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APPROXIMATION OF CUBIC MAPPINGS WITH n-VARIABLES IN β-NORMED LEFT BANACH MODULES ON BANACH ALGEBRAS

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Abstract. Let $M = \{1, 2, \dots, n\}$ and let $\mathcal{V} = \{I \subseteq M : 1 \in I\}$. Denote $M \setminus I$ by I^c for $I \in \mathcal{V}$. The goal of this paper is to investigate the solution and the stability using the alternative fixed point of generalized cubic functional equation

$$\sum_{I\in\mathcal{V}} f\Big(\sum_{i\in I} a_i x_i - \sum_{i\in I^c} a_i x_i\Big)$$
$$= 2^{n-2} a_1 \sum_{i=2}^{n} a_i^2 [f(x_1 + x_i) + f(x_1 - x_i)] + 2^{n-1} a_1 \Big(a_1^2 - \sum_{i=2}^{n} a_i^2\Big) f(x_1)$$

in β–Banach modules on Banach algebras, where $a_1, \dots, a_n \in \mathbb{Z} \setminus \{0\}$ with $a_1 \neq \pm 1$ and $a_n = 1$.

1. Introduction

We say a functional equation (ξ) is stable if any function g satisfying the equation (ξ) *approximately* is near to true solution of (ξ).

The first stability problem concerning group homomorphisms was raised by Ulam [26] in 1940 and affirmatively solved by Hyers [8]. Aoki [1] and Rassias [23] provided a generalization of the Hyers theorem for additive and linear functions, respectively, by allowing the Cauchy difference to be unbounded. In 1978, Th. M. Rassias [23] proved the following theorem.

Theorem 1.1. *Let $f : E \longrightarrow E'$ be a function from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

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for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then there exists a unique additive function $T : E \longrightarrow E'$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2 - 2^p}\|x\|^p \tag{1.2}$$

for all $x \in E$. If $p < 0$, then the inequality (1.1) holds for all $x, y \neq 0$, and (1.2) for $x \neq 0$. Also, if the function $t \mapsto f(tx)$ from $\mathbb{R}$ into E' is continuous in real t for each fixed $x \in E$, then T is linear.

In 1991, Z. Gajda [6] answered the question for the case $p > 1$, which was raised by Rassias. This new concept is known as Hyers–Ulam–Rassias stability of functional equations. During the last decades several stability problems of functional equations have been investigated in the spirt of Hyers–Ulam–Rassias. See [4, 7, 9, 15, 24, 25] for more detailed information on stability of functional equations.

Jun and Kim [10] introduced the following functional equation

$$f(2x_1 + x_2) + f(2x_1 - x_2) = 2f(x_1 + x_2) + 2f(x_1 - x_2) + 12f(x_1) \tag{1.1}$$

and they established the general solution and the generalized Hyers–Ulam–Rassias stability for the functional equation (1.1). They proved that a function f between real vector spaces X and Y is a solution of (1.1) if and only if there exits a unique function $C : X \times X \times X \longrightarrow Y$ such that $f(x) = C(x, x, x)$ for all $x \in X$, and C is symmetric for each fixed one variable and is additive for fixed two variables. It is easy to see that the function $f(x) = cx^3$ satisfies the functional equation (1.1), so it is natural to call (1.1) the cubic functional equation and every solution of the cubic functional equation (1.1) is said to be a cubic function.

Jun et al. [13] considered the following functional equation

$$f(ax_1+x_2)+f(ax_1-x_2) = a\big[f(x_1+x_2)+f(x_1-x_2)\big]+2a(a^2-1)f(x_1) \tag{1.2}$$

for any fixed integers a with $a \neq 0, \pm 1$. In this case, we see the equivalence of (1.1) and (1.2) (see [13]). Therefore, every solution of functional equations (1.1) and (1.2) is a cubic function (See Theorem 2.2 of [13]). For other cubic functional equations see [5], [12], [17]-[21].

Let $M = \{1, 2, \ldots, n\}$ and let $\mathcal{V} = \{I \subseteq M : 1 \in I\}$. Denote $M \backslash I$ by I^c for $I \in \mathcal{V}$. We will extend Eq. (1.2) to the general n-dimensional cubic functional equation for $n \geq 2$:

$$\begin{aligned} \sum_{I \in \mathcal{V}} f\Big(\sum_{i \in I} a_i x_i - \sum_{i \in I^c} a_i x_i\Big) &= 2^{n-2} a_1 \sum_{i=2}^{n} a_i^2 \big[f(x_1 + x_i) + f(x_1 - x_i)\big] \\ &\quad + 2^{n-1} a_1 \Big(a_1^2 - \sum_{i=2}^{n} a_i^2\Big) f(x_1), \end{aligned} \tag{1.3}$$

where $a_1, \ldots, a_n \in \mathbb{Z}\backslash\{0\}$ with $a_1 \neq \pm 1$ and $a_n = 1$. Moreover, we will study the stability of the given equation (1.3) in β–Banach module over Banach algebra via fixed point method.

As a special case, if $n = 2$ in (1.3), then we get the functional equation (1.2). Also, by putting $n = 3$ in (1.3), we obtain

$$\begin{aligned} &f(a_1x_1 - a_2x_2 - x_3) + f(a_1x_1 + a_2x_2 - x_3) \\ &\quad + f(a_1x_1 + x_3 - a_2x_2) + f(a_1x_1 + a_2x_2 + x_3) \\ &= 2a_1a_2^2\big[f(x_1 + x_2) + f(x_1 - x_2)\big] + 2a_1\big[f(x_1 + x_3) + f(x_1 - x_3)\big] \\ &\quad + 4a_1(a_1^2 - a_2^2 - 1)f(x_1). \end{aligned}$$

2. General solution

Let both X and Y be real vector spaces. We here present the general solution of (1.3).

Lemma 2.1. *Let $f : X \to Y$ be a cubic function. Then f satisfies*

$$\begin{aligned} &f(x_1 + x_2 + ax_3) + f(x_1 + x_2 - ax_3) + f(x_1 - x_2 + ax_3) + f(x_1 - x_2 - ax_3) \\ &= 2\big[f(x_1 + x_2) + f(x_1 - x_2)\big] + 2a^2\big[f(x_1 + x_3) + f(x_1 - x_3)\big] - 4a^2f(x_1) \end{aligned} \tag{2.1}$$

for all $x_1, x_2, x_3 \in X$, where a is an integer.

