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RANKS OF SUBMATRICES IN A GENERAL SOLUTION TO A QUATERNION SYSTEM WITH APPLICATIONS

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Abstract. Assume that X, partitioned into 2×2 block form, is a solution of the system of quaternion matrix equations $A_1XB_1 = C_1, A_2XB_2 = C_2$. We in this paper give the maximal and minimal ranks of the submatrices in X, and establish necessary and sufficient conditions for the submatrices to be zero, unique as well as independent. As applications, we consider the common inner inverse G, partitioned into 2×2 block form, of two quaternion matrices M and N. We present the formulas of the maximal and minimal ranks of the submatrices of G, and describe the properties of the submatrices of G as well. The findings of this paper generalize some known results in the literature.

1. Introduction

Throughout this paper, $\mathbb{R}$ stands for the real number field, $\mathbb{H}^{m\times n}$ represents the set of all $m \times n$ matrices over the quaternion algebra

$$\mathbb{H} = \{a_0 + a_1 i + a_2 j + a_3 k \mid i^2 = j^2 = k^2 = ijk = -1,\ a_0, a_1, a_2, a_3 \ \in \mathbb{R}\}.$$

For a matrix A over $\mathbb{H}$, we denote the transpose of A by A^T, the column right space, the row left space of A by $\mathcal{R}(A)$, $\mathcal{N}(A)$, respectively, an inner inverse of A by A^- which satisfies $AA^-A = A$. Moreover, R_A and L_A stand for the two projectors $L_A = I - A^-A$, $R_A = I - AA^-$ induced by A. In [2], $\dim\mathcal{R}(A) = \dim\mathcal{N}(A)$, which is called the rank of A and denoted by $r(A)$. I stands for the identity matrix with the appropriate size.

Consider the classical system of matrix equations

$$A_1XB_1 = C_1, A_2XB_2 = C_2, \tag{1.1}$$

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where $A_1, A_2 \in \mathbb{H}^{m\times k}, B_1, B_2 \in \mathbb{H}^{l\times n}$ and $C_1, C_2 \in \mathbb{H}^{m\times n}$ are known and X unknown. Partitioning a solution X of (1.1) into 2×2 block form, we have that

$$\begin{aligned} &[A_{11}, A_{12}] \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix} = C_1, \\ &[A_{21}, A_{22}] \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} \begin{bmatrix} B_{21} \\ B_{22} \end{bmatrix} = C_2, \end{aligned} \tag{1.2}$$

where $X_1 \in \mathbb{H}^{k_1\times l_1}, X_2 \in \mathbb{H}^{k_1\times l_2}, X_3 \in \mathbb{H}^{k_2\times l_1}$ and $X_4 \in \mathbb{H}^{k_2\times l_2}, k_1 + k_2 = k, l_1 + l_2 = l$.

We know that the system (1.1) has been investigated by many authors from different aspects (see, e.g., [4]-[15]). For example, Mitra [4] first considered it over the complex field in 1973. Özgüler and Akar [6] gave some solvability conditions for the consistency of the system over a principle domain in 1991. Wang [14] derived necessary and sufficient conditions for the existence and the expression for the general solution to the system over arbitrary regular rings with identity in 2004. Yan and Liao in 2008 investigated the least squares Hermitian solution to the system over $\mathbb{H}$. Wang [15] established necessary and sufficient conditions for the existence and the expressions for the general real solutions to (1.1) over $\mathbb{H}$. Moreover, Dehghan and Hajarian [1] in 2008 considered the generalized centro-symmetric solutions to (1.1) by an iterative algorithm.

Extreme ranks of general solutions to linear matrix equations have been actively ongoing for many years. For instance, Uhlig proposed maximal and minimal possible ranks of solutions of the equation $AX = B$ in [11]. Liu [3] gave some formulas for the extreme ranks of the submatrices in a solution X to the system of complex matrix equations $AX = C, XB = D$ in 2008. In 2009, Tian [7] gave some formulas for the extreme ranks of the submatrices in a solution X to the complex matrix equation $AXB = C$. Noticing that so far there has been little information on the extreme ranks, i.e., the maximal and minimal ranks of the submatrices in a solution X to (1.1), we in this paper aim to investigate the extremal ranks of submatrices in a solution to (1.1) over $\mathbb{H}$.

The paper is organized as follows. In Section 2, we first give formulas of the extreme ranks of matrices X_i $(i = 1, 2, 3, 4)$ in (1.2), and characterize structure of solutions to (1.2), then establish necessary and sufficient conditions for the uniqueness of the submatrices X_i $(i = 1, 2, 3, 4)$ in (1.2), and consider the independence of submatrices in solutions to (1.2). As applications, we in Section 3 give the maximal and minimal ranks of the submatrices of the common inner inverse G, partitioned into 2×2 block form, of quaternion matrices M and N, and describe the properties of the submatrices of G. The results in this paper generalize the results in [7] and [3].

2. Extreme ranks and the independence of submatrices in solutions to (1.2)

We begin with the following lemmas whose proofs are just like those over the complex field.

Lemma 2.1 ([5]). *Let A, B and C be $m \times n, m \times k, l \times n$ matrices over $\mathbb{H}$. Then*

(a) $r[A, B] = r(A) + r(R_A B) = r(B) + r(R_B A)$,

(b) $r \begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(CL_A) = r(C) + r(AL_C)$,

(c) $r \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r(B) + r(C) + r(R_B A L_C)$.

Lemma 2.2 ([8]). *Suppose that the system* (1.1) *is consistent. Then the general solution of* (1.1) *can be expressed as*

$$X = X_0 + L_A V_1 + V_2 R_B + L_{A_1} V_3 R_{B_2} + L_{A_2} V_4 R_{B_1}, \tag{2.1}$$

where X_0 is a particular solution to (1.1), $A = \left[\begin{smallmatrix} A_1 \\ A_2 \end{smallmatrix}\right], B = [B_1, B_2]$ *are known and $V_1 - V_4$ are arbitrary matrices with appropriate dimensions.*

Lemma 2.3 ([9]). *Let $p(X_1, X_2) = A - B_1 X_1 C_1 - B_2 X_2 C_2$ be given. Then*

$$\max_{X_1, X_2} r[p(X_1, X_2)] = \min \left\{ r[A, B_1, B_2], r \begin{bmatrix} A \\ C_1 \\ C_2 \end{bmatrix}, r \begin{bmatrix} A & B_1 \\ C_2 & 0 \end{bmatrix}, r \begin{bmatrix} A & B_2 \\ C_1 & 0 \end{bmatrix} \right\}, \tag{2.2}$$

and

$$\begin{aligned} &\min_{X_1, X_2} r[p(X_1, X_2)] \\ &= r[A, B_1, B_2] + r \begin{bmatrix} A \\ C_1 \\ C_2 \end{bmatrix} + \max \left\{ r \begin{bmatrix} A & B_1 \\ C_2 & 0 \end{bmatrix} - r \begin{bmatrix} A & B_1 & B_3 \\ C_2 & 0 & 0 \end{bmatrix} \right. \\ &\quad \left. - r \begin{bmatrix} A & B_1 \\ C_1 & 0 \\ C_2 & 0 \end{bmatrix}, r \begin{bmatrix} A & B_2 \\ C_1 & 0 \end{bmatrix} - r \begin{bmatrix} A & B_1 & B_2 \\ C_1 & 0 & 0 \end{bmatrix} - r \begin{bmatrix} A & B_2 \\ C_1 & 0 \\ C_2 & 0 \end{bmatrix} \right\}. \end{aligned} \tag{2.3}$$

Lemma 2.4 (see [9]). *Let $p(X_1, X_2) = A - B_1 X_1 C_1 - B_2 X_2 C_2$, and*

$$\mathcal{R}(B_1) \subseteq \mathcal{R}(B_2), \mathcal{R}(C_2^T) \subseteq \mathcal{R}(C_1^T)$$

be given. Then

$$\begin{aligned} &\min_{X_1, X_2} r[p(X_1, X_2)] \\ &= r[A, B_2] + r \begin{bmatrix} A \\ C_1 \end{bmatrix} + r \begin{bmatrix} A & B_1 \\ C_2 & 0 \end{bmatrix} - r \begin{bmatrix} A & B_1 \\ C_1 & 0 \end{bmatrix} - r \begin{bmatrix} A & B_2 \\ C_2 & 0 \end{bmatrix}. \end{aligned} \tag{2.4}$$

Lemma 2.5. *Let*

$$p(X_1, X_2, X_3, X_4) = A - B_1X_1C_1 - B_2X_2C_2 - B_3X_3C_3 - B_4X_4C_4 \tag{2.5}$$

be a linear matrix expression with four two-sided terms over $\mathbb{H}$*, and suppose that the given matrices satisfy the conditions*

$$\mathcal{R}(B_i) \subseteq \mathcal{R}(B_2), \ \mathcal{R}\left(C_j^T\right) \subseteq \mathcal{R}\left(C_1^T\right), i = 1, 3, 4, j = 2, 3, 4. \tag{2.6}$$

