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APPROXIMATION AND INTERPOLATION IN THE SPACE OF CONTINUOUS FUNCTIONS VANISHING AT INFINITY

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Abstract. We establish a result concerning simultaneous approximation and interpolation from certain uniformly dense subsets of the space of vector-valued continuous functions vanishing at infinity on locally compact Hausdorff spaces.

1. Introduction and preliminaries

Throughout this paper we shall assume, unless stated otherwise, that X is a locally compact Hausdorff space and $(E, \|\cdot\|)$ is a normed vector space over $\mathbb{K}$, where $\mathbb{K}$ denotes either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. We shall denote by E^* the topological dual of E and by $C(X; E)$ the vector space over $\mathbb{K}$ of all continuous functions from X into E.

A continuous function f from X to E is said to *vanish at infinity* if for every $\varepsilon > 0$ the set $\{x \in X : \|f(x)\| \geq \varepsilon\}$ is compact. Let $C_0(X; E)$ be the vector space of all continuous functions from X into E vanishing at infinity and equipped with the supremum norm. The vector subspace of all functions in $C(X; E)$ with compact support is denoted by $C_c(X; E)$.

Let A be a nonempty subset of $C_0(X; \mathbb{K})$. We denote by $A \bigotimes E$ the subset of $C_0(X; E)$ consisting of all functions of the form

$$f(x) = \sum_{i=1}^{n} \phi_i(x) v_i, \quad x \in X,$$

where $\phi_i \in A$, $v_i \in E$, $i = 1, \ldots, n$, $n \in \mathbb{N}$.

A subset $W \subset C_0(X; E)$ is an *interpolating family for* $C_0(X; E)$ if given any nonempty finite subset $S \subset X$ and any $f \in C_0(X; E)$, there exists $g \in W$ such that $g(x) = f(x)$ for all $x \in S$.

A nonempty subset B of $C_0(X; E)$ is said to have the approximation-interpolation property on finite subsets (in short, the *SAI property*) if for every $f \in C_0(X; E)$, every $\varepsilon > 0$ and every nonempty finite subset S of X, there exists $g \in B$ such that $\|f - g\| < \varepsilon$ and $f(x) = g(x)$ for all $x \in S$.

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The purpose of this paper is to present a result of simultaneous approximation and interpolation from certain subsets of $C_0(X;E)$. As a consequence, we obtain a generalization of a result by Prolla concerning simultaneous approximation and interpolation from vector subspaces of $C(X;E)$ when X is a compact Hausdorff space.

2. Main result

Walsh (Theorem 6.5.1 [2]) proved the following result.

Theorem 2.1. *Let K be a compact set in the complex plane and let $z_1, \dots, z_n$ be any set of n points in K. If the function f is defined on K and can be uniformly approximated by polynomials there, then f can be uniformly approximated by polynomials p which also satisfy the auxiliary conditions $p(z_i) = f(z_i)$, $i = 1, \dots, n$.*

Motivated by Walsh, we establish the result below.

Theorem 2.2. *Let A be an interpolating family for $C_0(X;\mathbb{K})$ and B an uniformly dense subset of $C_0(X;E)$. If $(A \bigotimes E) + B \subset B$, then B has the SAI property.*

Lemma 2.1. *If X is a locally compact Hausdorff space and $\{x_1, \dots, x_n\} \subset X$, then there exists $l_i \in C_c(X;\mathbb{R})$ such that $l_i(x_i) = 1$ and $l_i(x_j) = 0, j \neq i$.*

Proof. Since X is Hausdorff and $\{x_1, \dots, x_n\}$ is finite there exists an open neighborhood U_i of x_i such that $x_j \notin U_i$ for all $j \neq i,\ j \in \{1, \dots, n\}$. By Urysohn's Lemma [8] there exists $l_i \in C_c(X;\mathbb{R}),\ 0 \leq l_i \leq 1$, such that $l_i(x_i) = 1$ and $l_i(x) = 0$ if $x \notin U_i$, in particular, $l_i(x_j) = 0,\ j \neq i$. $\square$

Proof of Theorem 2.2. Let $S = \{x_1, \dots, x_n\}$ be a subset of X. Let $f \in C_0(X;E)$ and $\varepsilon > 0$.

