2}{3}, & \lambda = \lambda_2; \\ 0, & otherwise. \end{cases}$$

Let $\gamma \in I^X$ be defined by $\gamma(a) = \frac{1}{2}, \gamma(b) = 0$. Let $r = \frac{1}{2}$. Then $\gamma \leq C_T(I_T(C_T(\gamma, \frac{1}{2}), \frac{1}{2}), \frac{1}{2})$. Hence γ is $\frac{1}{2}$-fuzzy β-open. And also $\gamma \geq I_T(C_T (I_T(\gamma, \frac{1}{2}), \frac{1}{2}), \frac{1}{2})$. Hence γ is $\frac{1}{2}$-fuzzy β-closed. Thus γ is $\frac{1}{2}$-fuzzy β-regular.Here $\gamma = \lambda_1 \wedge \gamma$ where $T(\lambda_1) \geq r$ and γ is $\frac{1}{2}$-fuzzy β-regular. Hence γ is a $\frac{1}{2}$-weak fuzzy AB-set. Also, $I_T(\gamma, \frac{1}{2}) = I_T(C_T(I_T(\gamma, \frac{1}{2}), \frac{1}{2}), \frac{1}{2})$. Hence γ is a $\frac{1}{2}$-fuzzy α^*-set and thus it is a $\frac{1}{2}$-fuzzy C-set. But $I_T(\gamma, \frac{1}{2}) \neq I_T(C_T(\gamma, \frac{1}{2}), \frac{1}{2})$. Thus γ is not a $\frac{1}{2}$-fuzzy t-set and hence it is not a $\frac{1}{2}$-fuzzy B-set. Also, $\gamma \not\leq C_T(I_T(\gamma, \frac{1}{2}), \frac{1}{2})$ which implies γ is not $\frac{1}{2}$-fuzzy semiopen. And $\gamma \not\geq I_T(C_T(\gamma, \frac{1}{2}), \frac{1}{2})$. Therefore γ is not $\frac{1}{2}$-fuzzy semiclosed. Hence γ is not a $\frac{1}{2}$-fuzzy semiregular set.

Remark 2.3. By Examples 2.2 and 2.3, it follows that r-weak fuzzy AB-sets and r-fuzzy B-sets are independent notions.

Example 2.4. Let $X = \{a, b\}$. Let $\lambda_1, \lambda_2 \in I^X$, where$\lambda_1(a) = \frac{1}{2}, \lambda_1(b) = 1; \lambda_2(a) = 0, \lambda_2(b) = 1$. Define a smooth fuzzy topology $T : I^X \to I$ as follows :

$$T(\lambda) = \begin{cases} 1, & \lambda = \overline{0} \quad or \quad \overline{1}; \\ \frac{1}{2}, & \lambda = \lambda_1; \\ \frac{2}{3}, & \lambda = \lambda_2; \\ 0, & otherwise. \end{cases}$$

Let $r = \frac{1}{2}$. Then λ_1 is a $\frac{1}{2}$-fuzzy AB-set but not a $\frac{1}{2}$-fuzzy semiregular set.

Example 2.5. Let $X = \{a, b\}$. Let $\lambda_1, \lambda_2 \in I^X$, where $\lambda_1(a) = \frac{1}{2}, \lambda_1(b) = 1; \lambda_2(a) = 0, \lambda_2(b) = 1$. Define a smooth fuzzy topology $T : I^X \to I$ as follows :

$$T(\lambda) = \begin{cases} 1, & \lambda = \overline{0} \quad or \quad \overline{1}; \\ \frac{1}{2}, & \lambda = \lambda_1; \\ \frac{2}{3}, & \lambda = \lambda_2; \\ 0, & otherwise. \end{cases}$$

Let $r = \frac{1}{2}$. Then λ_1 is a $\frac{1}{2}$-weak fuzzy AB-set but not a β-regular set.

