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COMPACT MATRIX OPERATORS BETWEEN THE SPACES $m(\phi)$, $n(\phi)$ AND ℓ_p

Eberhard Malkowsky and Mohammad Mursaleen

Abstract. We give the characterizations of the classes of matrix transformations $(m(\phi), \ell_p)$, $(n(\phi), \ell_p)$ ([5, Theorem 2]), $(\ell_p, m(\phi))$ ([5, Theorem 1]) and $(\ell_p, n(\phi))$ for $1 \le p \le \infty$, establish estimates for the norms of the bounded linear operators defined by those matrix transformations, and characterize the corresponding subclasses of compact matrix operators.

1. Introduction and notations

If X and Y are Banach spaces, then $S_X = \{x \in X : \|x\| = 1\}$ and $\bar{B}_X = \{x \in X : \|x\| \le 1\}$ are the unit sphere and the closed unit ball in X, and $\mathcal{B}(X, Y)$ is the set of all bounded linear operators $L : X \to Y$ with the operator norm $\|\cdot\|$ defined by $\|L\| = \sup\{\|L(x)\| : x \in S_X\}$; $X^* = \mathcal{B}(X, \mathbb{C})$ is the continuous dual of X, that is, the space of all continuous linear functionals on X with the norm defined by $\|f\| = \sup\{|f(x)| : x \in S_X\}$ for all $f \in X^*$.

We write ω, and ℓ_∞ and ϕ for the sets of all complex sequences $x = (x_k)_{k=1}^\infty$, and of all bounded and finite sequences, respectively, and $\ell_p = \{x \in \omega : \sum_{k=1}^\infty |x_k|^p < \infty\}$ for $1 \le p < \infty$; furthermore, cs is the set of all convergent series. By e and $e^{(n)}$ $(n = 1, 2, \dots)$ we denote the sequences with $e_k = 1$ for all k, and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ $(k \ne n)$. If $x = (x_k)_{k=1}^\infty$ is a sequence and $n \in \mathbb{N}$ then we write $x^{[n]} = \sum_{k=1}^n x_k e^{(k)}$ for the n-section of x.

A BK space X is a Banach sequence space such that the coordinate maps $P_n : X \to \mathbb{C}$ with $P_n(x) = x_n$ $(x = (x_k)_{k=1}^\infty \in X)$ are continuous for each $n \in \mathbb{N}$. A BK space $X \supset \phi$ is said to have AK, if every sequence $x = (x_k)_{k=1}^\infty \in X$ has a unique representation $x = \lim_{n\to\infty} x^{[n]}$.

Let x and y be sequences, X and Y be subsets of ω and $A = (a_{nk})_{n,k=1}^\infty$ be an infinite matrix of complex numbers. Then $X^\beta = \{a \in \omega : (a_k x_k) \in cs$ for all $x \in X\}$ is the β-dual of X. By $A_n = (a_{nk})_{k=1}^\infty$ and $A^{(k)} = (a_{nk})_{n=1}^\infty$, we denote the sequences in the n^{th} row and the k^{th} column of the matrix A. We write

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$A_nx = \sum_{k=1}^{\infty} a_{nk}x_k$, $Ax = (A_nx)_{n=1}^{\infty}$ (provided all the series A_nx converge), and (X, Y) for the class of all matrices A such that $A_n \in X^{\beta}$ for all n and $Ax \in Y$ for all $x \in X$.

Let X and Y be BK spaces. Since matrix maps between BK spaces are continuous ([8, Theorem 4.2.8, p. 57]), we have $(X, Y) \subset \mathcal{B}(X, Y)$, that is, every matrix $A \in (X, Y)$ defines an operator $L_A \in \mathcal{B}(X, Y)$ where $L_A(x) = Ax$ for all $x \in X$.

The operator $\Delta : \omega \to \omega$ is defined by $\Delta x = (\Delta x_k) = (x_k - x_{k-1})_{k=1}^{\infty}$ $(x \in \omega)$ where we suppose $x_{-1} = 0$. Given any sequence x, we denote by $S(x)$ the class of all sequences that are rearrangements of x. Let $\mathcal{C}$ denote the set of all finite subsets of $\mathbb{N}$. Given any set $\sigma \in \mathcal{C}$, we denote by $c(\sigma)$ the sequence with

$$c_n(\sigma) = \begin{cases} 1 & (n \in \sigma) \\ 0 & (n \notin \sigma). \end{cases}$$

