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ON FUZZY PRIME SUBMODULES OF FUZZY MULTIPLICATION MODULES

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Abstract. In this paper, we will introduce the concept of fuzzy mulitplication module. We will define a new operation called a product on the family of all fuzzy submodules of a fuzzy mulitplication module. We will define a fuzzy subset of the idealization ring $R+M$ and find some relations with the product of fuzzy submodules and product of fuzzy ideals of the idealization ring $R+M$. Some properties of weakly fuzzy prime submoduels and fuzzy prime submodules which are defined by T.K.Mukherjee, M.K.Sen and D.Roy will be introduced. We will investigate some properties of fuzzy prime submodules of a fuzzy multiplication module.

1. Introduction

In this paper we will investigate some properties of fuzzy multiplication modules. In section 2, we will review some properties of fuzzy ideals of a ring and fuzzy submodules of a module.

Since A.Barnard investigated some properties of multiplication modules and ideals via the paper which was published in J of Algebra, several authors such as P. F. Smith, Z. El-Bast, R. Ameri and S. Atani found various properties of multiplication modules through some papers.

In section 3, We define a fuzzy multiplication module by using the concept of fuzzy submodules of an R-module M. An R-module M is called a fuzzy multiplication module iff for each fuzzy submodule U of M: there exists a fuzzy ideal A of R such that $U{=}A1_M$.

R. Ameri defined a product of two submodules on the family of all submodules of a multiplication module. In this paper we will define a product of fuzzy submodules on the family of all fuzzy submodules of M (denoted by $F(M)$) as $UV = (AB)1_M$ where U and V are fuzzy submodules of M and A and B are fuzzy ideals of a ring R such that $U = A1_M$ and $V = B1_M$. We will investigate some properties of this product and find some relations with respect to the ring which is the idealization ring given by $R + M$.

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In section4, we will introduce the concept of weakly fuzzy prime submodules and fuzzy prime submodules which are defined by T. K. Mukherjee, M. K. Sen and D. Roy. But we will show that some new statements which were not proved by them are true. Finally we will show that P is a fuzzy prime submodule of a fuzzy multiplication module M iff $UV \subseteq P$ implies $U \subseteq P$ or $V \subseteq P$.

2. Preliminaries

Throughout this paper, R denotes a commutative ring with identity and all related modules are unitary R-modules. We review some definitions about fuzzy ideals and fuzzy submodules, which are well known.

Definition 1. A fuzzy subset A of a ring R is called a fuzzy ideal of a ring R if and only if for each $x, y \in R$,

(i) $A(x-y) \geq \inf\{A(x), A(y)\}$,
(ii) $A(xy) \geq \sup\{A(x), A(y)\}$.

Definition 2. A fuzzy subset N of an R-module M is called a fuzzy submodule of M if and only if for each $a, b \in M$ and for each $r \in R$,

(i) $N(a-b) \geq \inf\{N(a), N(b)\}$,
(ii) $N(ra) \geq N(a)$.

Addition of two fuzzy subsets of a ring R(or of an R-module M) is defined as followings:

Definition 3. Let A and B be fuzzy subsets of a ring R (or fuzzy subsets of an R-module M). Define $A+B$ as followings:

$$A + B(a) = \sup_{a=b+c} \{\inf\{A(b), B(c)\}\}$$

for every $a \in R$ (or $a \in M$).

Product of a fuzzy subset of a ring R and a fuzzy subset of an R-module M (or a fuzzy subset of a ring R) is defined as followings:

Definition 4. Let A be a fuzzy subset of a ring R and U be a fuzzy subset of an R-module M(or a fuzzy subset of a ring R). For every $a \in M$ (or $a \in R$),

$$AU(a) = \sup_{a=\sum_{i=1}^{k} r_i x_i} \{\inf\{A(r_i), A(r_2) \cdots A(r_k), N(x_1), N(x_2) \cdots N(x_k)\}\}$$

where $r_i \in R$ and $x_i \in M$ (or $x_i \in R$).

The following statements are well known.

Proposition 2.1. *Let A and B be fuzzy ideals of a ring R and let U and V be fuzzy submodules of an R-module M. Then the followings are satisfied.*

(i) *$A+B$ is a fuzzy ideal of R,*
(ii) *$U+V$ is a fuzzy submodule of M,*
(iii) *AB is a fuzzy ideal of R,*

(iv) *AU is a fuzzy submodule of M,*
(v) $(A+B)U = AU + BU$,
(vi) $(AB)U = A(BU)$.