Example 2.6. Let $X = \{a, b\}$. Let $\lambda_1, \lambda_2 \in I^X$, where $\lambda_1(a) = \frac{1}{2}, \lambda_1(b) = 1; \lambda_2(a) = 0, \lambda_2(b) = 1$. Define a smooth fuzzy topology $T : I^X \to I$ as follows :

$$T(\lambda) = \begin{cases} 1, & \lambda = \overline{0} \quad or \quad \overline{1}; \\ \frac{1}{2}, & \lambda = \lambda_1; \\ \frac{2}{3}, & \lambda = \lambda_2; \\ 0, & otherwise. \end{cases}$$

Let $\gamma \in I^X$ be defined by $\gamma(a) = \frac{1}{2}, \gamma(b) = 0$. Let $r = \frac{1}{2}$. Then $I_T(\gamma, \frac{1}{2}) = I_T(C_T(\gamma, \frac{1}{2}), \frac{1}{2})$. Hence γ is a $\frac{1}{2}$-fuzzy t-set. Here $\gamma = \lambda_1 \wedge \gamma$ where $T(\lambda_1) \geq \frac{1}{2}$. Thus γ is a $\frac{1}{2}$-fuzzy B-set but not a $\frac{1}{2}$-fuzzy AB-set since γ is neither $\frac{1}{2}$-fuzzy semiopen nor $\frac{1}{2}$-fuzzy semiclosed.

Example 2.7. In Example 2.1, $\mu = \lambda_1 \wedge \mu$ where $T(\lambda_1) \geq r$ and μ is $\frac{1}{2}$-fuzzy β-regular but not $\frac{1}{2}$-fuzzy semiregular. Hence μ is a $\frac{1}{2}$-weak fuzzy AB-set but not a $\frac{1}{2}$-fuzzy AB-set.

Proposition 2.1. *Let (X, T) be a smooth fuzzy topological space. For $\lambda \in I^X$ and $r \in I_0$, the following statements are equivalent :*

(i) *λ is r-fuzzy β-regular.*
(ii) *λ is a r-fuzzy β-closed set and a r-weak fuzzy AB-set.*

Proof. The proof of $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (i)$ are clear from the definitions. □

Proposition 2.2. *Let (X, T) be a smooth fuzzy topological space. For $\lambda \in I^X$ and $r \in I_0$, the following statements are equivalent :*

(i) *$T(\lambda) \geq r$.*
(ii) *λ is a r-weak fuzzy AB-set and a r-fuzzy α-open set.*

Proof. The proof of $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (i)$ are clear from the definitions. □

3. Decomposition of fuzzy continuous functions and complete fuzzy continuous functions

In this section, fuzzy α-continuous and weak fuzzy AB-continuous functions are introduced. Decomposition of fuzzy continuous and complete fuzzy continuous functions are studied.

Definition 3.1. Let (X, T) and (Y, S) be any two smooth fuzzy topological spaces. A function $f : (X, T) \to (Y, S)$ is said to be weak fuzzy AB-continuous (resp. fuzzy AB-continuous) if for each $\lambda \in I^Y$ with $S(\lambda) \geq r$, $f^{-1}(\lambda) \in I^X$ is a r-weak fuzzy AB-set (resp. r-fuzzy AB-set).

Definition 3.2. Let (X, T) and (Y, S) be any two smooth fuzzy topological spaces. A function $f : (X, T) \to (Y, S)$ is said to be fuzzy α-continuous (resp. fuzzy C-continuous) if for each $\lambda \in I^Y$ with $S(\lambda) \geq r, f^{-1}(\lambda) \in I^X$ is r-fuzzy α-open (resp. r-fuzzy C-set).

Proposition 3.1. *Let (X, T) and (Y, S) be any two smooth fuzzy topological spaces. For any function $f : (X, T) \to (Y, S)$ the following statements are equivalent :*

(i) *f is fuzzy continuous.*
(ii) *f is weak fuzzy AB-continuous and fuzzy α-continuous.*

Proof. $(i) \Rightarrow (ii)$. The proof of $(i) \Rightarrow (ii)$ is clear from the definitions. $(ii) \Rightarrow (i)$. The proof of $(ii) \Rightarrow (i)$ follows from the Proposition 2.2. □

Proposition 3.2. *Let (X, T) and (Y, S) be any two smooth fuzzy topological spaces. For any function $f : (X, T) \to (Y, S)$ the following statements hold :*