For any $s \in \mathbb{N}$, $\mathcal{C}_s$ is the class of all $\sigma \in \mathcal{C}$ such that $\sum_{n=1}^{\infty} c_n(\sigma) \leq s$. The set Φ consists of all real sequences $(\phi_k)_{k=1}^{\infty}$ such that

$$\phi_1 > 0,\ \Delta\phi_k \geq 0 \text{ and } \Delta\left(\frac{\phi_k}{k}\right) \leq 0 \quad (k = 1, 2, \dots).$$

Sargent ([5]) defined and studied the following sequence spaces for $\phi \in \Phi$

$$m(\phi) = \left\{ x \in \omega : \sup_{s \geq 1} \sup_{\sigma \in \mathcal{C}_s} \left(\frac{1}{\phi_s} \sum_{k \in \sigma} |x_k| \right) < \infty \right\}$$

and

$$n(\phi) = \left\{ x \in \omega : \sup_{u \in S(x)} \left(\sum_{k=1}^{\infty} |u_k| \Delta\phi_k \right) < \infty \right\}$$

which are BK spaces ([5, (iii) and (iv), p. 162]) with their natural norms defined by

$$\|x\|_{m(\phi)} = \sup_{s \geq 1} \sup_{\sigma \in \mathcal{C}_s} \left(\frac{1}{\phi_s} \sum_{k \in \sigma} |x_k| \right) \text{ and } \|x\|_{n(\phi)} = \sup_{u \in S(x)} \left(\sum_{k=1}^{\infty} |u_k| \Delta\phi_k \right).$$

We give necessary and sufficient conditions for infinite matrices A to belong to any of the classes $(m(\phi), \ell_p)$, $(n(\phi), \ell_p)$ ([5, Theorem 2]), $(\ell_p, m(\phi))$ ([5, Theorem 1]) and $(\ell_p, n(\phi))$ for $1 \leq p \leq \infty$, and establish estimates for the norms of the corresponding operators L_A. Finally we characterize the compact operators L_A defined by the matrices A in the classes above, except for the cases $(\ell_1, m(\phi))$ and $(n(\phi), \ell_\infty)$.

2. Matrix transformations on and into $m(\phi)$ and $n(\phi)$

Here we give the characterizations of classes of matrix transformations A between the spaces ℓ_p $(1 \le p \le \infty)$ and $m(\phi)$ and $n(\phi)$ some of which can be found in [5, Theorems 1 and 2], and estimates for the operator norms of L_A.

Throughout let q be the conjugate number of p, that is, $q = \infty$ for $p = 1$, $q = p/(p-1)$ for $1 < p < \infty$ and $q = 1$ for $p = \infty$.

Let X be a BK space and $a \in \omega$. We write

$$\|a\|^* = \|a\|_X^* = \sup_{x \in S_X} \left| \sum_{k=1}^{\infty} a_k x_k \right|$$

provided the expression on the righthand side exists and is finite which is the case whenever $a \in X^\beta$ ([8, Theorem 7.2.9, p. 107]).

First we give the characterization of the class $(X, m(\phi))$ where X is any BK space, and establish an estimate for the operator norm of L_A.

Theorem 2.1. *Let X be a BK space. Then*

(a) *We have $A \in (X, m(\phi))$ if and only if*

$$\|A\|_{(X,m(\phi))} = \sup_{t \ge 1} \sup_{\tau \in \mathcal{C}_t} \left(\frac{1}{\phi_t} \left\| \sum_{n \in \tau} A_n \right\|_X^* \right) < \infty. \tag{2.1}$$

(b) *If $A \in (X, m(\phi))$, then*

$$\|A\|_{(X,m(\phi))} \le \|L_A\| \le 4 \cdot \|A\|_{(X,m(\phi))}. \tag{2.2}$$

Proof. (a) This is a special case of [2, Theorem 1].

(b) If $A \in (X, m(\phi))$, then $L_A \in \mathcal{B}(X, m(\phi))$, and so for all $x \in S_X$, $\tau \in \mathcal{C}_t$ and $t \ge 1$

$$\frac{1}{\phi_t} \left| \sum_{k=1}^{\infty} \left(\sum_{n \in \tau} a_{nk} \right) x_k \right| \le \frac{1}{\phi_t} \sum_{n \in \tau} |A_n x| \le \|L_A(x)\|_{m(\phi)} \le \|L_A\|.$$