3. Product of fuzzy submodules of a fuzzy multiplication module

We will define a fuzzy multiplication module using the concept of a multiplication module.

Definition 5. An R-module M is called a fuzzy multiplication module if and only if for every fuzzy submodule U of M, there exists a fuzzy ideal A of R such that $U = A1_M$ where $1_M(a) = 1$ for every $a \in M$.

Note that above fuzzy ideal A of R is called a presentation fuzzy ideal of a fuzzy submodule U (or for short, a presentation of U). It is clear that such presentation is not unique, that is, there exist some fuzzy ideals A and B of R such that $A1_M = B1_M$ even if $A \neq B$.

Definition 6. Let U and V be fuzzy submodules of a fuzzy multiplication module M where $U = A1_M$ and $V = B1_M$ for some fuzzy ideals A and B of a ring R. The product of U and V is denoted by UV and is defined by $UV = (AB)1_M$. Clearly, UV is a fuzzy submodule of M and contained in $U \cap V$.

Since a representation of a fuzzy submodule is not unique we must prove that the product is well defined that is the product is independent of presentations of two fuzzy submodules.

Proposition 3.1. *Let $U = A1_M$ and $V = B1_M$ be fuzzy submodules of a fuzzy multiplication module M where A and B are fuzzy ideals of R. Then the product of U and V is independent of presentations of U and V.*

Proof. Let $U = A1_M = A'1_M$ and $V = B1_M = B'1_M$ where A' and B' are fuzzy ideals of R. Then $(AB)1_M = A(B1_M) = A(B'1_M) = (AB')1_M = (B'A)1_M = B'(A1_M) = B'(A'1_M) = (B'A')1_M = (A'B')1_M$. Thus the product U and V is well defined. □

We can get the following proposition showing some properties of the product of fuzzy submodules of fuzzy multiplication modules.

Proposition 3.2. *Let M be a fuzzy multiplication module and let U and V be fuzzy submodules of M. Then the following statements are satisfied*

(i) *$F(M)$, the family of all fuzzy submodules of M with operation product on fuzzy submodules is a semigroup,*
(ii) *the product is distributive with respect to the sum on $F(M)$,*
(iii) $(U+V)(U \cap V) \subseteq UV$,
(iv) $U \cap V = UV$ *if* $U + V = 1_M$.

Proof. (i) Proposition 3.1 shows that the product is associative.

(ii) Let $U = A1_M$, $V = B1_M$ and $W = C1_M$ where A, B and C are fuzzy ideals of R. Then $U(V + W) = A1_M(B1_M + C1_M) = A1_M(B + C)1_M = (A(B + C))1_M = (AB + AC)1_M = (AB)1_M + (AC)1_M = UV + UW$.

(iii) $(U + V)(U \cap V) \leq U(U \cap V) + V(U \cap V) \leq UV$.

(iv) Since $UV \leq U \cap V$, it is clear by (iii). □

Since we defined the product on the family of all fuzzy submodules of a fuzzy multiplication module M, we can define a divisor of a fuzzy submodule of M. Let U and V be fuzzy submodules of a fuzzy multiplication module M. We say that U divides V, denoted by $U \mid V$, if there exists a fuzzy submodule W such that $V = UW$. Since there are three fuzzy ideals A, B and C of a ring R such that $U = A1_M$, $V = B1_M$ and $W = C1_M$, we get $V = WU = CA1_M = CU$. Thus we know that $U \mid V$ iff there exists a fuzzy ideal C such that $V = CU$.

Let M be an R-module. The idealization of R and M is the commutative ring with identity $R(M) = R + M$ with addition $(r, m) + (r', m') = (r + r', m + m')$ and multiplication $(r, m)(r', m') = (rr', rm' + r'm)$.

We can define a fuzzy subset of $R + M$ via a fuzzy subset of R and a fuzzy subset of M as following.

Definition 7. Let A be a fuzzy subset of a ring R and U be a fuzzy subset of an R-module M. Then a fuzzy subset of $R + M$, $A + U$ is defined as $(A + U)(r, m) = inf\{A(r), U(m)\}$.