(i) *Every fuzzy AB-continuous function is weak fuzzy AB-continuous.*
(ii) *Every weak fuzzy AB-continuous function is fuzzy C-continuous.*
(iii) *Every weak fuzzy AB-continuous function is fuzzy β-continous.*

Proof. (i) Since every r-fuzzy AB-set is r-weak fuzzy AB set, (i) holds. (ii) Since every r-weak fuzzy AB-set is r-fuzzy C-set, (ii) holds. (iii) Assume that f is weak fuzzy AB-continuous. Let $\lambda \in I^Y$ be such that $S(\lambda) \geq r$. Since f is weak fuzzy AB-continuous, $f^{-1}(\lambda)$ is a r-weak fuzzy AB-set. That is, $f^{-1}(\lambda) = \gamma \wedge \mu$ where $T(\gamma) \geq r$ and μ is r-fuzzy β-regular. Since $T(\gamma) \geq r$, γ is r-fuzzy β-open. Since μ is r-fuzzy β-regular, μ is r-fuzzy β-open. Hence $f^{-1}(\lambda)$ is r-fuzzy β-open and f is fuzzy β-continuous. □

Definition 3.3. Let (X, T) be a smooth fuzzy topological space. For $\lambda \in I^X$ and $r \in I_0$, λ is called

(1) r-fuzzy generalised semiclosed (briefly, r-fgs-closed) if $SC_T(\lambda, r) \leq \mu$ whenever $\lambda \leq \mu$ and $T(\mu) \geq r$.
(2) r-fsg^*-closed (resp. r-fuzzy semi $g\beta$-closed) if $SC_T(\lambda, r) \leq \mu$ (resp. β-$C_T(\lambda, r) leq \mu$) where $\lambda \leq \mu$ and μ is r-fuzzy semiopen.

Proposition 3.3. *Let (X, T) be a smooth fuzzy topological space. For $\lambda \in I^X$ and $r \in I_0$, the following relations hold : $(i) \Rightarrow (ii) \Rightarrow (iii)$ where*

(i) *λ is r-fuzzy regular open.*
(ii) *λ is r-fsg^*-closed and $T(\lambda) \geq r$.*
(iii) *λ is r-fuzzy semi $g\beta$-closed and $T(\lambda) \geq r$.*

Proof. $(i) \Rightarrow (ii)$. Let λ be any r fuzzy regular open. Then $T(\lambda) \geq r$ and $\lambda = I_T(C_T(\lambda, r), r)$. That is, $\lambda \geq I_T(C_T(\lambda, r), r)$. Therefore λ is r-fuzzy semiclosed and hence it is r-fsg^*-closed. $(ii) \Rightarrow (iii)$. Let λ be r-fsg^*-closed set with $T(\lambda) \geq r$. Since λ is r-fsg^*-closed, $SC_T(\lambda, r) \leq \mu$ whenever $\lambda \leq \mu$ and μ is r-fuzzy semiopen. Since every r-fuzzy semiclosed set is r-fuzzy β-closed, $SC_T(\lambda, r)$ is r-fuzzy β-closed. Therefore β-$C_T(\lambda, r) \leq SC_T(\lambda, r) \leq \mu$. Hence (iii) is proved. □

Proposition 3.4. *Let (X, T) be a smooth fuzzy topological space. For $\lambda \in I^X$ and $r \in I_0$, the following relations hold : $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v)$ where*

(i) *λ is r-fuzzy regular open.*
(ii) *λ is r-fuzzy semiregular and $T(\lambda) \geq r$.*
(iii) *λ is r-fuzzy semiclosed and $T(\lambda) \geq r$.*
(iv) *λ is r-fgs-closed and $T(\lambda) \geq r$.*
(v) *λ is r-fuzzy α-open and r-fgs-closed.*

Proof. $(i) \Rightarrow (ii)$. Let λ be a r-fuzzy regular open set. Then $T(\lambda) \geq r$ and $\lambda = I_T(C_T(\lambda, r), r)$. That is,

$$\lambda \geq I_T(C_T(\lambda, r), r). \tag{3.1}$$

and

$$\lambda \leq I_T(C_T(\lambda, r), r). \tag{3.2}$$

Since

$$\lambda \leq C_T(\lambda, r) \quad and \quad T(\lambda) \geq r, \lambda \leq C_T(I_T(\lambda, r), r). \tag{3.3}$$