This clearly implies

$$\frac{1}{\phi_t} \left\| \sum_{n \in \tau} A_n \right\|_X^* \le \|L_A\| \text{ for all } \tau \in \mathcal{C}_t \text{ and } \tau \ge 1,$$

and consequently $\|A\|_{(X,m(\phi))} \le \|L_A\|$, the first inequality in (2.2).
Furthermore, it follows from a well-known inequality ([4]) for all $x \in S_X$, $\tau \in \mathcal{C}_t$ and $\tau \ge 1$

$$\sum_{n \in \tau} |A_n x| \le 4 \cdot \max_{\tau' \subset \tau} \left| \sum_{n \in \tau'} A_n x \right| \le 4 \cdot \max_{\tau' \subset \tau} \left\| \sum_{n \in \tau'} A_n \right\|_X^*,$$

and this implies

$$\frac{1}{\phi_t} \sum_{n \in \tau} |A_n x| \le 4 \cdot \|A\|_{(X,m(\phi))},$$

hence

$$\|L_A(x)\| \le 4 \cdot \|A\|_{(X,m(\phi))} \text{ for all } x \in S_X,$$

and finally $\|L_A\| \le 4 \cdot \|A\|_{(X,m(\phi))}$, the second inequality in (2.2). □

Now we give the characterization of the class $(X, n(\phi))$ where X is any BK space, and establish an estimate for the operator norm of L_A. Given any matrix A, let $S(A)$ denote the class of all matrices that are obtained by rearranging the rows of A. We also write $\sup_N^*$ for the supremum taken over all finite subsets N of $\mathbb{N}$.

Theorem 2.2. *Let X be a BK space. Then*

(a) *We have $A \in (X, n(\phi))$ if and only if*

$$\|A\|_{(X,n(\phi))} = \sup_{B \in S(A)} \sup\nolimits_N^* \left\| \sum_{n \in N} B_n \Delta\phi_n \right\|_X^* < \infty. \tag{2.3}$$

(b) *If $A \in (X, n(\phi))$, then*

$$\|A\|_{(X,n(\phi))} \le \|L_A\| \le 4 \cdot \|A\|_{(X,n(\phi))}. \tag{2.4}$$

Proof. (a) This is a special case of [2, Theorem 2].

(b) If $A \in (X, n(\phi))$, then $L_A \in \mathcal{B}(X, n(\phi))$. Given $x \in X$, we write $y = Ax$ and observe that $v \in S(y)$ if and only if $v = Bx$ for some $B \in S(A)$, and so

$$\|Ax\|_{n(\phi)} = \sup_{B \in S(A)} \sum_{n=1}^{\infty} |B_n x| \Delta\phi_n.$$

If $A \in (X, n(\phi))$, then $L_A \in \mathcal{B}(X, n(\phi))$, and so for all $m \in \mathbb{N}$, all subsets N_m of $\{1, 2, \dots, m\}$, all $B \in S(A)$ and all $x \in S_X$

$$\left| \sum_{k=1}^{\infty} \left(\sum_{n \in N_m} b_{nk} \Delta\phi_n \right) x_k \right| \le \sum_{n=1}^{m} |B_n x \Delta\phi_n| \le \|L_A(x)\|_{n(\phi)} \le \|L_A\|.$$

This clearly implies

$$\left\| \sum_{n \in N_m} B_n \Delta\phi_n \right\|_X^* \le \|L_A\|$$

for all $m \in \mathbb{N}$, all subsets N_m of $\{1, 2, \dots, m\}$ and all $B \in S(A)$, and consequently

$$\|A\|_{(X,n(\phi))} = \sup_{B \in S(A)} \sup\nolimits_N^* \left\| \sum_{n \in N} B_n \Delta\phi_n \right\|_X^* \le \|L_A\|,$$

the first inequality in (2.4). Furthermore, it follows by the well-known inequality in ([4]) that

$$\sum_{n=1}^{m} |B_n x \Delta\phi_n| \le 4 \cdot \max_{N_m \subset \{1,\dots,m\}} \left| \sum_{n \in N_m} B_n x \Delta\phi_n \right|$$

$$\leq 4 \cdot \max_{N_m \subset \{1,\ldots,m\}} \left\| \sum_{n \in N_m} B_n \Delta\phi_n \right\|_X^* \leq 4 \cdot \|A\|_{(X,n(\phi))}$$

for all $m \in \mathbb{N}$, all $B \in S(A)$ and all $x \in S_X$. This implies $\|L_A(x)\|_{n(\phi)} \leq 4 \cdot \|A\|_{(X,n(\phi))}$ for all $x \in S_X$, and then $\|L_A\| \leq 4 \cdot \|A\|_{(X,n(\phi))}$, the second inequality in (2.4). □