Proof. If $a = 0$, then it is clear. If $a \in \{\pm 1\}$, the general solution of (2.1) is of the form $f(x) = C(x) + A(x) + f(0)$, where C is a cubic mapping and A is an additive mapping (see [11] and [14]). Hence f satisfies (2.1) since f is cubic. Now, let $a \neq 0, \pm 1$. Since f satisfies the functional equation (1.1), putting $x_1 = x_2 = 0$ in (1.1), we get $f(0) = 0$. Setting $x_1 = 0$ in (1.1) to get $f(-x_2) = -f(x_2)$ for all $x_2 \in X$. Letting $x_2 = 0$ in (1.1), we obtain that $f(2x_1) = 8f(x_1)$ for all $x_1 \in X$. Also, f satisfies the functional equation (1.2) for any integers a with $a \neq 0, \pm 1$. Letting $x_2 = 0$ in (1.2), we get $f(ax_1) = a^3f(x_1)$ for all $x_1 \in X$. If we replace x_2 by ax_2 in (1.2), it is easy to see that the equation (1.2) can be written in the following way,

$$f(x_1 + ax_2) + f(x_1 - ax_2) = a^2\big[f(x_1 + x_2) + f(x_1 - x_2)\big] + 2(1 - a^2)f(x_1) \tag{2.2}$$

for all $x_1, x_2 \in X$. Replacing x_1 and x_2 by $x_1 + x_2$ and $x_1 - x_2$ in (1.2), respectively, and using the identity $f(2x) = 8f(x)$, we have

$$\begin{aligned} &f\big((a + 1)x_1 + (a - 1)x_2\big) + f\big((a - 1)x_1 + (a + 1)x_2\big) \\ &= 8a\big[f(x_1) + f(x_2)\big] + 2a(a^2 - 1)f(x_1 + x_2) \end{aligned} \tag{2.3}$$

for all $x_1, x_2 \in X$. Replacing x_1 and x_2 by $x_1 + ax_3$ and $x_2 + ax_3$ in (2.3), respectively, we have

$$\begin{aligned} &f\big((a + 1)x_1 + (a - 1)x_2 + 2a^2x_3\big) + f\big((a - 1)x_1 + (a + 1)x_2 + 2a^2x_3\big) \\ &= 8a\big[f(x_1 + ax_3) + f(x_2 + ax_3)\big] + 2a^2(a^2 - 1)f(x_1 + x_2 + 2ax_3) \end{aligned} \tag{2.4}$$

for all $x_1, x_2, x_3 \in X$. Replacing x_3 by $-x_3$ in (2.4), we get

$$f\big((a+1)x_1 + (a-1)x_2 - 2a^2x_3\big) + f\big((a-1)x_1 + (a+1)x_2 - 2a^2x_3\big) = 8a\big[f(x_1 - ax_3) + f(x_2 - ax_3)\big] + 2a(a^2-1)f(x_1 + x_2 - 2ax_3) \tag{2.5}$$

for all $x_1, x_2, x_3 \in X$. Now, by adding (2.4) and (2.5), we have

$$\begin{aligned} &f\big((a+1)x_1 + (a-1)x_2 + 2a^2x_3\big) + f\big((a+1)x_1 + (a-1)x_2 - 2a^2x_3\big) \\ &\quad + f\big((a-1)x_1 + (a+1)x_2 + 2a^2x_3\big) + f\big((a-1)x_1 + (a+1)x_2 - 2a^2x_3\big) \\ &= 8a\big[f(x_1 + ax_3) + f(x_1 - ax_3) + f(x_2 + ax_3) + f(x_2 - ax_3)\big] \\ &\quad + 2a(a^2-1)\big[f(x_1 + x_2 + 2ax_3) + f(x_1 + x_2 - 2ax_3)\big] \end{aligned} \tag{2.6}$$

for all $x_1, x_2, x_3 \in X$. On the other hand, if we substitute x_1 by $x_1 + ax_3$ and x_2 by $x_2 - ax_3$ in (2.3), f satisfies

$$f\big((a+1)x_1 + (a-1)x_2 + 2ax_3\big) + f\big((a-1)x_1 + (a+1)x_2 - 2ax_3\big) = 8a\big[f(x_1 + ax_3) + f(x_2 - ax_3)\big] + 2a(a^2-1)f(x_1 + x_2) \tag{2.7}$$

for all $x_1, x_2, x_3 \in X$. Replacing x_3 by $-x_3$ in (2.7), we get

$$f\big((a+1)x_1 + (a-1)x_2 - 2ax_3\big) + f\big((a-1)x_1 + (a+1)x_2 + 2ax_3\big) = 8a\big[f(x_1 - ax_3) + f(x_2 + ax_3)\big] + 2a(a^2-1)f(x_1 + x_2) \tag{2.8}$$

for all $x_1, x_2, x_3 \in X$. Adding (2.7) to (2.8), we lead to

$$\begin{aligned} &f\big((a+1)x_1 + (a-1)x_2 + 2ax_3\big) + f\big((a+1)x_1 + (a-1)x_2 - 2ax_3\big) \\ &\quad + f\big((a-1)x_1 + (a+1)x_2 + 2ax_3\big) + f\big((a-1)x_1 + (a+1)x_2 - 2ax_3\big) \\ &= 8a\big[f(x_1 + ax_3) + f(x_1 - ax_3) + f(x_2 + ax_3) + f(x_2 - ax_3)\big] \\ &\quad + 4a(a^2-1)f(x_1 + x_2) \end{aligned} \tag{2.9}$$

for all $x_1, x_2, x_3 \in X$. Now, replacing x_3 by ax_3 in (2.9), we obtain

$$\begin{aligned} &f\big((a+1)x_1 + (a-1)x_2 + 2a^2x_3\big) + f\big((a+1)x_1 + (a-1)x_2 - 2a^2x_3\big) \\ &\quad + f((a-1)x_1 + (a+1)x_2 + 2a^2x_3) + f((a-1)x_1 + (a+1)x_2 - 2a^2x_3) \\ &= 8a\big[f(x_1 + a^2x_3) + f(x_1 - a^2x_3) + f(x_2 + a^2x_3) + f(x_2 - a^2x_3)\big] \\ &\quad + 4a(a^2-1)f(x_1 + x_2) \end{aligned} \tag{2.10}$$