From (3.1) and (3.3) if follows that λ is r-fuzzy semi closed and r-fuzzy semiopen. Hence λ is r-fuzzy semiregular. The proof of $(ii) \Rightarrow (iii)$, $(iii) \Rightarrow (iv)$ and $(iv) \Rightarrow (v)$ are clear from the definitions. □

Definition 3.4. Let (X, T) be a smooth fuzzy topological space. For $\lambda \in I^X$ and $r \in I_0$, λ is called r-fuzzy pre $g\beta$-closed (resp. r-fuzzy α-$g\beta$-closed) if $\beta C_T(\lambda, r) \leq \mu$ whenever $\lambda \leq \mu$ and μ is r-fuzzy preopen (resp. r-fuzzy α-open).

Remark 3.1. r-fuzzy pre $g\beta$-closed implies r-fuzzy α-$g\beta$-closed.

Remark 3.2. For some r-fuzzy sets defined above, we have the following implications:

$$\begin{array}{lll} r\text{–}fuzzy\ semiclosed \rightarrow & r\text{–}fsg^*\text{–}closed & \rightarrow r\text{–}fuzzy\ semi\ g\beta\text{–}closed \\ \qquad\downarrow & & \\ r\text{–}fuzzy\ \beta\text{–}closed & \rightarrow r\text{–}fuzzy\ preg\beta\text{–}closed & \end{array}$$

Example 3.1. In example 2.1 μ is $\frac{1}{2}$-fuzzy β-closed but not $\frac{1}{2}$-fuzzy semiclosed.

Example 3.2. Let $X = \{a, b\}$. Let $\lambda_1, \lambda_2 \in I^X$, where $\lambda_1(a) = \frac{1}{2}, \lambda_1(b) = \frac{1}{4}; \lambda_2(a) = \frac{1}{2}, \lambda_2(b) = \frac{2}{3}$. Define a smooth fuzzy topology $T : I^X \to I$ as follows :

$$T(\lambda) = \begin{cases} 1, & \lambda = \overline{0} \quad or \quad \overline{1}; \\ \frac{1}{2}, & \lambda = \lambda_1; \\ \frac{2}{3}, & \lambda = \lambda_2; \\ 0, & otherwise. \end{cases}$$

Let $\mu \in I^X$ be defined as $\mu(a) = \frac{1}{2}, \mu(b) = \frac{1}{2}$. Let $r = \frac{1}{2}$. Then β-$C_T(\mu, \frac{1}{2}) \leq \lambda$ whenever $\mu \leq \lambda$ and λ is $\frac{1}{2}$-fuzzy semiopen. Hence μ is $\frac{1}{2}$-fuzzy semi $g\beta$-closed but not $\frac{1}{2}$-fsg^*-closed since $SC_T(\mu, \frac{1}{2}) \not\leq \lambda$ whenever $\lambda \leq \lambda$ and λ is $\frac{1}{2}$-fuzzy semiopen.

Example 3.3. Let $X = \{a, b\}$. Let $\lambda_1, \lambda_2 \in I^X$, where $\lambda_1(a) = \frac{1}{3}, \lambda_1(b) = 0; \lambda_2(a) = \frac{1}{2}, \lambda_2(b) = \frac{1}{3}$. Define a smooth fuzzy topology $T : I^X \to I$ as follows :

$$T(\lambda) = \begin{cases} 1, & \lambda = \overline{0} \quad or \quad \overline{1}; \\ \frac{1}{2}, & \lambda = \lambda_1; \\ \frac{2}{3}, & \lambda = \lambda_2; \\ 0, & otherwise. \end{cases}$$

Let $\mu \in I^X$ be defined as $\mu(a) = \frac{1}{3}, \mu(b) = \frac{1}{2}$. Let $r = \frac{1}{2}$. Then β-$C_T(\mu, \frac{1}{2}) \leq \lambda$ whenever $\mu \leq \lambda$ and λ is $\frac{1}{2}$- fuzzy preopen. Hence μ is $\frac{1}{2}$-fuzzy pre-$g\beta$-closed but not $\frac{1}{2}$-fuzzy β-closed since $\mu \ngeq I_T(C_T(I_T(\mu, \frac{1}{2}), \frac{1}{2}), \frac{1}{2})$.