Then

$$\begin{aligned}
&\min_{X_i} r\left[p(X_1, X_2, X_3, X_4)\right] \\
&= r\begin{bmatrix} A & B_1 \\ C_2 & 0 \\ C_3 & 0 \\ C_4 & 0 \end{bmatrix} + r\begin{bmatrix} A & B_1 & B_3 & B_4 \\ C_2 & 0 & 0 & 0 \end{bmatrix} + r\begin{bmatrix} A \\ C_1 \end{bmatrix} + r\left[A, B_2\right] \\
&\quad - r\begin{bmatrix} A & B_1 \\ C_1 & 0 \end{bmatrix} - r\begin{bmatrix} A & B_2 \\ C_2 & 0 \end{bmatrix} \\
&\quad + \max\left\{ r\begin{bmatrix} A & B_1 & B_3 \\ C_2 & 0 & 0 \\ C_4 & 0 & 0 \end{bmatrix} - r\begin{bmatrix} A & B_1 & B_3 & B_4 \\ C_2 & 0 & 0 & 0 \\ C_4 & 0 & 0 & 0 \end{bmatrix} \right. \\
&\qquad - r\begin{bmatrix} A & B_1 & B_3 \\ C_2 & 0 & 0 \\ C_3 & 0 & 0 \\ C_4 & 0 & 0 \end{bmatrix}, \\
&\qquad r\begin{bmatrix} A & B_1 & B_4 \\ C_2 & 0 & 0 \\ C_3 & 0 & 0 \end{bmatrix} - r\begin{bmatrix} A & B_1 & B_3 & B_4 \\ C_2 & 0 & 0 & 0 \\ C_3 & 0 & 0 & 0 \end{bmatrix} \\
&\qquad \left. - r\begin{bmatrix} A & B_1 & B_4 \\ C_2 & 0 & 0 \\ C_3 & 0 & 0 \\ C_4 & 0 & 0 \end{bmatrix} \right\}
\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}
&\max_{X_i} r\left[p(X_1, X_2, X_3, X_4)\right] \\
&= \min\left\{ r\left[A, B_2\right], \ r\begin{bmatrix} A \\ C_1 \end{bmatrix}, r\begin{bmatrix} A & B_1 \\ C_2 & 0 \\ C_3 & 0 \\ C_4 & 0 \end{bmatrix}, r\begin{bmatrix} A & B_1 & B_3 & B_4 \\ C_2 & 0 & 0 & 0 \end{bmatrix}, \right. \\
&\qquad \left. r\begin{bmatrix} A & B_1 & B_3 \\ C_2 & 0 & 0 \\ C_4 & 0 & 0 \end{bmatrix}, r\begin{bmatrix} A & B_1 & B_4 \\ C_2 & 0 & 0 \\ C_3 & 0 & 0 \end{bmatrix} \right\}.
\end{aligned} \tag{2.8}$$

Proof. We only show (2.7). Under the assumption (2.6), substituting (2.4) into the two variant matrices X_1 and X_2 in (2.5) yields

(2.9)

$$\begin{aligned}
&\min_{X_i, X_2} r\left[p\left(X_1, X_2, X_3, X_4\right)\right] \\
&= r\left[A - B_3X_3C_3 - B_4X_4C_4, B_2\right] + r\begin{bmatrix} A - B_3X_3C_3 - B_4X_4C_4 \\ C_1 \end{bmatrix} \\
&\quad + r\begin{bmatrix} A - B_3X_3C_3 - B_4X_4C_4 & B_1 \\ C_2 & 0 \end{bmatrix} - r\begin{bmatrix} A - B_3X_3C_3 - B_4X_4C_4 & B_1 \\ C_1 & 0 \end{bmatrix} \\
&\quad - r\begin{bmatrix} A - B_3X_3C_3 - B_4X_4C_4 & B_2 \\ C_2 & 0 \end{bmatrix} \\
&= r\left[A, B_2\right] + r\begin{bmatrix} A \\ C_1 \end{bmatrix} - r\begin{bmatrix} A & B_1 \\ C_1 & 0 \end{bmatrix} - r\begin{bmatrix} A & B_2 \\ C_2 & 0 \end{bmatrix} \\
&\quad + r\begin{bmatrix} A - B_3X_3C_3 - B_4X_4C_4 & B_1 \\ C_2 & 0 \end{bmatrix}.
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
&\begin{bmatrix} A - B_3X_3C_3 - B_4X_4C_4 & B_1 \\ C_2 & 0 \end{bmatrix} \\
&= \begin{bmatrix} A & B_1 \\ C_2 & 0 \end{bmatrix} - \begin{bmatrix} B_3 \\ 0 \end{bmatrix} X_3\left[C_3, 0\right] - \begin{bmatrix} B_4 \\ 0 \end{bmatrix} X_4\left[C_4, 0\right].
\end{aligned}$$

In that case, applying (2.3) to it and then putting the corresponding results in (2.9) yields (2.7).

Similarly, we can prove (2.8). □

Using Lemma 2.5, we can get the following two lemmas easily.

Lemma 2.6. *Suppose that the system* (1.1) *has a solution. Then the maximal rank of* $C_3 - A_3XB_3$ *subject to* (1.1) *is the following*:

$$\begin{aligned}
&\max_{A_1XB_1=C_1, A_2XB_2=C_2} r(C_3 - A_3XB_3) \\
&= \min\left\{ r\left[C_3, A_3\right],\ r\begin{bmatrix} C_3 \\ B_3 \end{bmatrix}, s_1, s_2, s_3, s_4 \right\},
\end{aligned} \tag{2.10}$$

where

$$s_1 = r\begin{bmatrix} C_3 & 0 & 0 & A_3 \\ 0 & -C_1 & 0 & A_1 \\ 0 & 0 & -C_2 & A_2 \\ B_3 & B_1 & 0 & 0 \\ B_3 & 0 & B_2 & 0 \end{bmatrix} - r\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} - r\left(B_1\right) - r\left(B_2\right),$$

$$s_2 = r\begin{bmatrix} C_3 & 0 & 0 & A_3 & A_3 \\ 0 & -C_1 & 0 & A_1 & 0 \\ 0 & 0 & -C_2 & 0 & A_2 \\ B_3 & B_1 & B_2 & 0 & 0 \end{bmatrix} - r\left[B_1, B_2\right] - r\left(A_1\right) - r\left(A_2\right),$$

$$s_3 = r\begin{bmatrix} C_3 & 0 & A_3 \\ 0 & -C_1 & A_1 \\ B_3 & B_1 & 0 \end{bmatrix} - r\left(A_1\right) - r\left(B_1\right),$$

$$s_4 = r\begin{bmatrix} C_3 & 0 & A_3 \\ 0 & -C_2 & A_2 \\ B_3 & B_2 & 0 \end{bmatrix} - r\left(A_2\right) - r\left(B_2\right).$$

Lemma 2.7. *Suppose that the system* (1.1) *has a solution. Then the minimal rank of $C_3 - A_3XB_3$ subject to* (1.1) *is*

$$\begin{aligned} (2.11) \quad & \min_{A_1XB_1=C_1, A_2XB_2=C_2} r(C_3 - A_3XB_3) \\ &= r\begin{bmatrix} C_3 & 0 & 0 & A_3 \\ 0 & -C_1 & 0 & A_1 \\ 0 & 0 & -C_2 & A_2 \\ B_3 & B_1 & 0 & 0 \\ B_3 & 0 & B_2 & 0 \end{bmatrix} + r\begin{bmatrix} C_3 & 0 & 0 & A_3 & A_3 \\ 0 & -C_1 & 0 & A_1 & 0 \\ 0 & 0 & -C_2 & 0 & A_2 \\ B_3 & B_1 & B_2 & 0 & 0 \end{bmatrix} \\ &\quad - r\begin{bmatrix} C_3 & A_3 \\ B_3 & 0 \\ 0 & A_1 \\ 0 & A_2 \end{bmatrix} - r\begin{bmatrix} C_3 & A_3 & 0 & 0 \\ B_3 & 0 & B_1 & B_2 \end{bmatrix} + r\begin{bmatrix} C_3 \\ B_3 \end{bmatrix} + r\left[C_3, A_3\right] \\ &\quad + \max\left\{t_1, t_2\right\}, \end{aligned}$$

where

$$t_1 = r\begin{bmatrix} C_3 & 0 & A_3 \\ 0 & -C_1 & A_1 \\ B_3 & B_1 & 0 \end{bmatrix} - r\begin{bmatrix} C_3 & 0 & A_3 & A_3 \\ 0 & -C_1 & A_1 & 0 \\ B_3 & B_1 & 0 & 0 \\ 0 & 0 & 0 & A_2 \end{bmatrix} - r\begin{bmatrix} C_3 & 0 & A_3 & 0 \\ 0 & -C_1 & A_1 & 0 \\ B_3 & B_1 & 0 & 0 \\ B_3 & 0 & 0 & B_2 \end{bmatrix},$$

$$t_2 = r\begin{bmatrix} C_3 & 0 & A_3 \\ 0 & -C_2 & A_2 \\ B_3 & B_2 & 0 \end{bmatrix} - r\begin{bmatrix} C_3 & 0 & A_3 & A_3 \\ 0 & -C_2 & A_2 & 0 \\ B_3 & B_2 & 0 & 0 \\ 0 & 0 & 0 & A_1 \end{bmatrix} - r\begin{bmatrix} C_3 & 0 & A_3 & 0 \\ 0 & -C_2 & A_2 & 0 \\ B_3 & B_2 & 0 & 0 \\ B_3 & 0 & 0 & B_1 \end{bmatrix}.$$

For convenience, we adopt the following notations for the collections of the submatrices X_i $(i = 1, 2, 3, 4)$ in (1.2)

$$(2.12) \quad S_i = \left\{ X_i \left| \begin{array}{l} [A_{11}, A_{12}]\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}\begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix} = C_1, \\ [A_{21}, A_{22}]\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}\begin{bmatrix} B_{21} \\ B_{22} \end{bmatrix} = C_2 \end{array} \right. \right\}, \ i = 1, 2, 3, 4.$$

