

RADIAL SYMMETRY OF TOPOLOGICAL SOLUTIONS IN THE SELF-DUAL MAXWELL-CHERN-SIMONS GAUGED $O(3)$ SIGMA MODEL

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ABSTRACT. Using the moving plane method, we establish the radial symmetry of topological one-vortex solutions of a semilinear elliptic system arising from the self-dual Maxwell-Chern-Simons $O(3)$ model.

1. Introduction

In this paper we are interested in the following system of semilinear elliptic equations in \mathbb{R}^2 ;

$$\begin{aligned} \Delta u &= 2q \left(-N + s - \frac{1 - e^u}{1 + e^u} \right) + 4\pi \sum_{j=1}^{l_1} n_j \delta_{p_j} - 4\pi \sum_{j=1}^{l_2} m_j \delta_{q_j}, \\ \Delta N &= -\kappa^2 q^2 \left(-N + s - \frac{1 - e^u}{1 + e^u} \right) + q \frac{4e^u}{(1 + e^u)^2} N. \end{aligned} \quad (1.1)$$

Here, $P = \{p_1, \dots, p_{l_1}\}$ is a set of vortex points and $Q = \{q_1, \dots, q_{l_2}\}$ is a set of anti-vortex points. All of these are distinct points in \mathbb{R}^2 . The unknowns are $u : \mathbb{R}^2 \setminus (P \cup Q) \rightarrow \mathbb{R}$ and $N : \mathbb{R}^2 \rightarrow \mathbb{R}$. Furthermore, n_j 's and m_j 's are positive integers, κ and q are positive constants, and $-1 < s < 1$.

The system (1.1) is originated from the self-dual equations of the Maxwell-Chern-Simons gauged $O(3)$ model [6] which was suggested in order to break the scale invariance of solutions of the classical sigma model having the energy lower bound of a Bogomol'nyi type. In this model, the Maxwell and the Chern-Simons terms constitute the kinetic term for the gauge field, and the self-duality was attained by introducing a neutral scalar field N . For more information of this model and the derivation of (1.1), one can refer to [4].

We note that if (u, N) is a solution pair of (1.1) with $0 \leq s < 1$, then $(-u, -N)$ is also a solution for $-s$ with the change of roles of p_j 's and q_j 's.

Received June 7, 2010.

2010 *Mathematics Subject Classification.* 81T13, 35B40, 35J61.

Key words and phrases. Maxwell-Chern-Simons $O(3)$ model, radial symmetry of solutions, moving plane method, topological solution.

This research was supported by the Kyung Hee University Research Fund in 2007 (KHU-20070632).

Therefore, throughout this paper, we assume that $0 \leq s < 1$. There are two kinds of boundary conditions for (1.1); either

$$(1.2) \quad u \rightarrow \ln \frac{1-s}{1+s}, \quad N \rightarrow 0,$$

or

$$(1.3) \quad u \rightarrow -\infty, \quad N \rightarrow s-1.$$

The former is called topological and the latter nontopological. See [4] for further discussion for physical meaning of these boundary conditions. In this paper, we consider the topological condition (1.2). There are several results on (1.1) when the 't Hooft type periodic boundary conditions are given on a lattice (refer to [1, 2, 7, 8]). For the existence of solutions of (1.1) with a condition of (1.2) in the whole plane, there are two results as follows.

Theorem 1.1 ([4]). *There exists a constant κ_0 satisfying that for each $0 < \kappa < \kappa_0$, there is a constant $q_\kappa > 0$ such that a system of equations (1.1) with a condition of (1.2) admit a solution $(u, N) \in C^\infty(\mathbb{R}^2 \setminus (P \cup Q)) \times C^\infty(\mathbb{R}^2)$ for all $q > q_\kappa$. Moreover, the functions $u^2, N^2, |\nabla u|^2$, and $|\nabla N|^2$ decay exponentially at infinity.*

The constraints on κ and q in Theorem 1.1 are due to the method of construction of the solutions. In fact, such a restriction was necessary to find a subsolution in the super- and subsolution method and assures the iteration method being successful. This restriction can be removed when there appear only vortex points. In this case, we have

$$(1.4) \quad \begin{aligned} \Delta u &= 2q \left(-N + s - \frac{1-e^u}{1+e^u} \right) + 4\pi \sum_{j=1}^l n_j \delta_{p_j}, \\ \Delta N &= -\kappa^2 q^2 \left(-N + s - \frac{1-e^u}{1+e^u} \right) + q \frac{4e^u}{(1+e^u)^2} N. \end{aligned}$$

Here, we put $l = l_1$. The number $d = n_1 + \cdots + n_l$ is called the total vortex number. For the system (1.4), we have the following result.

