

## ON THE STABILITY OF BI-DERIVATIONS IN BANACH ALGEBRAS

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**ABSTRACT.** Let  $\mathcal{A}$  be a Banach algebra and let  $f : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$  be an approximate bi-derivation in the sense of Hyers-Ulam-Rassias. In this note, we prove the Hyers-Ulam-Rassias stability of bi-derivations on Banach algebras. If, in addition,  $\mathcal{A}$  is unital, then  $f : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$  is an exact bi-derivation. Moreover, if  $\mathcal{A}$  is unital, prime and  $f$  is symmetric, then  $f = 0$ .

### 1. Introduction

Let  $\mathcal{A}$  be an algebra over the real or complex field  $\mathbb{F}$ . An additive mapping  $d : \mathcal{A} \rightarrow \mathcal{A}$  is said to be a *ring derivation* if  $d(xy) = xd(y) + d(x)y$  holds for all  $x, y \in \mathcal{A}$ . A bi-additive mapping  $\Delta : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ , which is additive in both arguments, is called a *bi-derivation* if

$$\Delta(xy, z) = x\Delta(y, z) + \Delta(x, z)y$$

and

$$\Delta(x, yz) = y\Delta(x, z) + \Delta(x, y)z$$

hold for all  $x, y, z \in \mathcal{A}$ . In particular, if  $\Delta(x, y) = \Delta(y, x)$  is valid for all  $x, y \in \mathcal{A}$ , it is said that  $\Delta$  is symmetric. The concept of a symmetric bi-derivation was introduced by Gy. Maksa in [11]. It was shown in [9] that symmetric bi-derivations are related to general solutions of some functional equations.

Recently, T. Miura *et al.* [12] considered the stability of ring derivations on Banach algebras: Under suitable conditions, every approximate ring derivation  $f$  on a Banach algebra  $\mathcal{A}$  is an exact ring derivation. If  $\mathcal{A}$  is a commutative semisimple Banach algebra with the maximal ideal space without isolated points, then  $f$  is identically zero.

The study of stability problems originated from a question by S. M. Ulam [19] in 1940: *Under what condition does there exist a homomorphism near an approximate homomorphism?* In 1941, D. H. Hyers [8] gave a first affirmative

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answer to the question of Ulam for Banach spaces, which states that if  $\delta > 0$  and  $f : \mathcal{X} \rightarrow \mathcal{Y}$  is a mapping with  $\mathcal{X}$  a normed space,  $\mathcal{Y}$  a Banach space such that

$$\|f(x+y) - f(x) - f(y)\| \leq \delta$$

for all  $x, y \in \mathcal{X}$ , then there exists a unique additive mapping  $T : \mathcal{X} \rightarrow \mathcal{Y}$  such that

$$\|f(x) - T(x)\| \leq \delta$$

for all  $x \in \mathcal{X}$ .

A generalized version of the theorem of Hyers for approximately additive mappings was first given by T. Aoki [1] in 1950. In 1978, Th. M. Rassias [16] independently introduced the unbounded Cauchy difference and was the first to prove the stability of the linear mapping between Banach spaces: If there exist a  $\theta > 0$  and  $p < 1$  such that

$$(1.1) \quad \|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all  $x, y \in \mathcal{X}$ , then there exists a unique additive mapping  $T : \mathcal{X} \rightarrow \mathcal{Y}$  such that

$$(1.2) \quad \|f(x) - T(x)\| \leq \frac{2\theta}{2 - 2^p} \|x\|^p$$

for all  $x \in \mathcal{X}$ . Also, if  $f(tx)$  is continuous in all real  $t$  for each fixed  $x$  in  $\mathcal{X}$ , then  $T$  is linear. If  $p < 0$  and the inequality (1.1) holds for  $x, y \neq 0$ , then the inequality (1.2) for  $x \neq 0$ . In 1991, Z. Gajda [6] answered the question for the case  $p > 1$ , which was raised by Th. M. Rassias. He [6] also gave an example that the Rassias' stability result is not valid for  $p = 1$ .

During the last thirty years, a number of results concerning the stability have been obtained by various ways [5, 7, 10, 13, 17], and been applied to a number of functional equations and mappings. In particular, Badora [2], Bae and Park [3], Park [14], Rassias and Kim [15], Šemrl [18] have contributed works to the stability problem of derivations.

Suppose that  $\mathcal{A}$  is a Banach algebra. For a given mapping  $f : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ , we define

$$\begin{aligned} C_1 f(x, y, z) &= f(x+y, z) - f(x, z) - f(y, z), \\ C_2 f(x, y, z) &= f(x, y+z) - f(x, y) - f(x, z), \\ D_1 f(x, y, z) &= f(xy, z) - xf(y, z) - f(x, z)y, \\ D_2 f(x, y, z) &= f(x, yz) - yf(x, z) - f(x, y)z \end{aligned}$$

for all  $x, y, z \in \mathcal{A}$ .