Obviously, X_i $(i = 1, 2, 3, 4)$ in (1.1) can be written as

$$X_1 = [I_{k_1}, 0]\, X \begin{bmatrix} I_{l_1} \\ 0 \end{bmatrix} = P_1 X Q_1, X_2 = [I_{k_1}, 0]\, X \begin{bmatrix} 0 \\ I_{l_1} \end{bmatrix} = P_1 X Q_2, \tag{2.13}$$

$$X_3 = [0, I_{k_2}]\, X \begin{bmatrix} I_{l_1} \\ 0 \end{bmatrix} = P_2 X Q_1, X_4 = [0, I_{k_2}]\, X \begin{bmatrix} 0 \\ I_{l_2} \end{bmatrix} = P_2 X Q_2. \tag{2.14}$$

According to Lemma 2.2, the general solution to the consistent system (1.1) can be denoted as

$$X = X_0 + L_A V_1 + V_2 R_B + L_{A_1} V_3 R_{B_2} + L_{A_2} V_4 R_{B_1}.$$

Substituting it into (2.13) and (2.14), we can get the general expressions of X_i $(i = 1, 2, 3, 4)$ as follows

(2.15)
$$X_1 = P_1 X_0 Q_1 + P_1 L_A V_{11} + V_{21} R_B Q_1 + P_1 L_{A_1} V_3 R_{B_2} Q_1 + P_1 L_{A_2} V_4 R_{B_1} Q_1,$$
(2.16)
$$X_2 = P_1 X_0 Q_2 + P_1 L_A V_{12} + V_{21} R_B Q_2 + P_1 L_{A_1} V_3 R_{B_2} Q_2 + P_1 L_{A_2} V_4 R_{B_1} Q_2,$$
(2.17)
$$X_3 = P_2 X_0 Q_1 + P_2 L_A V_{11} + V_{22} R_B Q_1 + P_2 L_{A_1} V_3 R_{B_2} Q_1 + P_2 L_{A_2} V_4 R_{B_1} Q_1,$$
(2.18)
$$X_4 = P_2 X_0 Q_2 + P_1 L_A V_{12} + V_{22} R_B Q_2 + P_2 L_{A_1} V_3 R_{B_2} Q_2 + P_2 L_{A_2} V_4 R_{B_1} Q_2,$$

where X_0 is a particular common solution of (1.1), $A = \left[\begin{smallmatrix} A_1 \\ A_2 \end{smallmatrix}\right], B = [B_1, B_2]$ are known, and $V_1 = [V_{11}, V_{12}], V_2 = [V_{21}, V_{22}], V_3, V_4$ are arbitrary.

Theorem 2.8. *Assume that the system* (1.2) *has a solution. Then*

$$\max_{X_1 \in S_1} r(X_1) = \min \left\{ k_1, l_1, \widehat{s_1}, \widehat{s_2}, \widehat{s_3}, \widehat{s_4} \right\}, \tag{2.19}$$

$$\min_{X_1 \in S_1} r(X_1) = \widehat{t_1} + \widehat{t_2} + k_1 + l_1 + \max \left\{ \widehat{t_3}, \widehat{t_4} \right\}, \tag{2.20}$$

where

$$\widehat{s_1} = r \begin{bmatrix} -C_1 & 0 & A_{12} \\ 0 & -C_2 & A_{22} \\ B_{12} & 0 & 0 \\ -B_{21} & B_{21} & 0 \\ 0 & B_{22} & 0 \end{bmatrix} - r \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} - r \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix} - r \begin{bmatrix} B_{21} \\ B_{22} \end{bmatrix},$$

$$\widehat{s_2} = r \begin{bmatrix} -C_1 & 0 & A_{12} & -A_{11} & 0 \\ 0 & -C_2 & 0 & A_{21} & A_{22} \\ B_{12} & B_{22} & 0 & 0 & 0 \end{bmatrix} - r \begin{bmatrix} B_{11} & B_{21} \\ B_{12} & B_{22} \end{bmatrix}$$
$$- r [A_{11}, A_{12}] - r [A_{21}, A_{22}],$$

$$\widehat{s_3} = r \begin{bmatrix} -C_1 & A_{12} \\ B_{12} & 0 \end{bmatrix} - r [A_{11}, A_{12}] - r \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix},$$

$$\widehat{s_4} = r\begin{bmatrix} -C_2 & A_{22} \\ B_{22} & 0 \end{bmatrix} - r\left[A_{21}, A_{22}\right] - r\begin{bmatrix} B_{21} \\ B_{22} \end{bmatrix},$$

$$\widehat{t_1} = r\begin{bmatrix} -C_1 & 0 & A_{12} \\ 0 & -C_2 & A_{22} \\ B_{12} & 0 & 0 \\ -B_{21} & B_{21} & 0 \\ 0 & B_{22} & 0 \end{bmatrix} - r\begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix},$$

$$\widehat{t_2} = r\begin{bmatrix} -C_1 & 0 & A_{12} & -A_{11} & 0 \\ 0 & -C_2 & 0 & A_{21} & A_{22} \\ B_{12} & B_{22} & 0 & 0 & 0 \end{bmatrix} - r\left[B_{12}, B_{22}\right],$$

$$\widehat{t_3} = r\begin{bmatrix} -C_1 & A_{12} \\ B_{12} & 0 \end{bmatrix} - r\begin{bmatrix} -C_1 & A_{12} & -A_{11} & 0 \\ B_{12} & 0 & 0 & 0 \\ 0 & 0 & A_{21} & A_{22} \end{bmatrix}$$
$$- r\begin{bmatrix} -C_1 & A_{12} & 0 \\ B_{12} & 0 & 0 \\ -B_{11} & 0 & B_{21} \\ 0 & 0 & B_{22} \end{bmatrix} - k_1 - l_1$$

and

$$\widehat{t_4} = r\begin{bmatrix} -C_2 & A_{22} \\ B_{22} & 0 \end{bmatrix} - r\begin{bmatrix} -C_2 & A_{22} & -A_{21} & 0 \\ B_{22} & 0 & 0 & 0 \\ 0 & 0 & A_{11} & A_{12} \end{bmatrix}$$
$$- r\begin{bmatrix} -C_2 & A_{22} & 0 \\ B_{22} & 0 & 0 \\ -B_{21} & 0 & B_{11} \\ 0 & 0 & B_{12} \end{bmatrix} - k_1 - l_1.$$

Proof. It is easy to see that the maximal and minimal ranks of X_1 in (1.2) is the maximal and minimal ranks of P_1XQ_1 subject to the consistent system (1.1). Applying (2.10) and (2.11) to $X_1 = P_1XQ_1$ produces the following

$$\max_{A_1XB_1=C_1, A_2XB_2=C_2} r(P_1XQ_1) = \min\left\{r\left(P_1\right),\ r\left(Q_1\right), \widehat{s_1}, \widehat{s_2}, \widehat{s_3}, \widehat{s_4}\right\}, \tag{2.21}$$

$$\min_{A_1XB_1=C_1, A_2XB_2=C_2} r(P_1XQ_1) \tag{2.22}$$
$$= r\begin{bmatrix} 0 & 0 & 0 & P_1 \\ 0 & -C_1 & 0 & A_1 \\ 0 & 0 & -C_2 & A_2 \\ Q_1 & B_1 & 0 & 0 \\ Q_1 & 0 & B_2 & 0 \end{bmatrix} - r\begin{bmatrix} 0 & P_1 \\ Q_1 & 0 \\ 0 & A_1 \\ 0 & A_2 \end{bmatrix}$$
$$- r\begin{bmatrix} 0 & P_1 & 0 & 0 \\ Q_1 & 0 & B_1 & B_2 \end{bmatrix} + r\begin{bmatrix} 0 & 0 & 0 & P_1 & P_1 \\ 0 & -C_1 & 0 & A_1 & 0 \\ 0 & 0 & -C_2 & 0 & A_2 \\ Q_1 & B_1 & B_2 & 0 & 0 \end{bmatrix}$$

$$+ k_1 + l_1 + \max\left\{\widehat{t_1}, \widehat{t_2}\right\}.$$

Putting the given matrices $A_1 = [A_{11}, A_{12}]$, $B_1 = \left[\begin{smallmatrix} B_{11} \\ B_{12} \end{smallmatrix}\right]$, $A_2 = [A_{21}, A_{22}]$, $B_2 = \left[\begin{smallmatrix} B_{21} \\ B_{22} \end{smallmatrix}\right]$, P_1 and Q_1 in them and simplifying the ranks of the block matrices in (2.21) and (2.22), we have

$$r\begin{bmatrix} 0 & 0 & 0 & P_1 \\ 0 & -C_1 & 0 & A_1 \\ 0 & 0 & -C_2 & A_2 \\ Q_1 & B_1 & 0 & 0 \\ Q_1 & 0 & B_2 & 0 \end{bmatrix} = r\begin{bmatrix} 0 & 0 & 0 & I_{k_1} & 0 \\ 0 & -C_1 & 0 & A_{11} & A_{12} \\ 0 & 0 & -C_2 & A_{21} & A_{22} \\ I_{l_1} & B_{11} & 0 & 0 & 0 \\ 0 & B_{12} & 0 & 0 & 0 \\ I_{l_1} & 0 & B_{21} & 0 & 0 \\ 0 & 0 & B_{22} & 0 & 0 \end{bmatrix}$$
$$= r\begin{bmatrix} -C_1 & 0 & A_{12} \\ 0 & -C_2 & A_{22} \\ B_{12} & 0 & 0 \\ -B_{21} & B_{21} & 0 \\ 0 & B_{22} & 0 \end{bmatrix} + k_1 + l_1.$$

Similarly, we can obtain the desired formulas (2.19) and (2.20). □

Extreme ranks of the submatrices X_2, X_3 and X_4 in (1.2) can be derived by the similar approach. We omit them here for simplicity. The two rank equalities in (2.19) and (2.20) can help us to get the necessary and sufficient conditions for the existence of some special solutions to (1.2). We show them in the following.