Theorem 1.2 ([5]). *For any $\kappa, q > 0$, there exists a solution pair (u, N) of a system (1.4) with the boundary condition of (1.2).*

For simplicity, we define some notations as follows:

$$a = \frac{1+s}{1-s} \in [1, \infty), \quad w = u + \ln a, \quad f(t) = \frac{t-a}{t+a} + s = \frac{2a(t-1)}{(a+1)(t+a)},$$

where $f'(t) > 0$ and $f''(t) < 0$ on $[0, \infty)$. In particular, when the vortex points are the same point, say the origin, the system of equations (1.4) becomes

$$(1.5) \quad \begin{aligned} \Delta w &= 2q(-N + f(e^w)) + 4\pi d \delta_0, \\ \Delta N &= -\kappa^2 q^2(-N + f(e^w)) + 2q f'(e^w) e^w N, \end{aligned}$$

with the topological boundary condition

$$(1.6) \quad w, N \rightarrow 0 \quad \text{as } |x| \rightarrow \infty.$$

The goal of this paper is to show that every solution of (1.5) with (1.6) is radially symmetric about the origin. Though (1.5) is a system of equations, it has lots of maximum principle structures as we can see in the next section. This enables us to use the moving plane method to show the radial symmetry of solutions. For related works, it was shown that topological one-vortex solutions in the Abelian Maxwell-Chern-Simons-Higgs model are radially symmetric about the vortex point [3]. The main ingredient is to handle the difficulty arising from the sign changes of the derivatives of the nonlinear terms of the equations, from which we can apply the maximum principle. The statement of the main theorem is as follows.

Theorem 1.3. *Suppose that $s \in [1/3, 1)$, or equivalently $a \geq 2$. If (w, N) is a solution pair of (1.5) with (1.6), then (w, N) is radially symmetric about the origin.*

2. Radial symmetry of one-vortex solutions

This section is devoted to the proof of Theorem 1.3. To begin with, we use the maximum principle to get uniform signs of w and N .

Lemma 2.1. *If (w, N) is a solution pair of (1.5) with (1.6), then $w < 0$ in $\mathbb{R}^2 \setminus \{0\}$ and $N < 0$ in \mathbb{R}^2 .*

Proof. Assume N attains its positive maximum value at a point y . By the maximum principle to the second equation of (1.5), we have

$$0 < N \leq \frac{\kappa^2 q^2}{2qf'(e^w)e^w + \kappa^2 q^2} f(e^w)$$

at the maximum point y . Then we see that $f(e^{w(y)}) > 0$, which means $w(y) > 0$. Let z be a positive maximum point of w . Again by the maximum principle to the first equation of (1.5), $N(z) \geq f(e^{w(z)})$. Consequently, we have

$$f(e^{w(z)}) \leq N(z) \leq N(y) \leq \frac{\kappa^2 q^2 f(e^{w(y)})}{2qf'(e^{w(y)})e^{w(y)} + \kappa^2 q^2} < f(e^{w(y)}).$$

This leads to a contradiction since $w(z) \geq w(y) > 0$ and f is strictly increasing. Hence $N \leq 0$ for any point. On the other hand, the first equation of (1.5) becomes

$$\Delta w \geq 2qf(e^w) + 4\pi d\delta_0$$

in the sense of distribution since $N \leq 0$. By the strong maximum principle, we have $w < 0$ in $\mathbb{R}^2 \setminus \{0\}$. Let us rewrite the second equation of (1.5) as

$$\Delta N - (\kappa^2 q^2 + 2qf'(e^w)e^w)N = -\kappa^2 q^2 f(e^w)$$

with $N \rightarrow 0$ as $|x| \rightarrow \infty$. Again by the strong maximum principle, we find that $N < 0$ in \mathbb{R}^2 . \square