for all $x_1, x_2, x_3 \in X$. If we compare (2.6) with (2.10), we conclude that

$$\begin{aligned} &2a(a^2-1)\big[f(x_1 + x_2 + 2ax_3) + f(x_1 + x_2 - 2ax_3)\big] \\ &\quad + 8a\big[f(x_1 + ax_3) + f(x_1 - ax_3) + f(x_2 + ax_3) + f(x_2 - ax_3)\big] \\ &= 8a\big[f(x_1 + a^2x_3) + f(x_1 - a^2x_3) + f(x_2 + a^2x_3) + f(x_2 - a^2x_3)\big] \\ &\quad + 4a(a^2-1)f(x_1 + x_2) \end{aligned} \tag{2.11}$$

for all $x_1, x_2, x_3 \in X$. Since (2.2) holds for all integers a, it follows from (2.2) that

$$\begin{aligned} &f(x_1 + x_2 + 2ax_3) + f(x_1 + x_2 - 2ax_3) \\ &= 4\big[f(x_1 + x_2 + ax_3) + f(x_1 + x_2 - ax_3)\big] - 6f(x_1 + x_2) \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} &f(x_1 + a^2x_3) + f(x_1 - a^2x_3) + f(x_2 + a^2x_3) + f(x_2 - a^2x_3) \\ &= a^4\big[f(x_1 + x_3) + f(x_1 - x_3) + f(x_2 + x_3) + f(x_2 - x_3)\big] \\ &\quad + 2(1 - a^4)\big[f(x_1) + f(x_2)\big] \end{aligned} \tag{2.13}$$

for all $x_1, x_2, x_3 \in X$. It follows from (2.2), (2.11), (2.12) and (2.13) that

$$\begin{aligned} &f(x_1 + x_2 + ax_3) + f(x_1 + x_2 - ax_3) \\ &= a^2\big[f(x_1 + x_3) + f(x_1 - x_3) + f(x_2 + x_3) + f(x_2 - x_3)\big] \\ &\quad + 2f(x_1 + x_2) - 2a^2\big[f(x_1) + f(x_2)\big] \end{aligned} \tag{2.14}$$

for all $x_1, x_2, x_3 \in X$. If we replace x_2 by $-x_2$ in (2.14), we obtain by using the oddness of f that

$$\begin{aligned} &f(x_1 - x_2 + ax_3) + f(x_1 - x_2 - ax_3) \\ &= a^2\big[f(x_1 + x_3) + f(x_1 - x_3) - f(x_2 - x_3) - f(x_2 + x_3)\big] \\ &\quad + 2f(x_1 - x_2) - 2a^2\big[f(x_1) - f(x_2)\big] \end{aligned} \tag{2.15}$$

for all $x_1, x_2, x_3 \in X$. Adding (2.14) to (2.15), we obtain (2.1). □

Theorem 2.2. *A function $f : X \to Y$ satisfies the functional equation* (1.3) *if and only if the function $f : X \to Y$ is cubic.*

Proof. Let f be a function satisfying the functional equation (1.3). Putting $x_i = 0$ $(i = 1, \ldots, n)$ in (1.3), we have

$$2^{n-1}(a_1^3 - 1)f(0) = 0$$

that is, $f(0) = 0$ since $a_1 \neq \pm 1$. Setting $x_i = 0$ $(i = 2, \ldots, n-1)$ in (1.3) and then using $f(0) = 0$, we get

$$\begin{aligned} &2^{n-2}\big[f(a_1x_1 + x_n) + f(a_1x_1 - x_n)\big] \\ &= 2^{n-2}a_1\big[f(x_1 + x_n) + f(x_1 - x_n) + 2(a_1^2 - 1)f(x_1)\big] \end{aligned}$$

that is,

$$f(a_1x_1 + x_n) + f(a_1x_1 - x_n) = a_1\big[f(x_1 + x_n) + f(x_1 - x_n)\big] + 2a_1(a_1^2 - 1)f(x_1)$$

for all $x_1, x_n \in X$. Hence f satisfies (1.2). Thus f is cubic.

Conversely, suppose that f is cubic. Now, we are going to prove that f satisfies (1.3) by induction on $|M| = n \geq 2$. It holds for $n = 2$, since f satisfies

(1.2). Assume that it holds on the case where $|M| = n \geq 2$. Thus f satisfies (1.3). Since $a_n = 1$, it follows from (1.3) that

(2.16)
$$\sum_{\substack{I\in\mathcal{V}\\ n\in I}} f\Big(x_n + \sum_{\substack{i\in I\\ i\neq n}} a_i x_i - \sum_{i\in I^c} a_i x_i\Big) + \sum_{\substack{I\in\mathcal{V}\\ n\notin I}} f\Big(\sum_{i\in I} a_i x_i - \sum_{\substack{i\in I^c\\ i\neq n}} a_i x_i - x_n\Big)$$
$$= 2^{n-2} a_1 \sum_{i=2}^{n-1} a_i^2 \big[f(x_1 + x_i) + f(x_1 - x_i)\big] + 2^{n-1} a_1 \Big(a_1^2 - 1 - \sum_{i=2}^{n-1} a_i^2\Big) f(x_1)$$
$$+ 2^{n-2} a_1 \big[f(x_1 + x_n) + f(x_1 - x_n)\big]$$

for all $x_1, \ldots, x_n \in X$. Letting $b_i = a_i$ for all $1 \leq i \leq n-1$ and replacing x_n by $b_n x_n + x_{n+1}$ in (2.16), we obtain