Example 3.4. Let $X = \{a, b\}$. Let $\lambda_1, \lambda_2 \in I^X$, where $\lambda_1(a) = \frac{3}{4}, \lambda_1(b) = \frac{3}{4}; \lambda_2(a) = \frac{1}{4}, \lambda_2(b) = \frac{1}{2}$. Define a smooth fuzzy topology $T : I^X \to I$ as follows :

$$T(\lambda) = \begin{cases} 1, & \lambda = \overline{0} \quad or \quad \overline{1}; \\ \frac{1}{2}, & \lambda = \lambda_1; \\ \frac{2}{3}, & \lambda = \lambda_2; \\ 0, & otherwise. \end{cases}$$

Let $r = \frac{1}{2}$. Let $\lambda \in I^X$ be defined as $\lambda(a) = \frac{3}{4}; \lambda(b) = \frac{1}{4}$. Then $\lambda \ngeq I_T(C_T(\lambda, \frac{1}{2}), \frac{1}{2})$. Hence λ is not $\frac{1}{2}$-fuzzy semiclosed. Also, $SC_T(\lambda, \frac{1}{2}) \leq \mu$ whenever $\lambda \leq \mu$ and μ is $\frac{1}{2}$-fuzzy semiopen. Therefore λ is $\frac{1}{2}$-fsg^*-closed but not $\frac{1}{2}$-fuzzy semiclosed.

Proposition 3.5. *Let (X, T) be a smooth fuzzy topological space. For $\lambda \in I^X$, and $r \in I_0$, the following conditions are equivalent :*

(i) *λ is r-fuzzy regular open.*
(ii) *λ is r-fuzzy β-regular and $T(\lambda) \geq r$.*
(iii) *λ is r-fuzzy β-closed and $T(\lambda) \geq r$.*
(iv) *λ is r-fuzzy pre $g\beta$-closed and $T(\lambda) \geq r$.*
(v) *λ is r-fuzzy α-open and r-fuzzy pre $g\beta$-closed.*
(vi) *λ is r-fuzzy α-open and r-fuzzy α-$g\beta$-closed.*

Proof. $(i) \Rightarrow (ii)$. Since every r-fuzzy semiclosed set is r-fuzzy β-closed and every r-fuzzy semiopen set is r-fuzzy β-open with $T(\lambda) \geq r$, it follows that λ is r-fuzzy β-regular.

$(ii) \Rightarrow (iii), (iii) \Rightarrow (iv), (iv) \Rightarrow (v)$ and $(v) \Rightarrow (vi)$ are clear from the definitions.

$(vi) \Rightarrow (i)$. Let λ be any r-fuzzy α-open and r-fuzzy α-$g\beta$-closed set. Then β-$C_T(\lambda, r) \leq \lambda$. Hence λ is r-fuzzy β-closed.

That is, $I_T(C_T(I_T(\lambda, r), r), r) \leq \lambda$. Since every r-fuzzy α-open set is r-fuzzy semiopen. λ is r-fuzzy semiopen and hence $\lambda \leq C_T(I_T(\lambda, r), r)$. Since λ is r-fuzzy α-open,$\lambda \leq I_T(C_T(I_T(\lambda, r), r), r) \leq I_T(C_T(\lambda, r), r)$. Since λ is r-fuzzy β-closed and $\lambda \leq C_T(I_T(\lambda, r), r), C_T(\lambda, r) \leq C_T(I_T(\lambda, r), r)$ and $I_T(C_T(\lambda, r), r) \leq I_T(C_T(I_T(\lambda, r), r), r) \leq \lambda$. That is, $\lambda = I_T(C_T((\lambda, r), r)$. Therefore λ is r-fuzzy regular open. □

Definition 3.5. Let (X, T) and (Y, S) be any two smooth fuzzy topological spaces. A function $f : (X, T) \to (Y, S)$ is said to be fuzzy completely continuous (resp. fuzzy β - perfectly continuous) if for each $\lambda \in I^Y$ and $r \in I_0$ with $S(\lambda) \geq r, f^{-1}(\lambda) \in I^X$ is r-fuzzy regular open (resp. r-fuzzy β-regular).