For the proof of the radially symmetric property of solutions, we use the method of moving planes. For $\lambda < 0$ and $x = (x_1, x_2) \in \mathbb{R}^2$, we let

$$\begin{aligned}\Sigma_\lambda &= \{x \in \mathbb{R}^2 : x_1 < \lambda\}, \quad \Gamma_\lambda = \partial\Sigma_\lambda, \\ x_\lambda &= (2\lambda, 0), \quad \tilde{\Sigma}_\lambda = \Sigma \setminus \{x_\lambda\}.\end{aligned}$$

Then we define new functions

$$\begin{aligned}w_\lambda(x) &= w(2\lambda - x_1, x_2) \quad \text{for } x \in \tilde{\Sigma}_\lambda, \\ N_\lambda(x) &= N(2\lambda - x_1, x_2) \quad \text{for } x \in \Sigma_\lambda, \\ \gamma_\lambda(x) &= w_\lambda(x) - w(x) \quad \text{for } x \in \tilde{\Sigma}_\lambda, \\ \beta_\lambda(x) &= N_\lambda(x) - N(x) \quad \text{for } x \in \Sigma_\lambda.\end{aligned}$$

Although w_λ is not defined at x_λ , e^{w_λ} is well defined at x_λ such that $e^{w_\lambda(x_\lambda)} = 0$. Then, a short computation gives

$$\begin{aligned}(2.1) \quad \Delta\gamma_\lambda &= -2q\beta_\lambda + 2q\left(f(e^{w_\lambda}) - f(e^w)\right) \\ &= -2q\beta_\lambda + 2qf'(e^\zeta)e^\xi\gamma_\lambda \quad \text{on } \tilde{\Sigma}_\lambda, \\ \Delta\beta_\lambda &= \left(\kappa^2q^2 + 2qf'(e^{w_\lambda})e^{w_\lambda}\right)\beta_\lambda - \kappa^2q^2\left(f(e^{w_\lambda}) - f(e^w)\right) \\ (2.2) \quad &+ 2q\left(f'(e^{w_\lambda})e^{w_\lambda} - f'(e^w)e^w\right)N \quad \text{on } \Sigma_\lambda,\end{aligned}$$

where $\zeta(x)$ and $\xi(x)$ lie between $w_\lambda(x)$ and $w(x)$.

Lemma 2.2. *There exists a number $R_0 > 0$ such that if β_λ has a positive maximum value at y in Σ_λ , then $|y| \leq R_0$.*

Proof. Let us consider a polynomial

$$p(t) = \kappa^2q^2t(t+a)^2 + 4qat^2 + \kappa(a+1)^2(t-1) - \kappa^2q^2(a+1)^2.$$

Since $p(1) > 0$, we can choose $\eta \in (0, 1)$ so close to 1 that $p(\eta) > 0$. Since $w, N \rightarrow 0$ as $|x| \rightarrow \infty$, there exists a number $R_1 > 0$ such that

$$(2.3) \quad \ln \eta < w(x) < 0 \quad \text{and} \quad -\frac{\kappa}{2q}(1-\eta) < N(x) < 0 \quad \text{for all } |x| \geq R_1.$$

Let $\mu_1 = \max_{|x| \leq R_1} w(x) < 0$. Since $w \rightarrow 0$ as $|x| \rightarrow \infty$, we can find a number $R_2 > R_1$ such that

$$(2.4) \quad \mu_1 < w(x) < 0 \quad \text{for all } |x| \geq R_2.$$

For $a \geq 1$, we also choose $R_3 > R_2$ and $\mu_2 < 0$ so that

$$(2.5) \quad \kappa^2qa^3 + 4\mu_2(a+1)^2 > 0 \quad \text{and} \quad \mu_2 < N(x) < 0 \quad \text{for all } |x| \geq R_3.$$

We set $R_0 = 2R_3$.