Now we characterize the classes $(m(\phi), Y)$ and $(n(\phi), Y)$ where $Y = \ell_\infty$ or $Y = \ell_1$, and establish estimates for the operator norms $\|L_A\|$. Let N be finite subset of $\mathbb{N}$ and A be an infinite matrix then we write $b^{(A;N)}$ for the sequence with

$$b_k^{(A;N)} = \sum_{n \in N} a_{nk} \quad (k = 1, 2, \ldots).$$

Theorem 2.3. (a) *We have* $A \in (m(\phi), \ell_\infty)$ *if and only if*

$$\|A\|_{(m(\phi),\ell_\infty)} = \sup_n \left(\sup_{u \in S(A_n)} \sum_{k=1}^{\infty} |u_k| \Delta\phi_k \right) < \infty; \tag{2.5}$$

furthermore, if $A \in (m(\phi), \ell_\infty)$*, then*

$$\|L_A\| = \|A\|_{(m(\phi),\ell_\infty)}. \tag{2.6}$$

(b) *We have* $A \in (n(\phi), \ell_\infty)$ *if and only if*

$$\|A\|_{(n(\phi),\ell_\infty)} = \sup_n \left(\sup_{s \geq 1} \sup_{\sigma \in \mathcal{C}_s} \frac{1}{\phi_s} \sum_{k \in \sigma} |a_{nk}| \right) < \infty; \tag{2.7}$$

furthermore, if $A \in (n(\phi), \ell_\infty)$*, then*

$$\|L_A\| = \|A\|_{(n(\phi),\ell_\infty)}. \tag{2.8}$$

(c) *We have* $A \in (m(\phi), \ell_1)$ *if and only if*

(2.9)

$$\|A\|_{(m(\phi),\ell_1)} = \sup_N^* \left\| b^{(A;N)} \right\|_{n(\phi)} = \sup_N^* \left(\sup_{u \in S(b^{(A;N)})} \sum_{k=1}^{\infty} |u_k| \Delta\phi_k \right) < \infty;$$

furthermore, if $A \in (m(\phi), \ell_1)$*, then there are absolute constants* K_1 *and* K_2 *such that*

$$K_1 \cdot \|A\|_{(m(\phi),\ell_1)} \leq \|L_A\| \leq K_2 \cdot \|A\|_{(m(\phi),\ell_1)}. \tag{2.10}$$

(d) *We have* $A \in (n(\phi), \ell_1)$ *if and only if*

(2.11)

$$\|A\|_{(n(\phi),\ell_1)} = \sup_N^* \left\| b^{(A;N)} \right\|_{m(\phi)} = \sup_N^* \left(\sup_{s \geq 1} \sup_{\sigma \in \mathcal{C}_s} \left(\frac{1}{\phi_s} \sum_{k \in \sigma} \left| b_k^{(A;N)} \right| \right) \right) < \infty;$$

furthermore, if $A \in (n(\phi), \ell_1)$*, then there are absolute constants* K_1 *and* K_2 *such that*

$$K_1 \cdot \|A\|_{(n(\phi),\ell_1)} \leq \|L_A\| \leq K_2 \cdot \|A\|_{(n(\phi),\ell_1)}. \tag{2.12}$$

Proof. Since $m(\phi)$ and $n(\phi)$ are BK spaces, so are $(m(\phi))^\beta$ with $\|\cdot\|^*_{m(\phi)}$ and $(n(\phi))^\beta$ with $\|\cdot\|^*_{n(\phi)}$ ([8, Theorem 4.3.15, p. 64]); also since $(m(\phi))^\beta = n(\phi)$ ([5, Lemma 8]) and $(n(\phi))^\beta = m(\phi)$ ([5, Lemma 9]), the norms $\|\cdot\|^*_{m(\phi)}$ and $\|\cdot\|_{n(\phi)}$, and the norms $\|\cdot\|^*_{n(\phi)}$ and $\|\cdot\|_{m(\phi)}$ are equivalent on $(m(\phi))^\beta$), and on $(n(\phi))^\beta$ ([8, Corollary 4.2.4, p. 56]).

Thus Parts (a) and (b) are an immediate consequence of [3, Theorem 1.23, p. 155], and Parts (c) and (d) are an immediate consequence of [1, Satz 1]. □

We obtain the characterizations of the classes $(\ell_p, m(\phi))$ and $(\ell_p, n(\phi))$ for $1 \le p \le \infty$, and estimates for the operator norms of L_A as an immediate consequence of Theorems 2.1 and 2.2.