(2.17)
$$\sum_{\substack{I\in\mathcal{V}\\ n\in I}} f\Big(b_n x_n + x_{n+1} + \sum_{\substack{i\in I\\ i\neq n}} b_i x_i - \sum_{i\in I^c} b_i x_i\Big)$$
$$+ \sum_{\substack{I\in\mathcal{V}\\ n\notin I}} f\Big(\sum_{i\in I} b_i x_i - \sum_{\substack{i\in I^c\\ i\neq n}} b_i x_i - b_n x_n - x_{n+1}\Big)$$
$$= 2^{n-2} b_1 \sum_{i=2}^{n-1} b_i^2 \big[f(x_1 + x_i) + f(x_1 - x_i)\big] + 2^{n-1} b_1 \Big(b_1^2 - 1 - \sum_{i=2}^{n-1} b_i^2\Big) f(x_1)$$
$$+ 2^{n-2} b_1 \big[f(x_1 + b_n x_n + x_{n+1}) + f(x_1 - b_n x_n - x_{n+1})\big]$$

for all $x_1, \ldots, x_{n+1} \in X$ where b_n is a non-zero integer. Replacing x_n by $-x_n$ in (2.17), we have

(2.18)
$$\sum_{\substack{I\in\mathcal{V}\\ n\in I}} f\Big(-b_n x_n + x_{n+1} + \sum_{\substack{i\in I\\ i\neq n}} b_i x_i - \sum_{i\in I^c} b_i x_i\Big)$$
$$+ \sum_{\substack{I\in\mathcal{V}\\ n\notin I}} f\Big(\sum_{i\in I} b_i x_i - \sum_{\substack{i\in I^c\\ i\neq n}} b_i x_i + b_n x_n - x_{n+1}\Big)$$
$$= 2^{n-2} b_1 \sum_{i=2}^{n-1} b_i^2 \big[f(x_1 + x_i) + f(x_1 - x_i)\big] + 2^{n-1} b_1 \Big(b_1^2 - 1 - \sum_{i=2}^{n-1} b_i^2\Big) f(x_1)$$
$$+ 2^{n-2} b_1 \big[f(x_1 - b_n x_n + x_{n+1}) + f(x_1 + b_n x_n - x_{n+1})\big]$$

for all $x_1, \dots, x_{n+1} \in X$. Let $\mathcal{W} := \Big\{ J \subseteq \{1, 2, \dots, n+1\} : 1 \in J \Big\}$ and $b_{n+1} = 1$. Then

$$\begin{aligned}
&\sum_{\substack{I\in\mathcal{V}\\ n\in I}} \Big\{ f\Big(b_n x_n + x_{n+1} + \sum_{\substack{i\in I\\ i\neq n}} b_i x_i - \sum_{i\in I^c} b_i x_i \Big) + f\Big(-b_n x_n + x_{n+1} + \sum_{\substack{i\in I\\ i\neq n}} b_i x_i - \sum_{i\in I^c} b_i x_i \Big) \Big\} \\
&\quad + \sum_{\substack{I\in\mathcal{V}\\ n\notin I}} \Big\{ f\Big(\sum_{i\in I} b_i x_i - \sum_{\substack{i\in I^c\\ i\neq n}} b_i x_i - b_n x_n - x_{n+1} \Big) + f\Big(\sum_{i\in I} b_i x_i - \sum_{\substack{i\in I^c\\ i\neq n}} b_i x_i + b_n x_n - x_{n+1} \Big) \Big\} \\
&= \sum_{\substack{I\in\mathcal{V}\\ n\in I}} \Big\{ f\Big(x_{n+1} + \sum_{i\in I} b_i x_i - \sum_{i\in I^c} b_i x_i \Big) + f\Big(-b_n x_n + x_{n+1} + \sum_{\substack{i\in I\\ i\neq n}} b_i x_i - \sum_{i\in I^c} b_i x_i \Big) \Big\} \\
&\quad + \sum_{\substack{I\in\mathcal{V}\\ n\notin I}} \Big\{ f\Big(-x_{n+1} + \sum_{i\in I} b_i x_i - \sum_{i\in I^c} b_i x_i \Big) + f\Big(\sum_{i\in I} b_i x_i - \sum_{\substack{i\in I^c\\ i\neq n}} b_i x_i + b_n x_n - x_{n+1} \Big) \Big\} \\
&= \sum_{J\in\mathcal{W}} f\Big(\sum_{i\in J} b_i x_i - \sum_{i\in J^c} b_i x_i \Big).
\end{aligned}$$

Therefore adding (2.17) to (2.18), we get

$$\begin{aligned}
&\sum_{J\in\mathcal{W}} f\Big(\sum_{i\in J} b_i x_i - \sum_{i\in J^c} b_i x_i \Big) \\
&= 2^{n-1} b_1 \sum_{i=2}^{n-1} b_i^2 \big[f(x_1 + x_i) + f(x_1 - x_i) \big] + 2^n b_1 \Big(b_1^2 - 1 - \sum_{i=2}^{n-1} b_i^2 \Big) f(x_1) \\
&\quad + 2^{n-2} b_1 \big[f(x_1 + x_{n+1} - b_n x_n) + f(x_1 - x_{n+1} + b_n x_n) \\
&\quad + f(x_1 + x_{n+1} + b_n x_n) + f(x_1 - x_{n+1} - b_n x_n) \big]
\end{aligned} \tag{2.19}$$

for all $x_1, \dots, x_{n+1} \in X$. Finally, it follows from (2.1) and (2.19) that

$$\begin{aligned}
&\sum_{J\in\mathcal{W}} f\Big(\sum_{i\in J} b_i x_i - \sum_{i\in J^c} b_i x_i \Big) \\
&= 2^{n-1} b_1 \sum_{i=2}^{n+1} b_i^2 \big[f(x_1 + x_i) + f(x_1 - x_i) \big] + 2^n b_1 \Big(b_1^2 - \sum_{i=2}^{n+1} b_i^2 \Big) f(x_1)
\end{aligned}$$

for all $x_1, \dots, x_{n+1} \in X$. Hence (1.3) holds for $|M| = n+1$. This completes the proof. □

Remark 2.3. Using the proof of Theorem 2.2, we conclude that if $f : X \to Y$ is a cubic function, then f satisfies (1.3) for $a_1, \dots, a_n \in \mathbb{Z}$ with $a_n = 1$.