By Lemma 2.1, for each $x_i \in S$ there exists $l_i \in C_c(X;\mathbb{R})$ such that

$$\begin{aligned} l_i(x_i) &= 1 \\ l_i(x_j) &= 0;\ j \neq i,\ x_j \in S. \end{aligned}$$

Since A is an interpolanting family for $C_0(X;\mathbb{K})$, there exist $\phi_1, \dots, \phi_n \in A$ such that

$$\phi_i(x_j) = l_i(x_j); \quad 1 \leq i, j \leq n.$$

Since B is uniformly dense in $C_0(X;E)$ there exists $g \in B$ such that $\|f - g\| < \eta$ where $\eta := \varepsilon/(1 + \sum_{i=1}^n \|\phi_i\|)$.

The function $h : X \to E$ defined by

$$h(x) = \sum_{i=1}^{n} \phi_i(x)(f(x_i) - g(x_i))$$

belongs to $A \bigotimes E$ and $h(x_j) = f(x_j) - g(x_j)$ for $j = 1, \dots, n$.

Now the function $p = h + g$ belongs to B and $p(x_j) = f(x_j)$ for $j = 1, \ldots, n$. Moreover,

$$\|f - p\| \leq \|f - g\| + \|h\| < \eta + \eta \sum_{i=1}^{n} \|\phi_i\| = \varepsilon.$$

□

Example 2.1. The set of all continuous real-valued nowhere differentiable functions on $[a, b]$, denoted by $ND[a, b]$, has the SAI property. Indeed, let $P[a, b]$ be the set of all real polynomials on $[a, b]$. Note that

(a) $P[a, b]$ is an interpolating subset of $C([a, b]; \mathbb{R})$ (take the Lagrange polynomials);

(b) $ND[a, b]$ is uniformly dense in $C([a, b]; \mathbb{R})$;

(c) $(P[a, b] \bigotimes \mathbb{R}) + ND[a, b] = P[a, b] + ND[a, b] \subset ND[a, b]$.

Hence, it follows from Theorem 2.2 that $ND[a, b]$ has the SAI property.

Lemma 2.2. *Every uniformly dense vector subspace of $C_0(X; \mathbb{K})$ is an interpolating family for $C_0(X; \mathbb{K})$.*

Proof. Let $S = \{x_1, \ldots, x_n\}$ be a subset of X and G be an uniformly dense vector subspace of $C_0(X; \mathbb{K})$. Consider the following continuous linear mapping

$$\begin{array}{rcl} T : C_0(X; \mathbb{K}) & \to & \mathbb{R}^n \\ f & \mapsto & (f(x_1), \ldots, f(x_n)). \end{array}$$

Note that $T(G)$ is closed because it is a vector subspace of $\mathbb{R}^n$. Then by density of G and continuity of T, it follows that

$$T(C_0(X; \mathbb{K})) = T(\overline{G}) \subset \overline{T(G)} = T(G).$$

Therefore, for any $f \in C_0(X; \mathbb{K})$, there exists $g \in G$ such that $(f(x_1), \ldots, f(x_n)) = (g(x_1), \ldots, g(x_n))$. □

A subset M of $C_0(X; \mathbb{K})$ is *dense-lineable* or *algebraically generic* if $M \bigcup \{0\}$ contains a vector space dense in $C_0(X; \mathbb{K})$. For more information, see [1].

Corollary 2.1. *If M is a dense-lineable subset of $C_0(X; \mathbb{K})$, then $M \bigcup \{0\}$ has the SAI property. In particular, all dense vector subspaces of $C_0(X; \mathbb{K})$ have the SAI property.*

Proof. Since $M \bigcup \{0\}$ contains a vector space A dense in $C_0(X; \mathbb{K})$, it follows from Lemma 2.2 that A is an interpolating family for $C_0(X; \mathbb{K})$. Moreover,

$$(A \bigotimes \mathbb{K}) + A \subset A.$$

Then, by Theorem 2.2, it follows that A has the SAI property. Since $A \subset M \bigcup \{0\}$ we conclude that $M \bigcup \{0\}$ has the SAI property. □

The last corollary can also be proved by using Deutsch's result [3].

In order to give a criterion to identify vector subspaces of $C_0(X; E)$ which have the SAI property, we need the next two results.