Corollary 2.9. *Suppose that the system* (1.2) *has a solution. Then*
(a) (1.2) *has a solution with the form* $X = \left[\begin{smallmatrix} 0 & X_2 \\ X_3 & X_4 \end{smallmatrix}\right]$ *if and only if*

$$r\begin{bmatrix} -C_1 & 0 & A_{12} \\ 0 & -C_2 & A_{22} \\ B_{12} & 0 & 0 \\ -B_{21} & B_{21} & 0 \\ 0 & B_{22} & 0 \end{bmatrix} + r\begin{bmatrix} -C_1 & 0 & A_{11} & -A_{11} & 0 \\ 0 & -C_2 & 0 & A_{21} & A_{22} \\ B_{12} & B_{22} & 0 & 0 & 0 \end{bmatrix}$$
$$- r\begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} - r\,[B_{12}, B_{22}] + k_1 + l_1$$
$$= \max\left\{ r\begin{bmatrix} -C_1 & A_{12} & -A_{11} & 0 \\ B_{12} & 0 & 0 & 0 \\ 0 & 0 & A_{21} & A_{22} \end{bmatrix} + r\begin{bmatrix} -C_1 & A_{12} & 0 \\ B_{12} & 0 & 0 \\ -B_{11} & 0 & B_{21} \\ 0 & 0 & B_{22} \end{bmatrix} - r\begin{bmatrix} -C_1 & A_{12} \\ B_{12} & 0 \end{bmatrix} + k_1 + l_1, \right.$$
$$\left. r\begin{bmatrix} -C_1 & A_{22} & -A_{21} & 0 \\ B_{22} & 0 & 0 & 0 \\ 0 & 0 & A_{11} & A_{12} \end{bmatrix} + r\begin{bmatrix} -C_1 & A_{22} & 0 \\ B_{22} & 0 & 0 \\ -B_{21} & 0 & B_{11} \\ 0 & 0 & B_{12} \end{bmatrix} - r\begin{bmatrix} -C_1 & A_{22} \\ B_{22} & 0 \end{bmatrix} + k_1 + l_1 \right\}.$$

(b) *All the solutions of* (1.2) *have the form* $X = \left[\begin{smallmatrix} 0 & X_2 \\ X_3 & X_4 \end{smallmatrix}\right]$ *if and only if*

$$r\begin{bmatrix} -C_1 & 0 & A_{12} \\ 0 & -C_2 & A_{22} \\ B_{12} & 0 & 0 \\ -B_{21} & B_{21} & 0 \\ 0 & B_{22} & 0 \end{bmatrix} = r\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} + r\begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix} + r\begin{bmatrix} B_{21} \\ B_{22} \end{bmatrix},$$

or

$$\begin{aligned} & r\begin{bmatrix} -C_1 & 0 & A_{12} & -A_{11} & 0 \\ 0 & -C_2 & 0 & A_{21} & A_{22} \\ B_{12} & B_{22} & 0 & 0 & 0 \end{bmatrix} \\ = {} & r\begin{bmatrix} B_{11} & B_{21} \\ B_{12} & B_{22} \end{bmatrix} + r\left[A_{11}, A_{12}\right] + r\left[A_{21}, A_{22}\right], \end{aligned}$$

or

$$r\begin{bmatrix} -C_1 & A_{12} \\ B_{12} & 0 \end{bmatrix} = r\left[A_{11}, A_{12}\right] + r\begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix},$$

or

$$r\begin{bmatrix} -C_2 & A_{22} \\ B_{22} & 0 \end{bmatrix} = r\left[A_{21}, A_{22}\right] + r\begin{bmatrix} B_{21} \\ B_{22} \end{bmatrix}.$$

Theorem 2.10. *Assume that the system* (1.2) *has a solution. Then*
(a) (1.2) *has a solution with the form* $X = \left[\begin{smallmatrix} X_1 & 0 \\ X_3 & 0 \end{smallmatrix}\right]$ *if and only if*

$$\begin{aligned} & r\begin{bmatrix} -C_1 & 0 \\ 0 & -C_2 \\ B_{11} & 0 \\ 0 & B_{21} \\ -B_{12} & B_{22} \end{bmatrix} + r\begin{bmatrix} -C_1 & 0 & -A_{11} & -A_{12} \\ 0 & -C_2 & A_{21} & A_{22} \\ B_{11} & B_{21} & 0 & 0 \end{bmatrix} - r\left[B_{11}, B_{21}\right] \\ = {} & \max\left\{ r\begin{bmatrix} -C_1 & -A_{11} & -A_{12} \\ B_{11} & 0 & 0 \\ 0 & A_{21} & A_{22} \end{bmatrix} + r\begin{bmatrix} -C_1 & 0 \\ B_{11} & 0 \\ 0 & B_{21} \\ -B_{12} & B_{22} \end{bmatrix} - r\begin{bmatrix} -C_1 \\ B_{11} \end{bmatrix} + k + l_2, \right. \\ & \left. r\begin{bmatrix} -C_1 & -A_{21} & -A_{22} \\ B_{21} & 0 & 0 \\ 0 & A_{11} & A_{12} \end{bmatrix} + r\begin{bmatrix} -C_1 & 0 \\ B_{21} & 0 \\ 0 & B_{11} \\ -B_{22} & B_{12} \end{bmatrix} - r\begin{bmatrix} -C_1 \\ B_{21} \end{bmatrix} + k + l_2 \right\}. \end{aligned}$$

(b) (1.2) *has a solution with the form* $X = \left[\begin{smallmatrix} X_1 & X_2 \\ 0 & 0 \end{smallmatrix}\right]$ *if and only if*

$$\begin{aligned} & r\begin{bmatrix} -C_1 & 0 & A_{11} \\ 0 & -C_2 & A_{21} \\ -B_{11} & B_{21} & 0 \\ -B_{12} & B_{22} & 0 \end{bmatrix} + r\begin{bmatrix} -C_1 & 0 & A_{11} & 0 & -A_{12} \\ 0 & -C_2 & 0 & A_{21} & A_{22} \end{bmatrix} - r\begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} \\ = {} & \max\left\{ r\begin{bmatrix} -C_1 & A_{11} & 0 & -A_{12} \\ 0 & 0 & A_{21} & A_{22} \end{bmatrix} + r\begin{bmatrix} -C_1 & A_{11} & 0 \\ -B_{11} & 0 & B_{21} \\ -B_{12} & 0 & B_{22} \end{bmatrix} - r\left[-C_1, A_{11}\right] + l + k_2, \right. \end{aligned}$$

$$r\begin{bmatrix} -C_1 & A_{21} & 0 & -A_{22} \\ 0 & 0 & A_{11} & A_{12} \end{bmatrix} + r\begin{bmatrix} -C_1 & A_{21} & 0 \\ -B_{21} & 0 & B_{11} \\ -B_{22} & 0 & B_{12} \end{bmatrix} - r[-C_1, A_{21}] + l + k_2 \Bigg\}.$$

(c) (1.2) *has a solution with the form* $X = \left[\begin{smallmatrix} X_1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ *if and only if*

$$r\begin{bmatrix} -C_1 & 0 \\ 0 & -C_2 \\ B_{11} & 0 \\ 0 & B_{21} \\ -B_{12} & B_{22} \end{bmatrix} + r\begin{bmatrix} -C_1 & 0 & -A_{11} & -A_{12} \\ 0 & -C_2 & A_{21} & A_{22} \\ B_{11} & B_{21} & 0 & 0 \end{bmatrix} - r[B_{11}, B_{21}]$$

$$= \max\left\{ r\begin{bmatrix} -C_1 & -A_{11} & -A_{12} \\ B_{11} & 0 & 0 \\ 0 & A_{21} & A_{22} \end{bmatrix} + r\begin{bmatrix} -C_1 & 0 \\ B_{11} & 0 \\ 0 & B_{21} \\ -B_{12} & B_{22} \end{bmatrix} - r\begin{bmatrix} -C_1 \\ B_{11} \end{bmatrix} + k + l_2, \right.$$

$$\left. r\begin{bmatrix} -C_1 & -A_{21} & -A_{22} \\ B_{21} & 0 & 0 \\ 0 & A_{11} & A_{12} \end{bmatrix} + r\begin{bmatrix} -C_1 & 0 \\ B_{21} & 0 \\ 0 & B_{11} \\ -B_{22} & B_{12} \end{bmatrix} - r\begin{bmatrix} -C_1 \\ B_{21} \end{bmatrix} + k + l_2 \right\}$$

and

$$r\begin{bmatrix} -C_1 & 0 & A_{11} \\ 0 & -C_2 & A_{21} \\ -B_{11} & B_{21} & 0 \\ -B_{12} & B_{22} & 0 \end{bmatrix} + r\begin{bmatrix} -C_1 & 0 & A_{11} & 0 & -A_{12} \\ 0 & -C_2 & 0 & A_{21} & A_{22} \end{bmatrix} - r\begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix}$$

$$= \max\left\{ r\begin{bmatrix} -C_1 & A_{11} & 0 & -A_{12} \\ 0 & 0 & A_{21} & A_{22} \end{bmatrix} + r\begin{bmatrix} -C_1 & A_{11} & 0 \\ -B_{11} & 0 & B_{21} \\ -B_{12} & 0 & B_{22} \end{bmatrix} - r[-C_1, A_{11}] + l + k_2, \right.$$

$$\left. r\begin{bmatrix} -C_1 & A_{21} & 0 & -A_{22} \\ 0 & 0 & A_{11} & A_{12} \end{bmatrix} + r\begin{bmatrix} -C_1 & A_{21} & 0 \\ -B_{21} & 0 & B_{11} \\ -B_{22} & 0 & B_{12} \end{bmatrix} - r[-C_1, A_{21}] + l + k_2 \right\}.$$