Theorem 2.4. *Let $f : X \to Y$ be a cubic function and $b_1, \dots, b_n \in \mathbb{Z}$. Then*

$$\sum_{I\in\mathcal{V}} f\Big(\sum_{i\in I} b_i x_i - \sum_{i\in I^c} b_i x_i \Big) \tag{2.20}$$

$$= 2^{n-2}b_1\sum_{i=2}^{n} b_i^2\big[f(x_1+x_i)+f(x_1-x_i)\big]+2^{n-1}b_1\Big(b_1^2-\sum_{i=2}^{n}b_i^2\Big)f(x_1)$$

for all $x_1,\dots,x_n\in X$.

Proof. Since f is cubic, f satisfies (2.20) when $b_n=1$. That is

$$\sum_{\substack{I\in\mathcal{V}\\ n\in I}} f\Big(x_n+\sum_{\substack{i\in I\\ i\neq n}}b_ix_i-\sum_{i\in I^c}b_ix_i\Big)+\sum_{\substack{I\in\mathcal{V}\\ n\notin I}} f\Big(\sum_{i\in I}b_ix_i-\sum_{\substack{i\in I^c\\ i\neq n}}b_ix_i-x_n\Big) \tag{2.21}$$
$$= 2^{n-2}b_1\sum_{i=2}^{n-1} b_i^2\big[f(x_1+x_i)+f(x_1-x_i)\big]$$
$$+\,2^{n-2}b_1\big[f(x_1+x_n)+f(x_1-x_n)\big]+2^{n-1}b_1\Big(b_1^2-1-\sum_{i=2}^{n-1}b_i^2\Big)f(x_1)$$

for all $x_1,\dots,x_n\in X$. Replacing x_n by b_nx_n in (2.21), we get

$$\sum_{I\in\mathcal{V}} f\Big(\sum_{i\in I}b_ix_i-\sum_{i\in I^c}b_ix_i\Big) \tag{2.22}$$
$$= 2^{n-2}b_1\sum_{i=2}^{n-1} b_i^2\big[f(x_1+x_i)+f(x_1-x_i)\big]$$
$$+\,2^{n-2}b_1\big[f(x_1+b_nx_n)+f(x_1-b_nx_n)\big]$$
$$+\,2^{n-1}b_1\Big(b_1^2-1-\sum_{i=2}^{n-1}b_i^2\Big)f(x_1)$$

for all $x_1,\dots,x_n\in X$. Since f is cubic, f satisfies (2.2). Hence it follows from (2.2) and (2.22) that f satisfies (2.20). $\square$

Remark 2.5. Let $f:\mathbb{R}\to\mathbb{R}$ be a cubic function. If f is continuous at one point or f is measurable, then f is continuous on $\mathbb{R}$ and $f(x)=f(1)x^3$ for all $x\in\mathbb{R}$ (see [12]).

3. Generalized Hyers–Ulam stability

Before obtaining the main results in this section, we firstly introduce some useful concepts: we fix a real number β with $0<\beta\le 1$ and let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$. Let X be a linear space over $\mathbb{K}$. A real-valued function $\|\cdot\|_\beta$ is called a β–norm on X if and only if it satisfies

($\beta N1$) $\|x\|_\beta=0$ if and only if $x=0$;

($\beta N2$) $\|\lambda x\|_\beta=|\lambda|^\beta.\ \|x\|$ for all $\lambda\in\mathbb{K}$ and all $x\in X$;

($\beta N3$) $\|x+y\|_\beta\le\|x\|_\beta+\|y\|_\beta$ for all $x,y\in X$.

The pair $(X,\|\cdot\|_\beta)$ is called a β–normed space (see [2]). A β–Banach space is a complete β–normed space.

Throughout this section, let B be a unital Banach algebra with norm $|\cdot|$, $B_1=\{\,a\in B:|a|=1\,\}$, $\mathbb{X}$ be a β–normed left B-module and $\mathbb{Y}$ be a β–normed left Banach B-module. Using the fixed point alternative of Cădariu and Radu [3, 22], we will investigate the generalized Hyers–Ulam stability problem for the functional equation (1.3). Thus we find the condition that there exists a true cubic function near an approximately cubic function. We recall that $M = \{1, 2, \ldots, n\}$, $\mathcal{V} = \{I \subseteq M : 1 \in I\}$ and $I^c = M \setminus I$. Let $a_1, \ldots, a_n$ be non-zero integers with $a_1 \neq \pm 1, a_n = 1$ and $a_1^2 - \sum_{i=2}^{n} a_i^2 \neq 0$. For convenience, we use the following abbreviation for a given function $f : \mathbb{X} \to \mathbb{Y}$:

$$\begin{aligned} D_b f(x_1, \ldots, x_n) := &\sum_{I \in \mathcal{V}} f\Big(\sum_{i \in I} a_i b x_i - \sum_{i \in I^c} a_i b x_i\Big) \\ &- 2^{n-2} a_1 \sum_{i=2}^{n} a_i^2 \big[f(bx_1 + bx_i) + f(bx_1 - bx_i)\big] \\ &- 2^{n-1} a_1 \Big(a_1^2 - \sum_{i=2}^{n} a_i^2\Big) b^3 f(x_1) \end{aligned}$$

for all $x_1, \ldots, x_n \in \mathbb{X}$ and $b \in B_1$. We recall the following result by Margolis and Diaz [16].

Theorem 3.1. *Let (E, d) be a complete generalized metric space and let $J : E \to E$ be a strictly contractive function with Lipschitz constant $L < 1$. Then for each given element $x \in E$, either $d(J^n x, J^{n+1} x) = \infty$ for all nonnegative integers n or there exists a non-negative integer n_0 such that*

(1) *$d(J^n x, J^{n+1} x) < \infty$ for all $n \geq n_0$;*

(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*

(3) *y^* is the unique fixed point of J in the set $Y = \{y \in E : d(J^{n_0} x, y) < \infty\}$;*