Proposition 2.1. *The vector subspace $C_0(X;\mathbb{K})\bigotimes E$ is uniformly dense in $C_0(X;E)$.*

Proof. It follows from Corollary 6.4 [5] that $C_c(X;\mathbb{K})\bigotimes E$ is uniformly dense in $C_0(X;E)$. Since

$$C_c(X;\mathbb{K})\bigotimes E\subset C_0(X;\mathbb{K})\bigotimes E\subset C_0(X;E),$$

we conclude that $C_0(X;\mathbb{K})\bigotimes E$ is uniformly dense in $C_0(X;E)$. □

Lemma 2.3. *If A is an uniformly dense subset of $C_0(X;\mathbb{K})$, then $A\bigotimes E$ is uniformly dense in $C_0(X;E)$.*

Proof. By Proposition 2.1, $C_0(X;\mathbb{K})\bigotimes E$ is uniformly dense in $C_0(X;E)$. Then given $f\in C_0(X;E)$ and $\varepsilon>0$, there exists $g\in C_0(X;\mathbb{K})\bigotimes E$, say $g(x)=\sum_{i=1}^n\psi_j(x)v_j$, $\psi_j\in C_0(X;\mathbb{K})$, $v_j\in E$, $j=1,\dots,n$, such that $\|f-g\|<\varepsilon/2$.

Since A is uniformly dense in $C_0(X;\mathbb{K})$, there exists $a_j\in A$ such that

$$\|\psi_j-a_j\|<\frac{\varepsilon}{2(\sum_{j=1}^n\|v_j\|+1)}.$$

The function $h:=\sum_{j=1}^n a_jv_j\in A\bigotimes E$. Moreover,

$$\begin{aligned}\|f-h\|\le\|f-g\|+\|g-h\|&<\frac{\varepsilon}{2}+\sum_{j=1}^n\|\psi_j-a_j\|\|v_j\|\\&<\frac{\varepsilon}{2}+\frac{\varepsilon}{2(\sum_{j=1}^n\|v_j\|+1)}\sum_{j=1}^n\|v_j\|<\varepsilon.\end{aligned}$$ □

We obtain the following result.

Theorem 2.3. *If A is an uniformly dense vector subspace of $C_0(X;\mathbb{K})$ and B is a vector subspace of $C_0(X;E)$ such that $A\bigotimes E\subset B$, then B has the SAI property.*

Proof. It follows from Lemma 2.2, Lemma 2.3 and Theorem 2.2. □

Corollary 2.2 (Prolla [7], Theorem 7)**.** *Let X be a compact Hausdorff space and $B\subset C(X;E)$ a vector subspace such that $A:=\{\phi\circ g:\phi\in E^*,\ g\in B\}$ is uniformly dense in $C(X;\mathbb{K})$ and $A\bigotimes E\subset B$. Then B has the SAI property.*

Example 2.2. Let (X,d) be a compact metric space. A function $f:X\mapsto E$ is called Lipschitzian if there is some constant $K_f>0$ such that

$$\|f(x)-f(y)\|\le K_fd(x,y)$$

for every $x,y\in X$. We denote by $Lip(X;E)$ the subset of $C(X;E)$ of all such functions. By Theorem 9 [6], the vector subspace $Lip(X;\mathbb{K})$ is uniformly dense in $C(X;\mathbb{K})$. For any $f_1,\dots,f_n \in Lip(X;\mathbb{K})$, $v_1,\dots,v_n \in E$, there exist constants $k_1,\dots,k_n > 0$ such that

$$\|\sum_{j=1}^{n} f_j(x)v_j - f_j(y)v_j\| \le \sum_{j=1}^{n} |f_j(x) - f_j(y)|\|v_j\| \le \left(\sum_{j=1}^{n} k_j\|v_j\|\right) d(x,y)$$

for every $x, y \in X$. Hence, $Lip(X;\mathbb{K}) \bigotimes E \subset Lip(X;E)$. Then, by Theorem 2.3, $Lip(X;E)$ has the SAI property.

Example 2.3. Since $C_c(X;\mathbb{K})$ is uniformly dense in $C_0(X;\mathbb{K})$ (see Nachbin [4], p. 64) and $C_c(X;\mathbb{K}) \otimes E \subset C_c(X;E)$, it follows from Theorem 2.3 that $C_c(X;E)$ has the SAI property.

Example 2.4. Let X_i be a locally compact Hausdorff space for $i = 1,\dots,n$ and $X = X_1 \times \cdots \times X_n$.

Let A be the set of all finite sums of functions of the form

$$x = (x_1,\dots,x_n) \mapsto f(x) = g_1(x_1)\cdots g_n(x_n),$$

where $g_j \in C_0(X_j;\mathbb{K})$ for $j = 1,\dots,n$. By the weighted Dieudonné theorem ([4] Theorem 1 p. 68), A is an uniformly dense vector subspace of $C_0(X;\mathbb{K})$. From Theorem 2.3, $A \bigotimes E$ has the SAI property.

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