Even though A is a fuzzy ideal of a ring R and U is a fuzzy submodule of M, $A + U$ may be not a fuzzy ideal of the idealization ring $R(M)$. But we can get the following lemma.

Lemma 3.3. *If U is a fuzzy submodule of an R-module M and A is a fuzzy ideal of a ring R, then $0_1 + U$ and $A + 1_M$ are fuzzy ideals of $R(M)$. Moreover we get $(0_1 + U)(A + 1_M) = 0_1 + AU$.*

Proof. $0_1 + U$ is a fuzzy ideal by the followings.

(i) For r_1 and r_2 in R, m_1 and m_2 in M, if $r_1 + r_2 = 0$, then $0_1 + U((r_1 + r_2, m_1 + m_2) = U(m_1 + m_2) \geq \inf\{U(m_1), U(m_2)\} \geq \inf\{(0_1 + U)(r_1, m_1), 0_1 + U(r_2, m_2)\}$. If $r_1 + r_2$ is nonzero then either r_1 or r_2 is nonzero. Thus $0_1 + U(r_1 + r_2, m_1 + m_2) = 0 = \inf\{0_1 + U(r_1, m_1), 0_1 + N(r_2, m_2)\}$.

(ii) For $(r_1, m_1), (r_2, m_2)$ in $R + M$, If r_1 or r_2 is zero (or both are zero) then $r_1r_2 = 0$ and $0_1 + U((r_1, m_1)(r_2, m_2)) = 0_1 + U((r_1r_2, r_1m_2 + r_2m_1) = U(r_1m_2 + r_2m_1) \geq \inf\{U(r_1m_2), U(r_2m_1)\} \geq \inf\{U(m_2), U(m_1)\} \geq \inf\{0_1 + U(r_1, m_1), 0_1 + U(r_2, m_2)\}$. If r_i is nonzero for some $i = 1, 2$, clearly $0_1 + U((r_1, m_1)(r_2, m_2)) \geq \inf\{0_1 + U(r_1, m_1), 0_1 + U(r_2, m_2)\} = 0$. Thus $0_1 + U$ is a fuzzy ideal of a ring $R(N)$.

By the fact that $A + 1_M(r, m) = A(r)$ for every $(r, m) \in R(M)$ we can know easily that $A + 1_M$ is a fuzzy ideal of $R(M)$.

For every m in M, $(A+1_M)(0_1+U)(0, m) = \sup_{(0,m)=\sum(r_i,a_i)(s_i,b_i)}\{\inf\{A+1_M(r_1, a_1), A+1_M(r_2, a_2) \cdots A+1_M(r_k, a_k), 0_1+U(s_1, b_1) \cdots 0_1+U(s_k, b_k)\}\} =$

$\sup_{m=\sum r_i b_i}\{\inf\{(A(r_1), A(r_2)\cdots A(r_k), U(b_1), U(b_2)\cdots U(b_k)\}\} = AU(m) = 0_1 + AU(0,m)$. If r is nonzero then clearly for every m in M, $(A+1_M)(0_1 + U)(r,m) = 0_1 + AU(r,m) = 0$. Thus we know that $(A+1_M)(0_1+U) = 0_1 + AU$. □

Theorem 3.4. *Let U and V be fuzzy submodules of an R-module M. Then $U \mid V$ if and only if there exists a fuzzy ideal γ of a ring $R(M)$ such that $0_1 + V = \gamma(0_1 + U)$.*