Corollary 2.4. *Let $1 \le p \le \infty$. Then*

(a) *We have $A \in (\ell_p, m(\phi))$ if and only if*

(2.13)
$$\|A\|_{(\ell_p,m(\phi))} = \begin{cases} \sup_{t\ge 1}\sup_{\tau\in\mathcal{C}_t}\left(\dfrac{1}{\phi_t}\sup_k\left|\sum_{n\in\tau} a_{nk}\right|\right) < \infty & (p=1) \\ \sup_{t\ge 1}\sup_{\tau\in\mathcal{C}_t}\left(\dfrac{1}{\phi_t}\left(\sum_{k=1}^\infty\left|\sum_{n\in\tau} a_{nk}\right|^q\right)^{1/q}\right) < \infty & (1<p\le\infty); \end{cases}$$

furthermore, if $A \in (\ell_p, m(\phi))$, then

$$\|A\|_{(\ell_p,m(\phi))} \le \|L_A\| \le 4\cdot\|A\|_{(\ell_p,m(\phi))}. \tag{2.14}$$

(b) *We have $A \in (\ell_p, n(\phi))$ if and only if*

(2.15)
$$\|A\|_{(\ell_p,n(\phi))} = \begin{cases} \sup_{B\in S(A)}\sup{}^*_N\left(\sup_k\left|\sum_{n\in N} b_{nk}\Delta\phi_n\right|\right) < \infty & (p=1) \\ \sup_{B\in S(A)}\sup{}^*_N\left(\sum_{k=1}^\infty\left|\sum_{n\in N} b_{nk}\Delta\phi_n\right|^q\right)^{1/q} < \infty & (1<p\le\infty); \end{cases}$$

furthermore, if $A \in (\ell_p, n(\phi))$, then

$$\|A\|_{(\ell_p,n(\phi))} \le \|L_A\| \le 4\cdot\|A\|_{(\ell_p,n(\phi))}. \tag{2.16}$$

Proof. Since ℓ_p^* and ℓ_q are norm isomorphic for $1 \le p < \infty$, and $\|\cdot\|^*_{\ell_\infty} = \|\cdot\|_1$ on ℓ_∞^β, Parts (a) and (b) follow from Theorems 2.1 and 2.2 by replacing the norm $\|\cdot\|^*_X$ in (2.1)-(2.4) by $\|\cdot\|_q$ $(1 \le q \le \infty)$. □

Finally we characterize the classes $(m(\phi), \ell_p)$ and $(n(\phi), \ell_p)$ for $1 < p < \infty$, and give estimates for the norms of L_A. Given any infinite matrix A, we denote its transpose by A^t.

Corollary 2.5. *Let $1 < p < \infty$. Then*

(a) *We have* $A \in (m(\phi), \ell_p)$ *if and only if*

$$\|A\|_{(m(\phi),\ell_p)} = \sup_{B \in S(A^t)} \sup{}_K^* \left(\sum_{n=1}^{\infty} \left| \sum_{k \in K} b_{nk} \Delta\phi_k \right|^p \right)^{1/p} < \infty; \tag{2.17}$$

furthermore, if $A \in (m(\phi), \ell_p)$*, then*

$$\|A\|_{(m(\phi),\ell_p)} \le \|L_A\| \le 4 \cdot \|A\|_{(m(\phi),\ell_p)}. \tag{2.18}$$

(b) *We have* $A \in (n(\phi), \ell_p)$ *if and only if*

$$\|A\|_{(n(\phi),\ell_p)} = \sup_{s \ge 1} \sup_{\sigma \in \mathcal{C}_s} \left(\frac{1}{\phi_s} \left(\sum_{n=1}^{\infty} \left| \sum_{k \in \sigma} a_{nk} \right|^p \right)^{1/p} \right) < \infty; \tag{2.19}$$

furthermore, if $A \in (n(\phi), \ell_p)$*, then*

$$\|A\|_{(n(\phi),\ell_p)} \le \|L_A\| \le 4 \cdot \|A\|_{(n(\phi),\ell_p)}. \tag{2.20}$$

Proof. The necessity and sufficiency of the conditions in (2.17) and (2.19) follow from [5, Lemma 14] and those in (2.15) and (2.13) in Corollary 2.4, respectively. Also the estimates in (2.18) and (2.20) follow from [7, Lemma 2] and those in (2.16) and (2.14) in Corollary 2.4, respectively. □

Remark 2.6. The characterizations of the classes $(\ell_p, m(\phi))$ in Corollary 2.4(a) and of $(n(\phi), \ell_p)$ in Corollary 2.4(b) can be found in [5, Theorems 1 and 2].