Proof. According to (2.10) and (2.11), we have that

$$\min_{A_1XB_1=C_1, A_2XB_2=C_2} r\begin{bmatrix} X_2 \\ X_4 \end{bmatrix}$$
$$= \min_{A_1XB_1=C_1, A_2XB_2=C_2} r(XQ_2)$$
$$= r\begin{bmatrix} -C_1 & 0 \\ 0 & -C_2 \\ B_{11} & 0 \\ 0 & B_{21} \\ -B_{12} & B_{22} \end{bmatrix} + r\begin{bmatrix} -C_1 & 0 & -A_{11} & -A_{12} \\ 0 & -C_2 & A_{21} & A_{22} \\ B_{11} & B_{21} & 0 & 0 \end{bmatrix} - r[B_{11}, B_{21}]$$
$$+ \max\left\{ r\begin{bmatrix} -C_1 \\ B_{11} \end{bmatrix} - r\begin{bmatrix} -C_1 & -A_{11} & -A_{12} \\ B_{11} & 0 & 0 \\ 0 & A_{21} & A_{22} \end{bmatrix} - r\begin{bmatrix} -C_1 & 0 \\ B_{11} & 0 \\ 0 & B_{21} \\ -B_{12} & B_{22} \end{bmatrix} - k - l_2, \right.$$

$$r\begin{bmatrix} -C_1 \\ B_{21} \end{bmatrix} - r\begin{bmatrix} -C_1 & -A_{21} & -A_{22} \\ B_{21} & 0 & 0 \\ 0 & A_{11} & A_{12} \end{bmatrix} - r\begin{bmatrix} -C_1 & 0 \\ B_{21} & 0 \\ 0 & B_{11} \\ -B_{22} & B_{12} \end{bmatrix} - k - l_2 \Bigg\},$$

and

$$\min_{A_1XB_1=C_1, A_2XB_2=C_2} r\,[X_3, X_4]$$
$$= \min_{A_1XB_1=C_1, A_2XB_2=C_2} r\,(P_2X)$$
$$= r\begin{bmatrix} -C_1 & 0 & A_{11} \\ 0 & -C_2 & A_{21} \\ -B_{11} & B_{21} & 0 \\ -B_{12} & B_{22} & 0 \end{bmatrix} + r\begin{bmatrix} -C_1 & 0 & A_{11} & 0 & -A_{12} \\ 0 & -C_2 & 0 & A_{21} & A_{22} \end{bmatrix} - r\begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix}$$
$$+ \max\Bigg\{ r\begin{bmatrix} -C_1 & A_{11} \end{bmatrix} - r\begin{bmatrix} -C_1 & A_{11} & 0 & -A_{12} \\ 0 & 0 & A_{21} & A_{22} \end{bmatrix} - r\begin{bmatrix} -C_1 & A_{11} & 0 \\ -B_{11} & 0 & B_{21} \\ -B_{12} & 0 & B_{22} \end{bmatrix} - l - k_2,$$
$$r\,[-C_1, A_{21}] - r\begin{bmatrix} -C_1 & A_{21} & 0 & -A_{22} \\ 0 & 0 & A_{11} & A_{12} \end{bmatrix} - r\begin{bmatrix} -C_1 & A_{21} & 0 \\ -B_{21} & 0 & B_{11} \\ -B_{22} & 0 & B_{12} \end{bmatrix} - l - k_2 \Bigg\}.$$

Thus we have Part (a) and Part (b). Part (c) can be derived from Part (a) and Part (b). □

Now we show the uniqueness of the submatrices X_1, X_2, X_3 and X_4 in (1.2), which can be determined by (2.13) and (2.14).

Theorem 2.11. *Assume that the system* (1.2) *has a solution. Then the submatrix X_1 in* (1.2) *is unique if and only if* (1.2) *satisfies the following conditions:*

$$r\begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} = k_1, \mathcal{R}\begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} \cap \mathcal{R}\begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} = \{0\},$$
$$r\,[B_{11}, B_{21}] = l_1, \mathcal{R}\,[B_{11}, B_{21}]^T \cap \mathcal{R}\,[B_{12}, B_{22}]^T = \{0\},$$
$$r\,(A_{11}) = k_1, \mathcal{R}\,(A_{11}) \cap \mathcal{R}\,(A_{12}) = \{0\},$$

or

$$r\,(B_{21}) = l_1, \mathcal{R}\,(B_{21})^T \cap \mathcal{R}\,(B_{22})^T = \{0\}$$

and

$$r\,(A_{21}) = k_1, \mathcal{R}\,(A_{21}) \cap \mathcal{R}\,(A_{22}) = \{0\},$$

or

$$r\,(B_{11}) = l_1, \mathcal{R}\,(B_{11})^T \cap \mathcal{R}\,(B_{12})^T = \{0\}.$$

Proof. It follows from (2.15) that X_1 is unique if and only if $P_1L_A = 0, R_BQ_1 = 0, P_1L_{A_1} = 0$ or $R_{B_2}Q_1 = 0$ and $P_1L_{A_2} = 0$ or $R_{B_1}Q_1 = 0$. By Lemma 2.1,

$$P_1L_A = 0 \Longleftrightarrow r\begin{bmatrix} P_1 \\ A \end{bmatrix} = r\,(A)$$

$$\Longleftrightarrow r\begin{bmatrix} I_{k_1} & 0 \\ A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = r\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$\Longleftrightarrow k_1 + r\begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} = r\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$\Longleftrightarrow r\begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} = k_1 \text{ and } \mathcal{R}\begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} \cap \mathcal{R}\begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} = \{0\},$$

$$R_B Q_1 = 0 \Longleftrightarrow r[B_{11}, B_{21}] = l_1 \text{ and } \mathcal{R}[B_{11}, B_{21}]^T \cap \mathcal{R}[B_{12}, B_{22}]^T = \{0\},$$

$$P_1 L_{A_1} = 0 \Longleftrightarrow r(A_{11}) = k_1 \text{ and } \mathcal{R}(A_{11}) \cap \mathcal{R}(A_{12}) = \{0\},$$

$$R_{B_2} Q_1 = 0 \Longleftrightarrow r(B_{21}) = l_1 \text{ and } \mathcal{R}(B_{21})^T \cap \mathcal{R}(B_{22})^T = \{0\},$$

$$P_1 L_{A_2} = 0 \Longleftrightarrow r(A_{21}) = k_1 \text{ and } \mathcal{R}(A_{21}) \cap \mathcal{R}(A_{22}) = \{0\},$$

$$R_{B_1} Q_1 = 0 \Longleftrightarrow r(B_{11}) = l_1 \text{ and } \mathcal{R}(B_{11})^T \cap \mathcal{R}(B_{12})^T = \{0\}. \qquad \square$$

The following result concerns the independence of submatrices in solutions to (1.2).

Theorem 2.12. *Suppose that the system* (1.2) *has a solution with* $A_1 \neq 0, A_2 \neq 0, B_1 \neq 0$ *and* $B_2 \neq 0$.

(a) *Consider* S_1, S_2, S_3 *and* S_4 *in* (2.12) *as four independent matrix sets. Then*

(2.23)

$$\max_{X_i \in S_i} r\left[C_1 - [A_{11}, A_{12}]\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}\begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix}\right]$$
$$= \min\left\{ r(A_1), r(B_1), r\begin{bmatrix} 0 & A_1G_1 \\ H_1B_1 & 0 \\ H_2B_1 & 0 \\ H_3B_1 & 0 \end{bmatrix}, r\begin{bmatrix} 0 & A_1G_1 & A_1G_2 & A_1G_3 \\ H_1B_1 & 0 & 0 & 0 \end{bmatrix}, \right.$$
$$\left. r\begin{bmatrix} 0 & A_1G_1 & A_1G_2 \\ H_1B_1 & 0 & 0 \\ H_3B_1 & 0 & 0 \end{bmatrix}, r\begin{bmatrix} 0 & A_1G_1 & A_1G_3 \\ H_1B_1 & 0 & 0 \\ H_2B_1 & 0 & 0 \end{bmatrix} \right\},$$