Suppose that β_λ has a positive maximum value at $y \in \Sigma_\lambda$. In order to claim $|y| \leq R_0$, let us assume the contrary, that is, $|y| > R_0$. If $y = x_\lambda$, then we derive from the maximum principle to (2.2) that

$$0 \geq \kappa^2q^2\beta_\lambda(x_\lambda) - \kappa^2q^2(f(0) - f(e^{w(x_\lambda)})) - 2qf'(e^{w(x_\lambda)})e^{w(x_\lambda)}N(x_\lambda) > 0,$$

a contradiction. Here, we use the fact that f is strictly increasing for $t \geq 0$, $e^{w_\lambda(x_\lambda)} = 0$, and $N < 0$ on \mathbb{R}^2 . As a consequence, the positive maximum point y occurs in a set $S = \{x : |x| > R_0\} \cap \tilde{\Sigma}_\lambda$. Now we can rewrite (2.2) on $\tilde{\Sigma}_\lambda$ as

$$\begin{aligned} \Delta\beta_\lambda &= \left(\kappa^2 q^2 + 2qf'(e^{w_\lambda})e^{w_\lambda}\right)\beta_\lambda - \kappa^2 q^2 \left(f(e^{w_\lambda}) - f(e^w)\right) \\ &\quad + 2qf'(e^{w_\lambda})(e^{w_\lambda} - e^w)N + 2q\left(f'(e^{w_\lambda}) - f'(e^w)\right)e^w N \\ &= \left(\kappa^2 q^2 + 2qf'(e^{w_\lambda})e^{w_\lambda}\right)\beta_\lambda - \kappa^2 q^2 f'(e^\zeta)e^\xi \gamma_\lambda \\ (2.6) \quad &\quad + 2q\left(f'(e^{w_\lambda})e^\xi \gamma_\lambda N + f''(e^\tau)e^\xi e^w \gamma_\lambda N\right), \end{aligned}$$

where $\zeta(x), \xi(x)$ are the functions appearing in (2.1) and $\tau(x)$ lies between $w_\lambda(x)$ and $w(x)$. Applying the maximum principle, we deduce that

$$(2.7) \quad \beta_\lambda(y) \leq \frac{\kappa^2 q^2 f'(e^\zeta)e^\xi - 2qf'(e^{w_\lambda})e^\xi N - 2qf''(e^\tau)e^\xi e^w N}{\kappa^2 q^2 + 2qf'(e^{w_\lambda})e^{w_\lambda}} \gamma_\lambda \Big|_y.$$

Since $f'(t) > 0$ on $[0, \infty)$ and $N < 0$ on \mathbb{R}^2 , we infer that

$$\begin{aligned} &\kappa^2 q^2 f'(e^\zeta)e^\xi - 2qf'(e^{w_\lambda})e^\xi N - 2qf''(e^\tau)e^\xi e^w N \\ &> \kappa^2 q^2 f'(e^\zeta)e^\xi - 2qf''(e^\tau)e^\xi e^w N \\ &= \kappa^2 q^2 \frac{2a}{(e^\zeta + a)^2} e^\xi + \frac{8aq}{(e^\tau + a)^3} e^\xi e^w N > 2aq \left(\frac{\kappa^2 q}{(a+1)^2} + \frac{4\mu_2}{a^3} \right) e^\xi > 0 \end{aligned}$$

at the point y by the choice of a number $\mu_2 < 0$ in (2.5) for a fixed $a \geq 1$. As a consequence, we see that $\gamma_\lambda(y) > 0$. Furthermore, since $f''(t) < 0$ on $[0, \infty)$ and $N < 0$ on \mathbb{R}^2 , we deduce from (2.7) that

$$(2.8) \quad \beta_\lambda(y) \leq \frac{\kappa^2 q^2 e^{\xi(y)} f'(e^{\zeta(y)}) - 2qe^{\xi(y)} f'(e^{w_\lambda(y)})N(y)}{\kappa^2 q^2 + 2qf'(e^{w_\lambda(y)})e^{w_\lambda(y)}} \gamma_\lambda(y).$$