Remark 2.7. The proof of Corollary 2.5 extends to the case $p = 1$; hence we obtain alternative characterizations for the classes $(m(\phi), \ell_1)$ and $(n(\phi), \ell_1)$ and estimates for the operator norms from those given Theorem 2.3(c) and (d).

Remark 2.8. We observe that $\|A\|_{(X,\ell_p)} = \|A^t\|_{(\ell_q, X^\beta)}$ for $X = m(\phi)$ or $X = n(\phi)$ and $1 \le p \le \infty$ (by [5, Lemma 14] and [7, Lemma 2(4)]).

3. Compact operators

Here we give necessary and sufficient conditions for a matrix A to define a compact operator L_A between the spaces ℓ_p, $m(\phi)$ and $n(\phi)$.

We recall that a linear operator from a Banach space X into a Banach space Y is called *compact* if the domain of L is all of X and, for every bounded sequence $(x_n)_{n=1}^{\infty}$ in X, the sequence $(L(x_n))_{n=1}^{\infty}$ has a convergent subsequence in Y. We write $\mathcal{C}(X, Y)$ for the class of all compact operators from X into Y.

We note that the norms of the BK spaces ℓ_p for $1 \le p \le \infty$, $m(\phi)$ and $n(\phi)$ satisfy the condition

$$\|x\| = \sup_n \left\| x^{[n]} \right\| \text{ for all } x \in X; \tag{3.1}$$

this is trivial for ℓ_p, and the result for $m(\phi)$ and $n(\phi)$ can be found in [6, p. 64].

First we establish necessary and sufficient conditions on the entries of a matrix $A \in (m(\phi), \ell_1)$ or $A \in (n(\phi), \ell_1)$ for L_A to be a compact operator.

Given an infinite matrix $A = (a_{nk})_{n,k=1}^{\infty}$ and $m \in \mathbb{N}$, we write $A^{[m]} = (a_{nk}^{[m]})_{n,k=1}^{\infty}$ for the matrix with the rows $A_n^{[m]} = A_n$ for $1 \leq n \leq m$ and $A_n^{[m]} = 0$ for $n \geq m+1$; also let $C^{[m]} = A - A^{[m]}$. We denote by $\sup_{N_m}^*$ the supremum taken over all finite subsets of integers greater than or equal to $m+1$.

Theorem 3.1. (a) *If $A \in (m(\phi), \ell_1)$, then $L_A \in \mathcal{C}(m(\phi), \ell_1)$ if and only if*

$$\lim_{m\to\infty}\left(\sup{}_{N_m}^* \left\| b^{(A;N_m)} \right\|_{n(\phi)}\right) = \lim_{m\to\infty}\left(\sup{}_{N_m}^* \sup_{u \in S(b^{(A;N_m)})} \sum_{k=1}^{\infty} |u_k| \Delta\phi_k\right) = 0. \tag{3.2}$$

(b) *If $A \in (n(\phi), \ell_1)$, then $L_A \in \mathcal{C}(n(\phi), \ell_1)$ if and only if*

$$\begin{aligned}&\lim_{m\to\infty}\left(\sup{}_{N_m}^* \left\| b^{(A;N_m)} \right\|_{m(\phi)}\right) \\ &= \lim_{m\to\infty}\left(\sup{}_{N_m}^* \sup_{s\geq 1} \sup_{\sigma \in \mathcal{C}_s} \left(\frac{1}{\phi_s} \sum_{k\in\sigma} \left| \sum_{n\in N_m} a_{nk} \right|\right)\right) = 0.\end{aligned} \tag{3.3}$$

Proof. We assume $A \in (X, \ell_1)$ where $X = m(\phi)$ or $X = n(\phi)$. Since ℓ_1 has AK, $L_A \in \mathcal{C}(X, \ell_1)$ is equivalent to

$$\lim_{m\to\infty} \left\| C^{[m]} \right\|_{(X,\ell_1)} = \lim_{m\to\infty} \sup{}_N^* \left\| b^{(C^{[m]};N)} \right\|_{X^\beta} = 0 \tag{3.4}$$

by [7, Theorem 2(c), (8)] and (2.9)-(2.12) in Parts (c) and (d) of Theorem 2.3.