Definition 3.6. Let (X, T) and (Y, S) be any two smooth fuzzy topological spaces. A function $f : (X, T) \to (Y, S)$ is said to be fuzzy contra α-$g\beta$ continuous (resp. fuzzy contra pre-$g\beta$ continuous) if for each $\lambda \in I^Y$ and $r \in I_0$ with $S(\lambda) \geq r, f^{-1}(\lambda) \in I^X$ is r-fuzzy α-$g\beta$-closed (resp. r-fuzzy pre-$g\beta$-closed).

By Proposition 3.5, we obtain the following decompositions of complete fuzzy continuity.

Proposition 3.6. *Let (X, T) and (Y, S) be any two smooth fuzzy topological spaces. For a function $f : (X, T) \to (Y, S)$ the following conditions are equivalent :*

(i) *f is fuzzy completely continuous.*
(ii) *f is fuzzy continuous and fuzzy β-perfectly continuous.*
(iii) *f is fuzzy continuous and fuzzy contra pre-$g\beta$-continuous.*
(iv) *f is fuzzy α-continuous and fuzzy contra pre-$g\beta$-continuous.*
(v) *f is fuzzy α-continuous and fuzzy α-$g\beta$-continuous.*

4. Characterization of fuzzy extremally disconnected spaces

In this section, making use of the concept of r-weak fuzzy AB-set, characterization of fuzzy extremally disconnected spaces is established.

Proposition 4.1. *Let (X, T) be a smooth fuzzy topological space. For $\lambda \in I^X$ and $r \in I_0$ the following statements are equivalent :*

(i) *(X, T) is smooth fuzzy extremally disconnected.*
(ii) *$T = wfAB(X)$.*
(iii) *Every r-weak fuzzy AB-set $\lambda \in I^X, T(\lambda) \geq r$.*

Proof. $(i) \Rightarrow (ii)$. Let (X, T) be a smooth fuzzy extremally disconnected space. Let $\lambda \in I^X$ be $T(\lambda) \geq r$. Then $\lambda = \lambda \wedge 1_X$ where $T(\lambda) \geq r$ and 1_X is r-fuzzy β-regular. Then λ is r-weak fuzzy AB-set. Since every r-weak fuzzy AB-set is r-fuzzy C-set and hence $T(\lambda) \geq r$. This proves that $T = wfAB(X)$. $(ii) \Rightarrow (iii)$. Since $T = wfAB(X)$, for every r-weak fuzzy AB-set $\lambda \in I^X$,$T(\lambda) \geq r$. $(iii) \Rightarrow (i)$. Let $\lambda \in I^X$ be such that $T(\lambda) \geq r$. $C_T(\lambda, r) = 1_X \wedge C_T(\lambda, r)$. Since $T(\lambda) \geq r, \lambda = I_T(\lambda, r)$.Now, $C_T(\lambda, r) = C_T(I_T(\lambda, r), r) \leq C_T(I_T(C_T(\lambda, r), r), r) \leq C_T(I_T(C_T(C_T(\lambda, r), r), r), r)$. Therefore $C_T(\lambda, r)$ is r-fuzzy β-open. Since every $\mu \in I^X$ with $T(\overline{1} - \mu) \geq r$ is r-fuzzy β-closed,$C_T(\lambda, r)$ is r-fuzzy β-closed. Thus $C_T(\lambda, r)$ is r-fuzzy β-regular and hence is r-weak fuzzy AB-set. Since $\lambda \in I^X$ is r-weak fuzzy AB-set, $C_T(\lambda, r)$ satisfies $T(C_T(\lambda, r)) \geq r$. Therefore (X, T) is smooth fuzzy extremally disconnected. □

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