(4) *$d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.*

Theorem 3.2. *Let $f : \mathbb{X} \to \mathbb{Y}$ be a function with $f(0) = 0$ for which there exists a function $\varphi : \mathbb{X}^n = \underbrace{\mathbb{X} \times \mathbb{X} \times \cdots \times \mathbb{X}}_{n-times} \to [0, \infty)$ such that*

$$\|D_b f(x_1, \ldots, x_n)\|_\beta \leq \varphi(x_1, \ldots, x_n) \tag{3.1}$$

for all $x_1, \ldots, x_n \in \mathbb{X}$ and all $b \in B_1$. If there exists a constant $0 < L < 1$ such that

$$\varphi(a_1 x_1, \ldots, a_1 x_n) \leq |a_1|^{3\beta} L \, \varphi(x_1, \ldots, x_n) \tag{3.2}$$

for all $x_1, \ldots, x_n \in \mathbb{X}$, then there exists a unique cubic function $C : \mathbb{X} \to \mathbb{Y}$ such that

$$\|f(x) - C(x)\|_\beta \leq \frac{1}{2^{(n-1)\beta} |a_1|^{3\beta} (1 - L)} \varphi(x, \underbrace{0, \ldots, 0}_{n-1}) \tag{3.3}$$

for all $x \in \mathbb{X}$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathbb{X}$, then C is B-cubic, i.e., $C(bx) = b^3 C(x)$ for all $x \in \mathbb{X}$ and all $b \in B$.

Proof. It follows from (3.2) that

$$\lim_{k\to\infty} \frac{1}{|a_1|^{3k\beta}} \varphi(a_1^k x_1, \ldots, a_1^k x_n) = 0 \tag{3.4}$$

for all $x_1, \ldots, x_n \in \mathbb{X}$. Letting $x_1 = x$, $x_2 = x_3 = \cdots = x_n = 0$ and $b = 1$ in (3.1) and using $f(0) = 0$, we get

$$\|f(a_1 x) - a_1^3 f(x)\|_\beta \leq \frac{1}{2^{(n-1)\beta}} \varphi(x, 0, 0, \ldots, 0) \tag{3.5}$$

for all $x \in X$. Let E be the set of all functions $g : \mathbb{X} \to \mathbb{Y}$ with $g(0) = 0$ and introduce a generalized metric on E as follows:

$$d(g,h) := \inf\{\, K \in [0,\infty] : \|g(x)-h(x)\|_\beta \leq K\varphi(x,0,0,\ldots,0) \quad \text{for all } x \in \mathbb{X}\,\}.$$

It is easy to show that (E, d) is a generalized complete metric space (see the Theorem 2.5 of [3]). Now we consider the function $\Lambda : E \to E$ defined by

$$(\Lambda g)(x) = \frac{1}{a_1^3} g(a_1 x) \quad \text{for all } g \in E \text{ and } x \in \mathbb{X}.$$

Let $g, h \in E$ and let $K \in [0, \infty]$ be an arbitrary constant with $d(g, h) \leq K$. From the definition of d, we have

$$\|g(x) - h(x)\|_\beta \leq K\varphi(x, 0, 0, \ldots, 0)$$

for all $x \in \mathbb{X}$. By the assumption and the last inequality, we have

$$\begin{aligned}
\|(\Lambda g)(x) - (\Lambda h)(x)\|_\beta &= \frac{1}{|a_1|^{3\beta}} \|g(a_1 x) - h(a_1 x)\|_\beta \\
&\leq \frac{1}{|a_1|^{3\beta}} K\varphi(a_1 x, 0, 0, \ldots, 0) \leq KL\varphi(x, 0, 0, \ldots, 0)
\end{aligned}$$

for all $x \in \mathbb{X}$. So

$$d(\Lambda g, \Lambda h) \leq Ld(g, h)$$

for any $g, h \in E$. It follows from (3.5) that $d(\Lambda f, f) \leq \frac{1}{2^{(n-1)\beta}|a_1|^{3\beta}}$. Therefore according to Theorem 3.1, the sequence $\{\Lambda^k f\}$ converges to a fixed point C of Λ, i.e.,

$$C : \mathbb{X} \to \mathbb{Y}, \quad C(x) = \lim_{k\to\infty} (\Lambda^k f)(x) = \lim_{k\to\infty} \frac{1}{a_1^{3k}} f(a_1^k x)$$

and $C(a_1 x) = a_1^3 C(x)$ for all $x \in \mathbb{X}$. Also C is the unique fixed point of Λ in the set $E^* = \{g \in E : d(f, g) < \infty\,\}$ and

$$d(C, f) \leq \frac{1}{1-L} d(\Lambda f, f) \leq \frac{1}{2^{(n-1)\beta}|a_1|^{3\beta}(1-L)},$$

i.e., inequality (3.3) holds true for all $x \in \mathbb{X}$. It follows from the definition of C, (3.1) and (3.4) that

$$\|D_1 C(x_1, \ldots, x_n)\|_\beta = \lim_{k\to\infty} \frac{1}{|a_1|^{3k\beta}} \left\|D_1 f(a_1^k x_1, \ldots, a_1^k x_n)\right\|_\beta$$

$$\leq \lim_{k\to\infty} \frac{1}{|a_1|^{3k\beta}}\varphi(a_1^k x_1, \ldots, a_1^k x_n) = 0$$

for all $x_1, \ldots, x_n \in \mathbb{X}$. By Theorem 2.2, the function $C : \mathbb{X} \to \mathbb{Y}$ is cubic. Finally it remains to prove the uniqueness of C. Let $T : \mathbb{X} \to \mathbb{Y}$ be another cubic function satisfying (3.3). Since $d(f,T) \leq \frac{1}{2^{(n-1)\beta}|a_1|^{3\beta}(1-L)}$ and T is cubic, we get $T \in E^*$ and $(\Lambda T)(x) = \frac{1}{a_1^3}T(a_1 x) = T(x)$ for all $x \in \mathbb{X}$, i.e., T is a fixed point of Λ. Since C is the unique fixed point of Λ in E^*, then $T = C$.

Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathbb{X}$, then by the same reasoning as in the proof of [23] C is $\mathbb{R}$-cubic. Setting $x_1 = x$ and $x_2 = \cdots = x_n = 0$ in (3.1), we get

(3.6)
$$\Big\|f(a_1 bx) - a_1 \sum_{i=2}^{n} a_i^2 f(bx) - a_1(a_1^2 - \sum_{i=2}^{n} a_i^2)b^3 f(x)\Big\|_\beta \leq \frac{1}{2^{(n-1)\beta}}\varphi(x, 0, 0, \ldots, 0)$$

for all $x \in \mathbb{X}$ and all $b \in B_1$. By definition of C, (3.4) and (3.6), we obtain

$$C(a_1 bx) - a_1 \sum_{i=2}^{n} a_i^2 C(bx) - a_1(a_1^2 - \sum_{i=2}^{n} a_i^2)b^3 C(x) = 0$$

for all $x \in \mathbb{X}$ and all $b \in B_1$. Since C is cubic and $a_1^2 - \sum_{i=2}^n a_i^2 \neq 0$, we get $C(bx) = b^3 C(x)$ for all $x \in \mathbb{X}$ and all $b \in B_1 \cup \{0\}$. Now, let $b \in B \setminus \{0\}$. Since C is $\mathbb{R}$-cubic,

$$C(bx) = C\Big(|b|.\frac{b}{|b|}x\Big) = |b|^3 C\Big(\frac{b}{|b|}x\Big) = |b|^3.\Big(\frac{b}{|b|}\Big)^3 C(x) = b^3 C(x)$$

for all $x \in \mathbb{X}$ and all $b \in B$. This proves that C is B-cubic. □

Corollary 3.3. *Let $0 < r < 3$ and θ, δ be non-negative real numbers and let $f : \mathbb{X} \to \mathbb{Y}$ be a function with $f(0) = 0$ such that*

$$\|D_b f(x_1, \ldots, x_n)\|_\beta \leq \delta + \theta \sum_{i=1}^{n} \|x_i\|_\beta^r \tag{3.7}$$

for all $x_1, \ldots, x_n \in \mathbb{X}$ and all $b \in B_1$. Then there exists a unique cubic function $C : \mathbb{X} \to \mathbb{Y}$ such that

$$\|f(x) - C(x)\|_\beta \leq \frac{1}{2^{(n-1)\beta}(|a_1|^{3\beta} - |a_1|^{\beta r})}\ \delta + \frac{1}{2^{(n-1)\beta}(|a_1|^{3\beta} - |a_1|^{\beta r})}\ \theta\|x\|_\beta^r$$

for all $x \in \mathbb{X}$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathbb{X}$, then C is B-cubic.

Remark 3.4. Let $f : \mathbb{X} \to \mathbb{Y}$ be a function for which there exists a function $\varphi : \mathbb{X}^n \to [0, \infty)$ satisfying (3.1). Let $0 < L < 1$ be a constant such that $|a_1|^{3\beta}\varphi(x_1, \ldots, x_n) \leq L\varphi(a_1 x_1, \ldots, a_1 x_n)$ for all $x_1, \ldots, x_n \in \mathbb{X}$. $f(0) = 0$, since $\varphi(0,\ldots,0)=0$. By a similar method to the proof of Theorem 3.2, one can show that there exists a unique cubic function $C:\mathbb{X}\to\mathbb{Y}$ satisfying

$$\|f(x)-C(x)\|_\beta \le \frac{L}{2^{(n-1)\beta}|a_1|^{3\beta}(1-L)}\varphi(x,0,0,\ldots,0)$$

for all $x\in\mathbb{X}$. Moreover, if $f(tx)$ is continuous in $t\in\mathbb{R}$ for each fixed $x\in\mathbb{X}$, then C is B-cubic.

For the case $\varphi(x_1,\ldots,x_n):=\theta\sum_{i=1}^n\|x_i\|^r$ (where θ is a non-negative real number and $r>3$), there exists a unique cubic function $C:\mathbb{X}\to\mathbb{Y}$ satisfying

$$\|f(x)-C(x)\|_\beta \le \frac{1}{2^{(n-1)\beta}(|a_1|^{\beta r}-|a_1|^{3\beta})}\ \theta\|x\|_\beta^r$$

for all $x\in\mathbb{X}$.

Remark 3.5. Let $f:\mathbb{X}\to\mathbb{Y}$ be given and

$$\begin{aligned} D_b'f(x_1,\ldots,x_n):=&\sum_{I\in\mathcal{V}} f\Big(\sum_{i\in I}a_ibx_i-\sum_{i\in I^c}a_ibx_i\Big)\\ &-2^{n-2}a_1\sum_{i=2}^n a_i^2b^3\big[f(x_1+x_i)+f(x_1-x_i)\big]\\ &-2^{n-1}a_1\Big(a_1^2-\sum_{i=2}^n a_i^2\Big)b^3f(x_1)\end{aligned}$$

for all $x_1,\ldots,x_n\in\mathbb{X}$ and $b\in B_1$. Theorem 3.2 and Remark 3.4 hold true if we replace $D_bf(x_1,\ldots,x_n)$ by $D_b'f(x_1,\ldots,x_n)$ in (3.1). In this case we do not need the condition $a_1^2-\sum_{i=2}^n a_i^2\neq 0$.

The generalized Hyers–Ulam stability problem for the case of $r=3$ was excluded in Corollary 3.3 and Remarks 3.4, 3.5. In fact, the functional equation (1.3) is not stable for $r=3$ in (3.7) as we shall see in the following example, which is a modification of the example of Z. Gajda [6] for the additive functional inequality.

Example 3.6. Let $\phi:\mathbb{C}\to\mathbb{C}$ be defined by

$$\phi(x):=\begin{cases} x^3 & \text{for } |x|<1;\\ 1 & \text{for } |x|\ge 1.\end{cases}$$

Consider the function $f:\mathbb{C}\to\mathbb{C}$ by the formula

$$f(x):=\sum_{m=0}^{\infty}\alpha^{-3m}\phi(\alpha^m x),$$

where $\alpha>\max\{\,|a_1|,\ldots,|a_n|\,\}$ and $a_1,\ldots,a_n$ are non-zero integers. Let

$$D_\mu f(x_1,\ldots,x_n):=\sum_{I\in\mathcal{V}} f\Big(\sum_{i\in I}a_i\mu x_i-\sum_{i\in I^c}a_i\mu x_i\Big)$$

$$
\begin{aligned}
&\quad - 2^{n-2} a_1 \sum_{i=2}^{n} a_i^2 \big[f(\mu x_1 + \mu x_i) + f(\mu x_1 - \mu x_i) \big] \\
&\quad - 2^{n-1} a_1 \Big(a_1^2 - \sum_{i=2}^{n} a_i^2 \Big) \mu^3 f(x_1), \\
D'_\mu f(x_1, \ldots, x_n) :&= \sum_{I \in \mathcal{V}} f\Big(\sum_{i \in I} a_i \mu x_i - \sum_{i \in I^c} a_i \mu x_i \Big) \\
&\quad - 2^{n-2} a_1 \sum_{i=2}^{n} a_i^2 \mu^3 \big[f(x_1 + x_i) + f(x_1 - x_i) \big] \\
&\quad - 2^{n-1} a_1 \Big(a_1^2 - \sum_{i=2}^{n} a_i^2 \Big) \mu^3 f(x_1).
\end{aligned}
$$