Let $m \in \mathbb{N}$ be given, N be a finite subset of $\mathbb{N}$ and $N'_m = \{n \in N : n \geq m+1\}$. Then we obviously have

$$b_k^{(C^{[m]};N)} = \sum_{n\in N} c_{nk}^{[m]} = \sum_{n\in N'_m} a_{nk} = b_k^{(A;N_m)} \text{ for all } k,$$

hence

$$\sup{}_N^* \left\| b^{(C^{[m]};N)} \right\|_{X^\beta} = \sup{}_{N_m}^* \left\| b^{(A;N_m)} \right\|_{X^\beta}$$

and the conditions in (3.2) and (3.3) follow from (3.4). □

Theorem 3.2. *Let $1 < p < \infty$.*

(a) *If $A \in (m(\phi), \ell_p)$, then $L_A \in \mathcal{C}(m(\phi), \ell_p)$ if and only if*

$$\lim_{m\to\infty} \sup_{B\in S(A^t)} \sup{}_K^* \left(\sum_{n=m+1}^{\infty} \left| \sum_{k\in K} b_{nk} \Delta\phi_k \right|^p \right)^{1/p} = 0. \tag{3.5}$$

(b) *If $A \in (n(\phi), \ell_p)$, then $L_A \in \mathcal{C}(n(\phi), \ell_p)$ if and only if*

$$\lim_{m\to\infty} \left(\sup_{s\geq 1} \sup_{\sigma\in\mathcal{C}_s} \left(\frac{1}{\phi_s} \sum_{n=m+1}^{\infty} \left| \sum_{k\in\sigma} a_{nk} \right|^p \right)^{1/p} \right) = 0. \tag{3.6}$$

Proof. We assume $A \in (X, \ell_p)$ $(1 < p < \infty)$ where $X = m(\phi)$ or $X = n(\phi)$. Since ℓ_p has AK, again by [7, Theorem 2(c), (8)], $L_A \in \mathcal{C}(X, \ell_p)$ is equivalent to

$$\lim_{m\to\infty} \left\| C^{[m]} \right\|_{(X,\ell_p)} = 0. \tag{3.7}$$

We write $D^{[m]} = (C^{[m]})^t$. Then $D^{[m]}$ is the matrix with the columns $(D^{[m]})^{(k)} = 0$ for $1 \le k \le m$ and $(D^{[m]})^{(k)} = A_k = (a_{kn})_{n=1}^{\infty}$ for $k \ge m+1$. Now the conditions in (3.5) and (3.6) follow from (3.7) by Remark 2.8 and (2.19) and (2.20) in Corollary 2.5 for $X = m(\phi)$, and (2.17) and (2.18) in Corollary 2.5 for $X = n(\phi)$. □

Remark 3.3. It is obvious from Remark 2.7 that the result of Theorem 3.2 extends to $p = 1$ and so we obtain alternative characterizations for the classes $(m(\phi), \ell_1)$ and $(n(\phi), \ell_1)$ from those given in Theorem 3.1.

Now we establish necessary and sufficient conditions for the entries of a matrix $A \in (\ell_p, m(\phi))$ or $A \in (\ell_p, n(\phi))$ $(1 < p \le \infty)$ for L_A to be a compact operator.

Given an infinite matrix $A = (a_{nk})_{n,k=1}^{\infty}$ and $m \in \mathbb{N}$, we write $A^{\langle m\rangle} = (a_{nk}^{\langle m\rangle})_{n,k=1}^{\infty}$ for the matrix with the columns $(A^{\langle m\rangle})^{(k)} = A^{(k)}$ for $1 \le k \le m$ and $(A^{\langle m\rangle})^{(k)} = 0$ for $k \ge m+1$; also let $C^{\langle m\rangle} = A - A^{\langle m\rangle}$.

Theorem 3.4. *Let* $1 < p \le \infty$.

(a) *If* $A \in (\ell_p, m(\phi))$, *then* $L_A \in \mathcal{C}(\ell_p, m(\phi))$ *if and only if*

$$\lim_{m\to\infty} \left(\sup_{t\ge 1} \sup_{\tau\in\mathcal{C}_t} \frac{1}{\phi_t} \left(\sum_{k=m+1}^{\infty} \left| \sum_{n\in\tau} a_{nk} \right|^q \right)^{1/q} \right) = 0. \tag{3.8}$$

(b) *If* $A \in (\ell_p, n(\phi))$, *then* $L_A \in \mathcal{C}(\ell_p, n(\phi))$ *if and only if*

$$\lim_{m\to\infty} \left(\sup_{B\in S(A)} \sup{}_N^* \left(\sum_{k=m+1}^{\infty} \left| \sum_{n\in N} b_{nk}\Delta\phi_n \right|^q \right)^{1/q} \right) = 0. \tag{3.9}$$

Proof. We assume $A \in (\ell_p, Y)$ where $Y = m(\phi)$ or $Y = n(\phi)$.