On the other hand, since $\gamma_\lambda(x) \rightarrow 0$ as $|x| \rightarrow \infty$ on $\tilde{\Sigma}_\lambda$ and $\gamma_\lambda(x) \rightarrow -\infty$ as $x \rightarrow x_\lambda$, we conclude that γ_λ has a maximum point on $\tilde{\Sigma}_\lambda$. Let z be a maximum point of γ_λ in $\tilde{\Sigma}_\lambda$. Then $\gamma_\lambda(z) \geq \gamma_\lambda(y) > 0$. The maximum principle on the equation (2.1) and (2.8) yield that

$$\begin{aligned} f'(e^{\zeta(z)})e^{\xi(z)}\gamma_\lambda(y) &\leq f'(e^{\zeta(z)})e^{\xi(z)}\gamma_\lambda(z) \leq \beta_\lambda(z) \leq \beta_\lambda(y) \\ &\leq \frac{\kappa^2 q^2 e^{\xi(y)} f'(e^{\zeta(y)}) - 2qe^{\xi(y)} f'(e^{w_\lambda(y)})N(y)}{\kappa^2 q^2 + 2qf'(e^{w_\lambda(y)})e^{w_\lambda(y)}} \gamma_\lambda(y) \\ (2.9) \quad &\leq \frac{\kappa^2 q^2 - 2qN(y)}{\kappa^2 q^2 + 2qf'(e^{w_\lambda(y)})e^{w_\lambda(y)}} e^{\xi(y)} f'(e^{\zeta(y)}) \gamma_\lambda(y), \end{aligned}$$

where the last inequality comes from the decreasing property of f' on $w(y) \leq \zeta(y) \leq w_\lambda(y)$. We also note that

$$e^{\xi(y)} f'(e^{\zeta(y)}) \leq f'(e^{\zeta(y)}) \leq f'(\eta)$$

on $\ln \eta < \zeta(y), \xi(y) < 0$. Since the function $tf'(t)$ is increasing on $0 < t < 1$,

$$f'(e^{w_\lambda(y)})e^{w_\lambda(y)} \geq f'(e^{w(y)})e^{w(y)} > f'(\eta)\eta$$

on $\ln \eta < w(y) < w_\lambda(y) < 0$. From (2.3) and (2.9), we have

$$\begin{aligned} \eta \frac{2a}{(a+1)^2} &< f'(e^{\zeta(z)})e^{\xi(z)} < \frac{\kappa^2 q^2 - 2qN(y)}{\kappa^2 q^2 + 2qf'(e^{w_\lambda(y)})e^{w_\lambda(y)}} e^{\xi(y)} f'(e^{\zeta(y)}) \\ &\leq \frac{\kappa^2 q^2 + \kappa(1-\eta)}{\kappa^2 q^2 + 2q\eta f'(\eta)} \cdot \frac{2a}{(a+\eta)^2} \end{aligned}$$

on $\ln \eta < w(j) < \xi(j), \zeta(j) < w_\lambda(j) < 0$ for $j = y, z$. Furthermore, this leads to a contradiction;

$$1 < \frac{\kappa^2 q^2 + \kappa(1-\eta)}{\kappa^2 q^2 + 2q\eta f'(\eta)} \cdot \frac{(a+1)^2}{\eta(a+\eta)^2} < \frac{\kappa^2 q^2 + \kappa(1-\eta)}{\kappa^2 q^2 + 2q\frac{2a\eta}{(a+\eta)^2}} \cdot \frac{(a+1)^2}{\eta(a+\eta)^2} < 1$$

by the choice of η . Hence, we complete the proof. \square

Corollary 2.3. *If $\lambda \leq -R_0$, then $\gamma_\lambda \leq 0$ in $\tilde{\Sigma}_\lambda$ and $\beta_\lambda \leq 0$ in Σ_λ .*

Proof. Assume that $\beta_\lambda(y) > 0$ at some $y \in \Sigma_\lambda$ for $\lambda \leq -R_0$. Then $y \notin \{|x| \leq R_0\}$, which clearly contradicts to the previous lemma. Hence $\beta_\lambda \leq 0$ in Σ_λ .