Then f satisfies

$$
\|D_\mu f(x_1, \ldots, x_n)\| \leq \frac{n 2^n \alpha^{12}}{M^3(\alpha^3 - 1)} \sum_{i=1}^{n} |x_i|^3, \tag{3.8}
$$

$$
\|D'_\mu f(x_1, \ldots, x_n)\| \leq \frac{n 2^n \alpha^{12}}{M^3(\alpha^3 - 1)} \sum_{i=1}^{n} |x_i|^3 \tag{3.9}
$$

for all $\mu \in \mathbb{T} := \{ \lambda \in \mathbb{C} : |\lambda| = 1 \}$, all $M \in (0, \frac{1}{n})$ and all $x_1, \ldots, x_n \in \mathbb{C}$, and the range of $|f(x) - C(x)|/|x|^3$ for $x \neq 0$ is unbounded for each cubic function $C : \mathbb{C} \to \mathbb{C}$.

Proof. It is enough to prove that f satisfies (3.8) and we have a similar proof for (3.9). It is clear that f is bounded by $\frac{\alpha^3}{\alpha^3 - 1}$ on $\mathbb{C}$. Let $0 < M < \frac{1}{n}$. If $\sum_{i=1}^{n} |a_i x_i|^3 = 0$ or $\sum_{i=1}^{n} |a_i x_i|^3 \geq \frac{M^3}{\alpha^3}$, then

$$
\begin{aligned}
\big|D_\mu f(x_1, \ldots, x_n)\big| &\leq 2^{n-1} \Big[1 + |a_1| \sum_{i=2}^{n} |a_i|^2 + |a_1| \Big| a_1^2 - \sum_{i=2}^{n} a_i^2 \Big| \Big] \frac{\alpha^3}{\alpha^3 - 1} \\
&\leq n 2^n \frac{\alpha^6}{\alpha^3 - 1} \\
&\leq \frac{n 2^n \alpha^{12}}{M^3(\alpha^3 - 1)} \sum_{i=1}^{n} |x_i|^3.
\end{aligned}
$$

Now suppose that $0 < \sum_{i=1}^{n} |a_i x_i|^3 < \frac{M^3}{\alpha^3}$. Then there exists an integer $k \geq 1$ such that

$$
\frac{M^3}{\alpha^{3(k+1)}} \leq \sum_{i=1}^{n} |a_i x_i|^3 < \frac{M^3}{\alpha^{3k}}. \tag{3.10}
$$

Therefore

$$
\alpha^m \Big| \sum_{i \in I} a_i \mu x_i - \sum_{i \in I^c} a_i \mu x_i \Big|, \alpha^m |\mu x_1 \pm \mu x_i|, \alpha^m |x_1| < 1
$$

for all $m = 0, 1, \ldots, k-1,\ i = 2, \ldots, n$ and all $I \in \mathcal{V}$. From the definition of f and (3.10), we have

$$\begin{aligned}
&\big|D_\mu f(x_1, \ldots, x_n)\big| \\
&= \bigg| \sum_{I\in\mathcal{V}} \sum_{m=k}^{\infty} \alpha^{-3m} \phi\Big(\sum_{i\in I} \alpha^m a_i \mu x_i - \sum_{i\in I^c} \alpha^m a_i \mu x_i \Big) \\
&\qquad - 2^{n-2} a_1 \sum_{i=2}^{n} a_i^2 \sum_{m=k}^{\infty} \alpha^{-3m} \big[\phi(\alpha^m \mu x_1 + \alpha^m \mu x_i) + \phi(\alpha^m \mu x_1 - \alpha^m \mu x_i)\big] \\
&\qquad - 2^{n-1} a_1 \Big(a_1^2 - \sum_{i=2}^{n} a_i^2\Big) \mu^3 \sum_{m=k}^{\infty} \alpha^{-3m} \phi(\alpha^m x_1) \bigg| \\
&\le 2^{n-1} \Big[1 + |a_1| \sum_{i=2}^{n} |a_i|^2 + |a_1| \Big|a_1^2 - \sum_{i=2}^{n} a_i^2\Big|\Big] \frac{\alpha^3}{\alpha^{3k}(\alpha^3 - 1)} \\
&\le n2^n \frac{\alpha^6}{\alpha^{3k}(\alpha^3-1)} \le \frac{n2^n\alpha^9}{M^3(\alpha^3-1)} \sum_{i=1}^{n} |a_i x_i|^3 \le \frac{n2^n\alpha^{12}}{M^3(\alpha^3-1)} \sum_{i=1}^{n} |x_i|^3.
\end{aligned}$$

Therefore f satisfies (3.8). Let $C : \mathbb{C} \to \mathbb{C}$ be a cubic function such that

$$|f(x) - C(x)| \le \beta |x|^3$$

for all $x \in \mathbb{C}$. Then there exists a constant $\gamma \in \mathbb{C}$ such that $C(x) = \gamma x^3$ for all rational numbers x. So we have

$$|f(x)| \le (\beta + |c|)|x|^3 \tag{3.11}$$

for all rational numbers x. Let $m \in \mathbb{N}$ with $m > \beta + |\gamma|$. If x is a rational number in $(0, \alpha^{1-m})$, then $\alpha^k x \in (0,1)$ for all $k = 0, 1, \ldots, m-1$. So

$$f(x) \ge \sum_{k=0}^{m-1} \alpha^{-3k} \phi(\alpha^k x) = m x^3 > (\beta + |\gamma|) x^3$$

which contradicts (3.11). □

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