Proof. Suppose that there exists a fuzzy ideal γ of a ring $R(M)$ such that $0_1 + V = \gamma(0_1+U)$. We can define a new fuzzy subset A of a ring R as following: for every r in R, $A(r) = \sup_{m\in M}\{\gamma(r,m)\}$. At first we will show that A is a fuzzy ideal of R. Suppose that $A(r_1+r_2) = \sup_{m\in M}\{\gamma(r_1+r_2,m)\} < \inf\{A(r_1), A(r_2)\} = \inf\{\sup_{m\in M}\{\gamma(r_1,m)\}, \sup_{m\in M}\{\gamma(r_2,m)\}$. Then by the property of supremum, there exist m_1, m_2 in M such that $A(r_1+r_2) < \gamma(r_1,m_1)$ and $A(r_1+r_2) < \gamma(r_2,m_2)$. But the fact that $\inf\{\gamma(r_1,m_1),\gamma(r_2,m_2)\} \le \gamma((r_1+r_2, m_1+m_2)) \le \sup_{m\in M}\{\gamma((r_1+r_2,m))\} = A(r_1+r_2)$ shows that the assumption is impossible. So for every r_1, r_2 in R, $A(r_1+r_2) \ge \inf\{A(r_1), A(r_2)\}$. Suppose that for some r_1, r_2 in R $A(r_1r_2) < A(r_1)$ or $A(r_1r_2) < A(r_2)$. Without the loss of generality we can assume that $A(r_1r_2) < A(r_1)$. Then there exist $a \in M$ such that $A(r_1r_2) < \gamma(r_1,a)$. But it is impossible since $\gamma((r_1,a)(r_2,0)) = \gamma((r_1r_2, r_2a) \le \sup_{m\in M}\{\gamma((r_1r_2,m))\} = A(r_1r_2)$. Thus for every r_1, r_2 in R, $A(r_1r_2) \ge A(r_i)$ for every $i = 1,2$. So A is a fuzzy ideal of R. Next we will show that $V = A1_MU$. Since $0_1+V = \gamma(0_1+U)$, $0_1+V(0,m) = \inf\{1, V(m)\} = V(m) = \gamma(0_1+U)(0,m) = \sup_{(0,m)=\sum(r_i,m_i)(s_i,m'_i)}\{\inf\{\gamma(r_1,m_1)\cdots\gamma(r_k,m_k), 0_1+U(s_1,m'_1)\cdots 0_1+U(s_k, m'_k)\}\} = \sup_{m=\sum r_i m'_i}\{\inf\{\gamma(r_1,m_1)\cdots\gamma(r_k,m'_k), U(m'_1)\cdots U(m'_k)\}\} \le$

$\sup_{m=\sum r_i m'_i}\{\inf\{A(r_1), A(r_2)\cdots A(r_k), U(m'_1)\cdots U(m'_k)\}\} = AU(m)$. The sixth equality is satisfied by the fact that if s_j is nonzero then $0_1+U(s_j, m'_j) = 0$. Thus $V \subseteq AU$. Suppose that there exists $m \in M$ such that $V(m) < AU(m)$, then there exist $r_i \in R$ and $m_i \in M$ such that $V(m) < \inf\{A(r_1), A(r_2)\cdots A(r_l), U(m_1), U(m_2)\cdots U(m_l)\}$ where $m = \sum r_i m_i$. But this is impossible since $\inf\ A(r_1), A(r_2)\cdots A(r_l), U(m_1), U(m_2)\cdots U(m_l) \le inf\{\sup_{a\in M}\{\gamma(r_1,a)\}, \sup_{a\in M}\{\gamma(r_2,a)\}\cdots\sup_{a\in M}\{\gamma(r_k,a)\}, U(m_1), ..., U(m_k)\} = \sup_{(a_1,\cdots,a_k)\in M^k}\{\inf\{\gamma(r_1,a_1),\gamma(r_2,a_2)\cdots\gamma(r_k,a_k), U(m_1), U(m_2)\cdots U(m_k)\}\} \le \gamma(0_1+U))(0,m) = V(m)$.

Conversely suppose that $V = UW$ where $W = A1_M$ for some a fuzzy ideal $A$of R(i.e., $V = AU$). Let γ be $A + 1_M$. Then clearly $(A+1_M)(0_1+U) = 0_1 + AU = 0_1 + V$. □

4. Fuzzy prime submodule of a fuzzy multiplication module

Generally a submodule N of an R-module M is said to be prime iff $rx \in N$ and $x \notin N$ implies $rm \in N$ for all $m \in M$. T. K. Mukherjee, M. K. Sen and

D. Roy defined weakly fuzzy prime and fuzzy prime submodules respectively as the concept like similar to prime submodules.

Definition 8. A fuzzy submodule U of an R-module M is called a weakly fuzzy prime (primary) submodule if and only if $U(rx) > U(x)$ for $r \in R$, $x \in M$ implies that $U(rm) \geq U(rx)$ for all $m \in M$ ($U(rx) > U(x)$ for $r \in R$, $x \in M$ implies that there exist a positive integer n such that $U(r^n m) \geq U(rx)$ for all $m \in M$).