(2.24)

$$\max_{X_i \in S_i} r\left[C_2 - [A_{21}, A_{22}]\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}\begin{bmatrix} B_{21} \\ B_{22} \end{bmatrix}\right]$$
$$= \min\left\{ r(A_2), r(B_2), r\begin{bmatrix} 0 & A_2G_1 \\ H_1B_2 & 0 \\ H_2B_2 & 0 \\ H_3B_2 & 0 \end{bmatrix}, r\begin{bmatrix} 0 & A_2G_1 & A_2G_2 & A_2G_3 \\ H_1B_2 & 0 & 0 & 0 \end{bmatrix}, \right.$$
$$\left. r\begin{bmatrix} 0 & A_2G_1 & A_2G_2 \\ H_1B_2 & 0 & 0 \\ H_3B_2 & 0 & 0 \end{bmatrix}, r\begin{bmatrix} 0 & A_2G_1 & A_2G_3 \\ H_1B_2 & 0 & 0 \\ H_2B_2 & 0 & 0 \end{bmatrix} \right\},$$

where

$$G_1 = \operatorname{diag}\left[\begin{array}{cc} P_1 L_A & P_2 L_A \end{array}\right], G_2 = \operatorname{diag}\left[\begin{array}{cc} P_1 L_{A_1} & P_2 L_{A_1} \end{array}\right],$$
$$G_3 = \operatorname{diag}\left[\begin{array}{cc} P_1 L_{A_2} & P_2 L_{A_2} \end{array}\right], H_1 = \operatorname{diag}\left[\begin{array}{cc} R_B Q_1 & R_B Q_2 \end{array}\right],$$
$$H_2 = \operatorname{diag}\left[\begin{array}{cc} R_{B_2} Q_1 & R_{B_2} Q_2 \end{array}\right], H_3 = \operatorname{diag}\left[\begin{array}{cc} R_{B_1} Q_1 & R_{B_1} Q_2 \end{array}\right].$$

(b) *The four submatrices X_1, X_2, X_3 and X_4 in* (1.2) *are independent, that is, for any choice of $X_i \in S_i (i = 1, 2, 3, 4)$, the corresponding matrix $X = \left[\begin{smallmatrix} X_1 & X_2 \\ X_3 & X_4 \end{smallmatrix}\right]$ is a solution of* (1.2), *if and only if*

$$r\begin{bmatrix} A_{11} & 0 \\ 0 & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = r\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, r\begin{bmatrix} 0 & B_{11} & B_{21} \\ B_{12} & 0 & B_{22} \end{bmatrix} = r\begin{bmatrix} B_{11} & B_{21} \\ B_{12} & B_{22} \end{bmatrix},$$

$$r\begin{bmatrix} B_{11} & B_{21} & 0 \\ 0 & B_{22} & 0 \\ 0 & 0 & B_{21} \\ B_{12} & 0 & B_{22} \end{bmatrix} = 2r\begin{bmatrix} B_{21} \\ B_{22} \end{bmatrix}, r(B_{11}) + r(B_{12}) = r\begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix}$$

or

$$r\begin{bmatrix} A_{11} & 0 \\ 0 & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = r\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, r\begin{bmatrix} 0 & B_{11} & B_{21} \\ B_{12} & 0 & B_{22} \end{bmatrix} = r\begin{bmatrix} B_{11} & B_{21} \\ B_{12} & B_{22} \end{bmatrix},$$

$$r(A_{11}) + r(A_{12}) = r[A_{11}, A_{12}], r\begin{bmatrix} A_{11} & 0 & 0 & A_{12} \\ A_{21} & A_{22} & 0 & 0 \\ 0 & 0 & A_{21} & A_{22} \end{bmatrix} = 2r[A_{21}, A_{22}]$$

or

$$r\begin{bmatrix} A_{11} & 0 \\ 0 & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = r\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, r\begin{bmatrix} 0 & B_{11} & B_{21} \\ B_{12} & 0 & B_{22} \end{bmatrix} = r\begin{bmatrix} B_{11} & B_{21} \\ B_{12} & B_{22} \end{bmatrix},$$

$$r(A_{11}) + r(A_{12}) = r[A_{11}, A_{12}], r\begin{bmatrix} B_{11} & B_{21} & 0 \\ 0 & B_{22} & 0 \\ 0 & 0 & B_{21} \\ B_{12} & 0 & B_{22} \end{bmatrix} = 2r\begin{bmatrix} B_{21} \\ B_{22} \end{bmatrix}$$

or

$$r\begin{bmatrix} A_{11} & 0 \\ 0 & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = r\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, r\begin{bmatrix} 0 & B_{11} & B_{21} \\ B_{12} & 0 & B_{22} \end{bmatrix} = r\begin{bmatrix} B_{11} & B_{21} \\ B_{12} & B_{22} \end{bmatrix},$$

$$r(B_{11}) + r(B_{12}) = r\begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix}, r\begin{bmatrix} A_{11} & 0 & 0 & A_{12} \\ A_{21} & A_{22} & 0 & 0 \\ 0 & 0 & A_{21} & A_{22} \end{bmatrix} = 2r\left[\begin{array}{cc} A_{21} & A_{22} \end{array}\right]$$

and

$$r\begin{bmatrix} A_{21} & 0 \\ 0 & A_{22} \\ A_{11} & A_{12} \end{bmatrix} = r\begin{bmatrix} A_{21} & A_{22} \\ A_{11} & A_{12} \end{bmatrix}, r\begin{bmatrix} 0 & B_{21} & B_{11} \\ B_{22} & 0 & B_{12} \end{bmatrix} = r\begin{bmatrix} B_{21} & B_{11} \\ B_{22} & B_{12} \end{bmatrix},$$

$$r\begin{bmatrix} B_{21} & B_{11} & 0 \\ 0 & B_{12} & 0 \\ 0 & 0 & B_{11} \\ B_{22} & 0 & B_{12} \end{bmatrix} = 2r\begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix}, r\left(B_{21}\right) + r\left(B_{22}\right) = r\begin{bmatrix} B_{21} \\ B_{22} \end{bmatrix}$$

or

$$r\begin{bmatrix} A_{21} & 0 \\ 0 & A_{22} \\ A_{11} & A_{12} \end{bmatrix} = r\begin{bmatrix} A_{21} & A_{22} \\ A_{11} & A_{12} \end{bmatrix}, r\begin{bmatrix} 0 & B_{21} & B_{11} \\ B_{22} & 0 & B_{12} \end{bmatrix} = r\begin{bmatrix} B_{21} & B_{11} \\ B_{22} & B_{12} \end{bmatrix},$$

$$r\left(A_{21}\right) + r\left(A_{22}\right) = r\left[A_{21}, A_{22}\right], r\begin{bmatrix} A_{21} & 0 & 0 & A_{22} \\ A_{11} & A_{12} & 0 & 0 \\ 0 & 0 & A_{11} & A_{12} \end{bmatrix} = 2r\left[A_{11}, A_{12}\right]$$

or

$$r\begin{bmatrix} A_{21} & 0 \\ 0 & A_{22} \\ A_{11} & A_{12} \end{bmatrix} = r\begin{bmatrix} A_{21} & A_{22} \\ A_{11} & A_{12} \end{bmatrix}, r\begin{bmatrix} 0 & B_{21} & B_{11} \\ B_{22} & 0 & B_{12} \end{bmatrix} = r\begin{bmatrix} B_{21} & B_{11} \\ B_{22} & B_{12} \end{bmatrix},$$

$$r\left(A_{21}\right) + r\left(A_{22}\right) = r\left[A_{21}, A_{22}\right], r\begin{bmatrix} B_{21} & B_{11} & 0 \\ 0 & B_{12} & 0 \\ 0 & 0 & B_{11} \\ B_{22} & 0 & B_{12} \end{bmatrix} = 2r\begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix}$$

or

$$r\begin{bmatrix} A_{21} & 0 \\ 0 & A_{22} \\ A_{11} & A_{12} \end{bmatrix} = r\begin{bmatrix} A_{21} & A_{22} \\ A_{11} & A_{12} \end{bmatrix}, r\begin{bmatrix} 0 & B_{21} & B_{11} \\ B_{22} & 0 & B_{12} \end{bmatrix} = r\begin{bmatrix} B_{21} & B_{11} \\ B_{22} & B_{12} \end{bmatrix},$$

$$r\left(B_{21}\right) + r\left(B_{22}\right) = r\begin{bmatrix} B_{21} \\ B_{22} \end{bmatrix}, r\begin{bmatrix} A_{21} & 0 & 0 & A_{22} \\ A_{11} & A_{12} & 0 & 0 \\ 0 & 0 & A_{11} & A_{12} \end{bmatrix} = 2r\left[A_{11}, A_{12}\right].$$

Proof. Based on (2.15), (2.16), (2.17) and (2.18), the general expressions of $X_1 - X_4$ in $S_1 - S_4$ can independently be written as

$$\begin{aligned}
X_1 &= P_1X_0Q_1 + P_1L_AV_{11} + V_{21}R_BQ_1 + P_1L_{A_1}V_{31}R_{B_2}Q_1 + P_1L_{A_2}V_{41}R_{B_1}Q_1,\\
X_2 &= P_1X_0Q_2 + P_1L_AV_{12} + V_{22}R_BQ_2 + P_1L_{A_1}V_{32}R_{B_2}Q_2 + P_1L_{A_2}V_{42}R_{B_1}Q_2,\\
X_3 &= P_2X_0Q_1 + P_2L_AV_{13} + V_{23}R_BQ_1 + P_2L_{A_1}V_{33}R_{B_2}Q_1 + P_2L_{A_2}V_{43}R_{B_1}Q_1,\\
X_4 &= P_2X_0Q_2 + P_1L_AV_{14} + V_{24}R_BQ_2 + P_2L_{A_1}V_{34}R_{B_2}Q_2 + P_2L_{A_2}V_{44}R_{B_1}Q_2,
\end{aligned}$$

where X_0 is a particular solution of (1.1), $V_{11} - V_{14}, V_{21} - V_{24}, V_{31} - V_{34}$, and $V_{41} - V_{44}$ are arbitrary.