It is clear that

$$\gamma_\lambda \leq 0 \quad \text{on } B_{R_1}(x_\lambda) \setminus \{x_\lambda\}$$

for $\lambda \leq -R_0$. Let us consider the equation (2.1)

$$\Delta \gamma_\lambda - 2qf'(e^\xi)e^\xi \gamma_\lambda = -2q\beta_\lambda \geq 0$$

with the boundary condition $\gamma_\lambda \leq 0$ on $\partial(\tilde{\Sigma}_\lambda \setminus B_{R_1}(x_\lambda))$. By the maximum principle, we have $\gamma_\lambda \leq 0$ on $\tilde{\Sigma}_\lambda \setminus B_{R_1}(x_\lambda)$. Therefore, $\gamma_\lambda \leq 0$ in $\tilde{\Sigma}_\lambda$. \square

As a consequence of the above corollary, we define a number

$$\lambda_0 = \sup\{\lambda < 0 : \gamma_\nu \leq 0 \text{ on } \tilde{\Sigma}_\nu, \beta_\nu \leq 0 \text{ on } \Sigma_\nu \text{ for all } \nu < \lambda\}.$$

Lemma 2.4. *For $a \geq 2$, $\lambda_0 = 0$.*

Proof. Suppose that $\lambda_0 \neq 0$. For a sequence $\lambda_k \in (\lambda_0, 0)$ such that $\lambda_k \rightarrow \lambda_0$, we have $\gamma_{\lambda_k}(z_k) > 0$ or $\beta_{\lambda_k}(y_k) > 0$, where z_k and y_k are maximum points of γ_{λ_k} and β_{λ_k} , respectively. If $\gamma_{\lambda_k}(z_k) > 0$, then by the maximum principle to (2.1), it is clear that $\beta_{\lambda_k}(y_k) > 0$. So it is obvious $\beta_{\lambda_k}(y_k) > 0$ no matter what the sign of $\gamma_{\lambda_k}(z_k)$ is. Setting $r_0 > \max\{R_0, -\lambda_0\}$, we can say $|y_k| \leq r_0$ by Lemma 2.2.

Passing to a subsequence, we may assume that y_k converges to a point y . Obviously,

$$y \in \Sigma_{\lambda_0} \cup \Gamma_{\lambda_0}, \quad \beta_{\lambda_0}(y) \geq 0, \quad \text{and} \quad \nabla \beta_{\lambda_0}(y) = 0.$$

Since $a \geq 2$ and $w_{\lambda_0} \leq \tau \leq w$, we see that from (2.6)

$$\begin{aligned} &\Delta \beta_{\lambda_0} - (\kappa^2 q^2 + 2qf'(e^{w_{\lambda_0}})e^{w_{\lambda_0}})\beta_{\lambda_0} \\ &\geq 2qNe^\xi(f'(e^{w_{\lambda_0}}) + f''(e^\tau)e^w)\gamma_{\lambda_0} \end{aligned}$$

$$\begin{aligned}
 &= 2qNe^\xi \left(\frac{2a}{(e^{w_{\lambda_0}} + a)^2} - e^w \frac{4a}{(e^\tau + a)^3} \right) \gamma_{\lambda_0} \\
 &\geq 4aqNe^\xi \frac{e^{w_{\lambda_0}} + a - 2e^w}{(e^\tau + a)^3} \gamma_{\lambda_0} > 0
 \end{aligned}$$

on Σ_{λ_0} . Then it follows from the strong maximum principle that β_{λ_0} cannot attain its maximum value in Σ_{λ_0} . So $y \in \Gamma_{\lambda_0}$. Then it comes from the Hopf Lemma that $(\partial\beta_{\lambda_0}/\partial x_1)(y) > 0$, which contradicts to the fact that $\nabla\beta_{\lambda_0}(y) = 0$. Consequently, we come to the conclusion of $\lambda_0 = 0$. \square

Since $\lambda_0 = 0$, we have for $x_1 < 0$

$$w(-x_1, x_2) \leq w(x_1, x_2) \quad \text{and} \quad N(-x_1, x_2) \leq N(x_1, x_2).$$

By the standard moving plane argument, w and N are symmetric in the plane about the origin. Therefore, we complete the proof of Theorem 1.3.

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