Definition 9. A fuzzy submodule U of an R-module M is called fuzzy prime (primary) submodule iff for a fuzzy ideal A of a ring R and a fuzzy submodule V of M, $AV \subseteq U$ implies that $V \subseteq U$ or $A \subseteq Q_U$ where Q_U is a fuzzy subset of R defined such as $Q_U(r) = \inf_{m \in M}\{U(rm)\}$ for every $r \in R$ ($AV \subseteq U$ implies $V \subseteq U$ or $A \subseteq \sqrt{Q_U}$ where $\sqrt{Q_U}(r)$ is defined as $\sqrt{Q_U}(r) = \sup_n\{Q_U(r^n))\}$.

Clearly a fuzzy prime (primary) submodule is a weakly fuzzy prime(primary) submodule. It is easily proved that Q_U is a fuzzy ideal of R if U is a fuzzy submodule of M. Also we easily proved that $Q_{C_N} = (N : M) = \{r \in R \mid rM \subseteq N\}$ where C_N is the characteristic function where N is a submodule of M. The following propositions were proved in [4].

Proposition 4.1. *Let N be a submodule of an R-module M. Its characteristic function C_N is a weakly fuzzy prime(primary) submodule of M iff N is a prime (primary) submodule of M.*

Proposition 4.2. *Let U be a fuzzy submodule of an R-module M. Then U is weakly fuzzy prime(primary) iff $U^t = \{x \in M \mid U(x) \geq t\}$ is a prime(primary) submodule of M for all $t \in [0, 1]$.*

Proposition 4.3. *Let U be a (weakly) fuzzy primary submodule of an R-module M, then Q_U is a (weakly) fuzzy primary ideal of R.*

The following proposition shows that if a fuzzy submodule is a characteristic function then a weakly fuzzy prime submodule is a fuzzy prime submodule.

Proposition 4.4. *A fuzzy submodule P is a prime submodule of an R-module M if and only if the characteristic function C_P is a fuzzy prime submodule of M.*

Proof. Since fuzzy prime is weakly fuzzy prime, the necessary condition is satisfied. Suppose that a submodule P is prime and $AU \subseteq C_P$ where A is a fuzzy ideal of a ring R and U is a fuzzy submodule of M. Let $U \nsubseteq C_P$. Then there exists $m \in M$ such that $U(m) > C_P(m)$. This inequality says that $C_P(m) = 0$ (i.e., $m \notin P$) and $U(m)$ is not zero. For every $r \in R$, clearly $C_P(rm) = 0$ or 1. If $C_P(rm) = 1$, then $rm \in P$ and $ra \in P$ for all $a \in M$ since P is prime. In this case we know that $Q_{C_P}(r) = 1$ since $C_P(ra) = 1$ for every $a \in M$. If $C_P(rm) = 0$, then $\inf\{A(r), U(m)\} \leq \sup_{rm=\sum r_i x_i}\{\inf\{A(r_1), A(r_2) \cdots A(r_k), U(x_1) \cdots U(x_k)\}\} = AU(rm) \leq C_P$

$(rm) = 0$ implies that $A(r) = 0$. Thus $A(r) \le Q_{C_P}(r)$ for all $r \in R$. Thus $A \subseteq Q_{C_P}$. □

By the above proposition the characteristic function of a submodule of M is weakly fuzzy prime iff fuzzy prime. But we don't know that every weakly fuzzy prime submodule is fuzzy prime.

Lemma 4.5. *Let U be a fuzzy submodule of a fuzzy multiplication module M. Then $U = Q_U 1_M$ where $Q_U(r) = \inf_{m \in M}\{f(rm)\}$.*