Since $\ell_p^{\beta} = \ell_q$ has AK for $1 < p \le \infty$, that is, for $1 \le q < \infty$, it follows from [7, Corollary, p. 84] that $L_A \in \mathcal{C}(\ell_p, Y)$ if and only if

$$\lim_{m\to\infty} \left\| C^{\langle m\rangle} \right\|_{(\ell_p,Y)} = 0. \tag{3.10}$$

Now the conditions in (3.8) and (3.9) are immediate consequences of (2.13)-(2.16) in Corollary 2.4. □

Remark 3.5. Let $1 < p < \infty$ and $X = m(\phi)$ or $X = n(\phi)$. It follows from [5, Lemma 14] and [7, Theorem 3] by [7, Lemma 2(4)] that if $A \in (X, \ell_p)$, then $L_A \in \mathcal{C}(X, \ell_p)$ if and only if $L_{A^t} \in C(\ell_q, X^{\beta})$; also $\|L_{A^t}\| = \|L_A\|$ by Remark

2.8. Thus the conditions in (3.5) and (3.6) can immediately be obtained from those in (3.9) and (3.8), respectively, and vice versa.

In the sequel we always assume that

$$\phi_k \to \infty \text{ and } \frac{k}{\phi_k} \to \infty \quad (k \to \infty), \tag{3.11}$$

since $m(\phi) = \ell_1$ (and consequently $n(\phi) = \ell_\infty$) if and only if $\lim_{k\to\infty} \phi_k < \infty$, and $m(\phi) = \ell_\infty$ (and consequently $n(\phi) = \ell_1$) if and only if $\lim_{k\to\infty}(k/\phi_k) = 0$ ([5, Lemma 5]).

Theorem 3.6. (a) *If $A \in (m(\phi), \ell_\infty)$, then $L_A \in \mathcal{C}(m(\phi), \ell_\infty)$ if and only if*

$$\lim_{m\to\infty} \left(\sup_n \sup_{u \in S(A_n)} \sum_{k=m+1}^{\infty} |u_k|\phi_k \right) = 0. \tag{3.12}$$

(b) *If $A \in (\ell_1, n(\phi))$, then $L_A \in \mathcal{C}(\ell_1, n(\phi))$ if and only if*

$$\lim_{m\to\infty} \left(\sup_k \sup_{u \in S(A^{(k)})} \sum_{n=m+1}^{\infty} |u_n|\phi_n \right) = 0. \tag{3.13}$$

Proof. (a) Since we assume (3.11), $(m(\phi))^\beta = n(\phi)$ has AK by [6, Theorem 8 (c)], and it follows from [7, Corollary, p. 84] and (2.6) in Theorem 2.3(a) that $L_A \in \mathcal{C}(m(\phi), \ell_\infty)$ if and only if

$$\lim_{m\to\infty} \left\| C^{\langle m \rangle} \right\|_{(m(\phi),\ell_\infty)} = 0. \tag{3.14}$$

Now (3.12) is an immediate consequence of (3.14) and (2.5) in Theorem 2.3 (a).

(b) Let $A \in (\ell_1, n(\phi))$. As in Remark 3.5 it follows that $L_A \in \mathcal{C}(\ell_1, n(\phi))$ if and only if $L_{A^t} \in \mathcal{C}(m(\phi), \ell_\infty)$; also $\|L_A\| = \|L_{A^t}\|$. So (3.13) is obtained from (3.12) with A replaced by A^t and n and k interchanged. □

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Eberhard Malkowsky
Department of Mathematics
University of Giessen
Arndtstrasse 2, D-35392 Giessen, Germany
and
Department of Mathematics
Faculty of Science
Fatih University
34500 Büyükçekmece, Istanbul, Turkey
E-mail address: Eberhard.Malkowsky@math.uni-giessen.de, ema@Bankerinter.net

Mohammad Mursaleen
Department of Mathematics
Aligarh Muslim University
Aligarh–202002, India
E-mail address: mursaleenm@gmail.com