Substituting them into X yields

(2.25)

$$\begin{aligned}
\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} &= \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} X_0 \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} + \begin{bmatrix} P_1 L_A & 0 \\ 0 & P_2 L_A \end{bmatrix} \begin{bmatrix} V_{11} & V_{12} \\ V_{13} & V_{14} \end{bmatrix} \\
&\quad + \begin{bmatrix} V_{21} & V_{22} \\ V_{23} & V_{24} \end{bmatrix} \begin{bmatrix} R_B Q_1 & 0 \\ 0 & R_B Q_2 \end{bmatrix} \\
&\quad + \begin{bmatrix} P_1 L_{A_1} & 0 \\ 0 & P_2 L_{A_1} \end{bmatrix} \begin{bmatrix} V_{31} & V_{32} \\ V_{33} & V_{34} \end{bmatrix} \begin{bmatrix} R_{B_2} Q_1 & 0 \\ 0 & R_{B_2} Q_2 \end{bmatrix} \\
&\quad + \begin{bmatrix} P_1 L_{A_2} & 0 \\ 0 & P_2 L_{A_2} \end{bmatrix} \begin{bmatrix} V_{41} & V_{42} \\ V_{43} & V_{44} \end{bmatrix} \begin{bmatrix} R_{B_1} Q_1 & 0 \\ 0 & R_{B_1} Q_2 \end{bmatrix} \\
&= X_0 + G_1 V_1 + V_2 H_1 + G_2 V_3 H_2 + G_3 V_4 H_3,
\end{aligned}$$

where

$$\begin{aligned}
&G_1 = \operatorname{diag}\begin{bmatrix} P_1 L_A & P_2 L_A \end{bmatrix}, G_2 = \operatorname{diag}\begin{bmatrix} P_1 L_{A_1} & P_2 L_{A_1} \end{bmatrix}, \\
&G_3 = \operatorname{diag}\begin{bmatrix} P_1 L_{A_2} & P_2 L_{A_2} \end{bmatrix}, H_1 = \operatorname{diag}\begin{bmatrix} R_B Q_1 & R_B Q_2 \end{bmatrix}, \\
&H_2 = \operatorname{diag}\begin{bmatrix} R_{B_2} Q_1 & R_{B_2} Q_2 \end{bmatrix}, H_3 = \operatorname{diag}\begin{bmatrix} R_{B_1} Q_1 & R_{B_1} Q_2 \end{bmatrix}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\max_{X_i \in S_i} r\left(C_1 - [A_{11}, A_{12}] \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix} \right) \\
=&\max_{G_1, G_2, G_3, H_1, H_2, H_3} r\left(A_1 G_1 V_1 B_1 + A_1 V_2 H_1 B_1 + A_1 G_2 V_3 H_2 B_1 + A_1 G_3 V_4 H_3 B_1 \right),
\end{aligned}$$

$$\begin{aligned}
&\max_{X_i \in S_i} r\left(C_2 - [A_{21}, A_{22}] \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} \begin{bmatrix} B_{21} \\ B_{22} \end{bmatrix} \right) \\
=&\max_{G_1, G_2, G_3, H_1, H_2, H_3} r\left(A_2 G_1 V_1 B_2 + A_2 V_2 H_1 B_2 + A_2 G_2 V_3 H_2 B_2 + A_2 G_3 V_4 H_3 B_2 \right).
\end{aligned}$$

According to Lemma 2.6, we get (2.23) and (2.24).
On the other hand,

$$\begin{aligned}
r(A_1 G_1) = r[A_{11} P_1 L_A, A_{12} P_2 L_A] &= r \begin{bmatrix} A_{11} P_1 & A_{12} P_2 \\ A & 0 \\ 0 & A \end{bmatrix} - 2r(A) \\
&= r \begin{bmatrix} A_{11} & 0 \\ 0 & A_{12} \\ A_{21} & A_{22} \end{bmatrix} - r \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},
\end{aligned}$$

$$r(H_1 B_1) = r \begin{bmatrix} 0 & B_{11} & B_{21} \\ B_{12} & 0 & B_{22} \end{bmatrix} - r \begin{bmatrix} B_{11} & B_{21} \\ B_{12} & B_{22} \end{bmatrix}.$$

Similarly, we can simplify the other matrices in (2.23) and (2.24). The result in Part (b) is a direct consequence of (2.23) and (2.24). □

3. The common inner inverse of two arbitrary matrices

Suppose that

$$M=\begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix}, N=\begin{bmatrix} N_1 & N_2 \\ N_3 & N_4 \end{bmatrix} \tag{3.1}$$

are two partitioned matrices over $\mathbb{H}$, and have common inner inverse

$$G=\begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix}, \tag{3.2}$$

where $M_1, N_1 \in \mathbb{H}^{m\times n}$, $M_2, N_2 \in \mathbb{H}^{m\times k}$, $M_3, N_3 \in \mathbb{H}^{l\times n}$, $M_4, N_4 \in \mathbb{H}^{l\times k}$, $G_1 \in \mathbb{H}^{n\times m}$.

Let

$$T_i=\left\{G_i \middle| \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix} \in \{M^-\}\cap\{N^-\}\right\}, i=1,2,3,4. \tag{3.3}$$

It is obvious that G is a solution to the pair of matrix equations $MXM = M, NXN = N$. Thus applying Theorem 2.8 to (3.1) and (3.2), we have the following.

Theorem 3.1. *Let M, N and G be given by* (3.1) *and* (3.2). *Then*

$$\max_{G_1\in T_1} r(G_1)=\min\{m,n,\widetilde{s_1},\widetilde{s_2},\widetilde{s_3},\widetilde{s_4}\},$$

$$\min_{G_1\in T_1} r(G_1)=\widetilde{t_1}+\widetilde{t_2}+m+n+\max\{\widetilde{t_3},\widetilde{t_4}\},$$

where

$$\widetilde{s_1}=r\begin{bmatrix} -M_1 & -M_2 & 0 & 0 & M_2 \\ -M_3 & -M_4 & 0 & 0 & M_4 \\ 0 & 0 & -N_1 & -N_2 & N_2 \\ 0 & 0 & -N_3 & -N_4 & N_4 \\ M_3 & M_4 & 0 & 0 & 0 \\ -N_1 & -N_2 & N_1 & N_2 & 0 \\ 0 & 0 & N_3 & N_4 & 0 \end{bmatrix} - r\begin{bmatrix} M \\ N \end{bmatrix} - r(M) - r(N),$$

$$\widetilde{s_2}=r\begin{bmatrix} -M_1 & -M_2 & 0 & 0 & M_2 & -M_1 & 0 \\ -M_3 & -M_4 & 0 & 0 & M_4 & -M_3 & 0 \\ 0 & 0 & -N_1 & -N_2 & 0 & N_1 & N_2 \\ 0 & 0 & -N_3 & -N_4 & 0 & N_3 & N_4 \\ M_3 & M_4 & N_3 & N_4 & 0 & 0 & 0 \end{bmatrix}$$
$$- r[M,N] - r(M) - r(N),$$

$$\widetilde{s_3}=r\begin{bmatrix} -M_1 & 0 & M_2 \\ -M_3 & 0 & M_4 \\ M_3 & M_4 & 0 \end{bmatrix} - 2r(M),$$

$$\widetilde{s_4}=r\begin{bmatrix} -N_1 & 0 & N_2 \\ -N_3 & 0 & N_4 \\ N_3 & N_4 & 0 \end{bmatrix} - 2r(N),$$

$$\widetilde{t_1} = r\begin{bmatrix} -M_1 & -M_2 & 0 & 0 & M_2 \\ -M_3 & -M_4 & 0 & 0 & M_4 \\ 0 & 0 & -N_1 & -N_2 & N_2 \\ 0 & 0 & -N_3 & -N_4 & N_4 \\ M_3 & M_4 & 0 & 0 & 0 \\ -N_1 & -N_2 & N_1 & N_2 & 0 \\ 0 & 0 & N_3 & N_4 & 0 \end{bmatrix} - r\left[M_3, M_4, N_3, N_4\right],$$

$$\widetilde{t_2} = r\begin{bmatrix} -M_1 & -M_2 & 0 & 0 & M_2 & -M_1 & 0 \\ -M_3 & -M_4 & 0 & 0 & M_4 & -M_3 & 0 \\ 0 & 0 & -N_1 & -N_2 & 0 & N_1 & N_2 \\ 0 & 0 & -N_3 & -N_4 & 0 & N_3 & N_4 \\ M_3 & M_4 & N_3 & N_4 & 0 & 0 & 0 \end{bmatrix} - r\begin{bmatrix} M_2 \\ M_4 \\ N_2 \\ N_4 \end{bmatrix},$$

$$\begin{aligned} \widetilde{t_3} = -r&\begin{bmatrix} -M_1 & 0 & M_2 & -M_1 & 0 \\ -M_3 & 0 & M_4 & -M_3 & 0 \\ M_3 & M_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & N_1 & N_2 \\ 0 & 0 & 0 & N_3 & N_4 \end{bmatrix} \\ -r&\begin{bmatrix} -M_1 & 0 & M_2 & 0 & 0 \\ -M_3 & 0 & M_4 & 0 & 0 \\ M_3 & M_4 & 0 & 0 & 0 \\ -M_1 & -M_2 & 0 & N_1 & N_2 \\ 0 & 0 & 0 & N_3 & N_4 \end{bmatrix} \\ +r&\begin{bmatrix} -M_1 & 0 & M_2 \\ -M_3 & 0 & M_4 \\ M_3 & M_4 & 0 \end{bmatrix} - m - n, \end{aligned}$$

and

$$\begin{aligned} \widetilde{t_4} = r&\begin{bmatrix} -N_1 & 0 & N_2 \\ -N_3 & 0 & N_4 \\ N_3 & N_4 & 0 \end{bmatrix} - r\begin{bmatrix} -N_1 & 0 & N_2 & -N_1 & 0 \\ -N_3 & 0 & N_4 & -N_3 & 0 \\ N_3 & N_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & M_1 & M_2 \\ 0 & 0 & 0 & M_3 & M_4 \end{bmatrix} \\ -r&\begin{bmatrix} -N_1 & 0 & N_2 & 0 & 0 \\ -N_3 & 0 & N_4 & 0 & 0 \\ N_3 & N_4 & 0 & 0 & 0 \\ -N_1 & -N_2 & 0 & M_1 & M_2 \\ 0 & 0 & 0 & M_3 & M_4 \end{bmatrix} - m - n. \end{aligned}$$