Proof. Let $m \in M$. If $m = \sum r_i a_i$ where $r_i \in R$ and $a_i \in M$, then for every i, $Q_U(r_i) \le U(r_i a_i)$. Thus $\inf\{Q_U(r_i)\} \le \inf\{U(r_1 a_1) \cdots U(r_k a_k)\}) \le U(m)$ since U is a fuzzy submodule. Thus we know that $Q_U 1_M \subseteq U$ even if M is not a fuzzy multiplication module. Suppose that $Q_U 1_M(a) < U(a) = A1_M(a)$ for some a in M where A is a fuzzy ideal such that $U = A1_M$. Then there exist $r_i \in R$ and $a_i \in M$ such that $a = \sum r_i a_i (1 \le i \le k)$ and $Q_U 1_M(a) < \inf\{A(r_1), A(r_2) \cdots A(r_k)\} \le \inf\{U(r_1 b), U(r_2 b) \cdots U(r_k b)\}$ for every b in M. But it is impossible since $\inf_{b \in M}\{U(r_1 b), U(r_2 b) \cdots U(r_k b)\} = inf\{Q_U(r_i)\} \le Q_U 1_M(a)$. Thus $Q_U 1_M = A1_M$. □

Proposition 4.6. *If P is a fuzzy prime submodule of an R-module M, then Q_P is a fuzzy prime ideal of a ring R.*

Proof. Let $AB \subseteq Q_P$ and $B \nsubseteq Q_P$. Then there exists $r \in R$ such that $B(r) > Q_P(r) = \inf_{a \in M}\{P(ra)\}$, so there exists $m \in M$ such that $B(r) > P(rm)$. Since $B1_M(rm) \ge B(r) > P(rm)$, $B1_M \nsubseteq P$. Since $(AB)1_M \subset Q_P 1_M \subset P$ by the definition of fuzzy prime submodule we get $A \subseteq Q_P$. □

We don't know that the converse of above Proposition is either true or not true. But if M is a fuzzy multiplication module, we know that the converse is true.

Theorem 4.7. *Let U be a fuzzy submodule of a fuzzy multiplication module M. Then if Q_U is a fuzzy prime ideal of a ring R, U is a fuzzy prime submodule of M.*

Proof. Let $AV \subseteq U$ where A is a fuzzy ideal of a ring R and V is a fuzzy submodule of M and $V \nsubseteq U$. Since M is a fuzzy multiplication module there exists a fuzzy ideal B of a ring R such that $V = B1_M$. Then there exists an $m \in M$ such that $B1_M(m) > Q_U 1_M(m)$. This implies $B \nsubseteq Q_U$. But $AB1_M \subseteq Q_U 1_M$ implies $AB \subset Q_U$ since if there exists $r \in R$ such that $AB(r) > Q_U(r)$ then $AB(r) > U(rm)$ for every $m \in M$ and thus $AB1_M \nsubseteq Q_U 1_M$. Thus $A \subseteq Q_U$ since Q_U is fuzzy prime. □

We can get the following theorem.

Theorem 4.8. *Let M be a fuzzy multiplication R-module. Then a fuzzy submodule P of M is a fuzzy prime submodule if and only if for fuzzy submodules U and V of M, $UV \subseteq P$ implies $U \subseteq P$ or $V \subseteq P$.*

Proof. Let P be a fuzzy prime submodule of M. Suppose that $UV \subseteq P$ and $V \nsubseteq P$. Since M is a fuzzy multiplication R- module there exist some fuzzy ideals A and B of a ring R such that $U = A1_M$ and $V = B1_M$. Then $UV = A(B1_M) \subseteq P$ and $B1_M \nsubseteq P$. Since P is fuzzy prime, $A \subseteq Q_P$, i.e., for every $r \in R$, $A(r) \leq P(rm)$ for every $m \in M$. Thus we know that $A1_M = U \subseteq P$ since $A1_M(m) = sup_{m=\sum r_i x_i}\{\inf\{A(r_1), ...A(r_k)\}\} \leq \inf\{P(r_1x_1)\cdots P(r_kx_k)\} \leq P(m)$ for every $m \in M$. Conversely we assume that $AU \subseteq P$ and $U \nsubseteq P$ where A is a fuzzy ideal of R and U is a fuzzy submodule of M. We can find a fuzzy ideal B such that $U = B1_M$. Then $A1_MB1_M = AB1_M \subseteq P$ implies $A1_M \subseteq P$ since $B1_M \nsubseteq P$. Thus $A \subseteq Q_P$ since for every $r \in R$, $A(r) \leq A1_M(rm) \leq P(rm)$ for every $m \in M$ implies $A(r) \leq Q_P(r)$ for every $r \in R$. □

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