Proof. It just follows from (2.19) and (2.20). □

Corollary 3.2. *Let* M, N *and* G *be given by* (3.1) *and* (3.2). *Then*

(a) M, N *has a common inner inverse with the form* $G = \left[\begin{smallmatrix} 0 & G_2 \\ G_3 & G_4 \end{smallmatrix}\right]$ *if and only if*

$$\widetilde{t_1} + \widetilde{t_2} + m + n + \max\left\{\widetilde{t_3}, \widetilde{t_4}\right\} = 0.$$

(b) *All the common inner inverses of* M, N *have the form* $G = \begin{bmatrix} 0 & G_2 \\ G_3 & G_4 \end{bmatrix}$ *if and only if*

$$\widetilde{s_1} = 0 \text{ or } \widetilde{s_2} = 0 \text{ or } \widetilde{s_3} = 0 \text{ or } \widetilde{s_4} = 0,$$

where $\widetilde{s_1} - \widetilde{s_4}$ *and* $\widetilde{t_1} - \widetilde{t_4}$ *are defined as in Theorem 3.1.*

Corollary 3.3. *Let* M, N *and* G *be given by* (3.1) *and* (3.2). *Then*

(a) M, N *have a common inner inverse with the form* $G = \begin{bmatrix} G_1 & 0 \\ G_3 & 0 \end{bmatrix}$ *if and only if*

$$\begin{aligned}
&2r(M) + 2(N) + r[N_1 - M_1, N_2 - M_2] - r[M_1, M_2, N_1, N_2] \\
&= \max\left\{ r\begin{bmatrix} -M_1 & -M_2 \\ N_1 & N_2 \\ N_3 & N_4 \end{bmatrix} + r(M) - r(N) - n - k - l,\ r\begin{bmatrix} -M_1 & -M_2 \\ -M_3 & -M_4 \\ N_1 & N_2 \end{bmatrix} \right. \\
&\left. -r\begin{bmatrix} -M_1 & -M_2 & -N_1 & -N_2 \\ -M_3 & -M_4 & -N_3 & -N_4 \\ N_1 & N_2 & 0 & 0 \\ 0 & 0 & M_1 & M_2 \\ 0 & 0 & M_3 & M_4 \end{bmatrix} - r\begin{bmatrix} -M_1 & -M_2 & 0 & 0 \\ -M_3 & -M_4 & 0 & 0 \\ N_1 & N_2 & 0 & 0 \\ 0 & 0 & M_1 & M_2 \\ -N_3 & -N_4 & M_3 & M_4 \end{bmatrix} - n - k - l \right\}.
\end{aligned}$$

(b) M, N *have a common inner inverse with the form* $G = \begin{bmatrix} G_1 & G_2 \\ 0 & 0 \end{bmatrix}$ *if and only if*

$$\begin{aligned}
&2r(M) + 2r(N) + r\begin{bmatrix} N_1 - M_1 \\ N_3 - M_3 \end{bmatrix} - r\begin{bmatrix} M_1 \\ M_3 \\ N_1 \\ N_3 \end{bmatrix} \\
&= \max\left\{ r\begin{bmatrix} M_1 & 0 & -M_2 \\ M_3 & 0 & -M_4 \\ 0 & N_1 & N_2 \\ 0 & N_3 & N_4 \end{bmatrix} + r\begin{bmatrix} -M_1 & N_1 & N_2 \\ -M_3 & N_3 & N_4 \end{bmatrix} + k + m + n, \right. \\
&\left. r\begin{bmatrix} N_1 & 0 & -N_2 \\ N_3 & 0 & -N_4 \\ 0 & M_1 & M_2 \\ 0 & M_3 & M_4 \end{bmatrix} + r\begin{bmatrix} -N_1 & M_1 & M_2 \\ -N_3 & M_3 & M_4 \end{bmatrix} + k + m + n \right\}.
\end{aligned}$$

(c) M, N *have a common inner inverse with the form* $G = \begin{bmatrix} G_1 & 0 \\ 0 & 0 \end{bmatrix}$ *if and only if*

$$\begin{aligned}
&2r(M) + 2(N) + r[N_1 - M_1, N_2 - M_2] - r[M_1, M_2, N_1, N_2] \\
&= \max\left\{ r\begin{bmatrix} -M_1 & -M_2 \\ N_1 & N_2 \\ N_3 & N_4 \end{bmatrix} + r(M) - r(N) - n - k - l,\ r\begin{bmatrix} -M_1 & -M_2 \\ -M_3 & -M_4 \\ N_1 & N_2 \end{bmatrix} \right.
\end{aligned}$$

$$-r\begin{bmatrix} -M_1 & -M_2 & -N_1 & -N_2 \\ -M_3 & -M_4 & -N_3 & -N_4 \\ N_1 & N_2 & 0 & 0 \\ 0 & 0 & M_1 & M_2 \\ 0 & 0 & M_3 & M_4 \end{bmatrix} - r\begin{bmatrix} -M_1 & -M_2 & 0 & 0 \\ -M_3 & -M_4 & 0 & 0 \\ N_1 & N_2 & 0 & 0 \\ 0 & 0 & M_1 & M_2 \\ -N_3 & -N_4 & M_3 & M_4 \end{bmatrix} - n - k - l \Bigg\}$$

and

$$2r(M) + 2r(N) + r\begin{bmatrix} N_1 - M_1 \\ N_3 - M_3 \end{bmatrix} - r\begin{bmatrix} M_1 \\ M_3 \\ N_1 \\ N_3 \end{bmatrix}$$

$$= \max\left\{ r\begin{bmatrix} M_1 & 0 & -M_2 \\ M_3 & 0 & -M_4 \\ 0 & N_1 & N_2 \\ 0 & N_3 & N_4 \end{bmatrix} + r\begin{bmatrix} -M_1 & N_1 & N_2 \\ -M_3 & N_3 & N_4 \end{bmatrix} + k + m + n, \right.$$

$$\left. r\begin{bmatrix} N_1 & 0 & -N_2 \\ N_3 & 0 & -N_4 \\ 0 & M_1 & M_2 \\ 0 & M_3 & M_4 \end{bmatrix} + r\begin{bmatrix} -N_1 & M_1 & M_2 \\ -N_3 & M_3 & M_4 \end{bmatrix} + k + m + n \right\}.$$

Corollary 3.4. *Let M, N and G be given by* (3.1) *and* (3.2)*. Then the submatrix G_1 in* (3.2) *is unique if and only if M, N satisfy the following conditions:*

$$r\begin{bmatrix} M_1 \\ M_3 \\ N_1 \\ N_3 \end{bmatrix} = n, \mathcal{R}\begin{bmatrix} M_1 \\ M_3 \\ N_1 \\ N_3 \end{bmatrix} \cap \mathcal{R}\begin{bmatrix} M_2 \\ M_4 \\ N_2 \\ N_4 \end{bmatrix} = \{0\},$$

$$r[M_1, M_2, N_1, N_2] = m, \mathcal{R}[M_1, M_2, N_1, N_2]^T \cap \mathcal{R}[M_2, M_4, N_2, N_4]^T = \{0\},$$

$$r\begin{bmatrix} M_1 \\ M_3 \end{bmatrix} = n, \mathcal{R}\begin{bmatrix} M_1 \\ M_3 \end{bmatrix} \cap \mathcal{R}\begin{bmatrix} M_2 \\ M_4 \end{bmatrix} = \{0\}$$

or

$$r[N_1, N_2] = m, \mathcal{R}[N_1, N_2]^T \cap \mathcal{R}[N_2, N_4]^T = \{0\}$$

and

$$r\begin{bmatrix} N_1 \\ N_3 \end{bmatrix} = n, \mathcal{R}\begin{bmatrix} N_1 \\ N_3 \end{bmatrix} \cap \mathcal{R}\begin{bmatrix} N_2 \\ N_4 \end{bmatrix} = \{0\}$$

or

$$r[M_1, M_2] = m, \mathcal{R}[M_1, M_2]^T \cap \mathcal{R}[M_3, M_4]^T = \{0\}.$$

Remark 3.1. Clearly, the results in [7] and [3] are the special cases of those in this paper.

4. Conclusion

In this paper, we have given formulas of the extreme ranks of matrices X_i $(i = 1, 2, 3, 4)$ in (1.2), and characterized the structure of solutions to (1.2). We have established necessary and sufficient conditions for the uniqueness of the submatrices X_i $(i = 1, 2, 3, 4)$ in (1.2), and considered the independence of submatrices in solutions to (1.2). As applications, we have presented the maximal and minimal ranks of the submatrices of the common inner inverse G, partitioned into 2×2 block form, of quaternion matrices M and N, and described the properties of the submatrices of G. The results in [7] and [3] can be regarded as the special cases of those in this paper.

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