In this note, we will deal with the following type of approximate bi-derivations in the sense of Hyers-Ulam-Rassias, that is, let  $p, q, \theta, \varepsilon$  be real numbers with  $p, q \neq 1$  and  $\theta, \varepsilon > 0$ . We consider a mapping  $f : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$  with the properties

$$(1.3) \quad \|C_1 f(x, y, z)\| \leq \theta(\|x\|^p + \|y\|^p)\|z\|^q,$$

$$(1.4) \quad \|C_2 f(x, y, z)\| \leq \theta \|x\|^p (\|y\|^p + \|z\|^q),$$

$$(1.5) \quad \|D_1 f(x, y, z)\| \leq \varepsilon \|x\|^p \|y\|^p \|z\|^q,$$

$$(1.6) \quad \|D_2 f(x, y, z)\| \leq \varepsilon \|x\|^p \|y\|^q \|z\|^q$$

for all  $x, y, z \in \mathcal{A}$ . In addition, we will investigate approximate bi-derivations which become zero.

## 2. Stability of bi-derivations

In this section,  $\mathbb{R}$ ,  $\mathbb{Q}$  and  $\mathbb{N}$  will denote the set of the real, the rational and the natural numbers, respectively.

**Theorem 2.1.** *Let  $\mathcal{A}$  be a Banach algebra. Suppose that  $f : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$  is a mapping satisfying the inequalities (1.3)  $\sim$  (1.6) for some  $\theta, \varepsilon > 0$  and some  $p, q \in \mathbb{R} \setminus \{1\}$ . If  $p, q < 1$  or  $p, q > 1$ , then there exists a unique bi-derivation  $\Delta : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$  such that*

$$(2.1) \quad \|\Delta(x, z) - f(x, z)\| \leq K(p, q, \theta) \|x\|^p \|z\|^q$$

for all  $x, z \in \mathcal{A}$ , where  $K(p, q, \theta) = \frac{\theta}{2} \left( \frac{1}{|2-2^p|} + \frac{1}{|2-2^q|} \right)$ . If  $p, q < 0$  and the inequalities (1.3)  $\sim$  (1.6) hold for  $x, y, z \neq 0$ , then the inequality (2.1) holds for  $x, z \neq 0$ .

*Proof.* Assume that  $\tau = 1$  if  $p, q < 1$  and  $\tau = -1$  if  $p, q > 1$ . By (1.3), for each fixed  $z \in \mathcal{A}$ , the function  $f_z(x) = f(x, z)$  satisfies the inequality

$$(2.2) \quad \|f_z(x+y) - f_z(x) - f_z(y)\| \leq \theta (\|x\|^p + \|y\|^p) \|z\|^q$$

for all  $x, y \in \mathcal{A}$ . So, from the results of Rassias [16] and Gajda [6], the inequality (2.2) guarantees that there exists a unique additive mapping  $A_z : \mathcal{A} \rightarrow \mathcal{A}$  defined by  $A_z(x) = \lim_{n \rightarrow \infty} 2^{-\tau n} f_z(2^{\tau n} x)$  for all  $x \in \mathcal{A}$  such that

$$\|A_z(x) - f_z(x)\| \leq \frac{\theta}{|2-2^p|} \|x\|^p \|z\|^q$$

for all  $x \in \mathcal{A}$ . Now, let us define  $A_1(x, z) = A_z(x)$  for all  $x, z \in \mathcal{A}$ . Then  $A_1$  is additive in the first variable, i.e.,  $C_1 A_1(x, y, z) = 0$  for all  $x, y, z \in \mathcal{A}$ .

By (1.4), for each fixed  $x \in \mathcal{A}$ , the function  $f_x(z) = f(x, z)$  satisfies the inequality

$$(2.3) \quad \|f_x(y+z) - f_x(y) - f_x(z)\| \leq \theta \|x\|^p (\|y\|^q + \|z\|^q)$$

for all  $y, z \in \mathcal{A}$ . From the results of Rassias [16] and Gajda [6], the inequality (2.3) implies that there exists a unique additive mapping  $A_x : \mathcal{A} \rightarrow \mathcal{A}$  defined by  $A_x(z) = \lim_{n \rightarrow \infty} 2^{-\tau n} f_x(2^{\tau n} z)$  for all  $z \in \mathcal{A}$  such that

$$\|A_x(z) - f_x(z)\| \leq \frac{\theta}{|2-2^p|} \|x\|^p \|z\|^q$$

for all  $z \in \mathcal{A}$ . Let  $A_2(x, z) = A_z(x)$  for all  $x, z \in \mathcal{A}$ . Then  $A_2$  is additive in the second variable, i.e.,  $C_2 A_2(x, y, z) = 0$  for all  $x, y, z \in \mathcal{A}$ . By (1.3), (1.4) and the definitions of  $A_1$  and  $A_2$ , we get

$$\begin{aligned} C_2 A_1(x, y, z) &= 0, \\ C_1 A_2(x, y, z) &= 0 \end{aligned}$$

for all  $x, y, z \in \mathcal{A}$ . Indeed, we see that

$$\begin{aligned} &\|2^{-\tau n} f(2^{\tau n} x, y + z) - 2^{-\tau n} f(2^{\tau n} x, y) - 2^{-\tau n} f(2^{\tau n} x, z)\| \\ &\leq 2^{\tau n(p-1)} \theta \|x\|^p (\|y\|^q + \|z\|^q) \rightarrow 0 \quad \text{as } n \rightarrow \infty \end{aligned}$$

for all  $x, y, z \in \mathcal{A}$ . Hence we get

$$A_1(x, y + z) - A_1(x, y) - A_1(x, z) = 0,$$

i.e.,  $C_2 A_1(x, y, z) = 0$  for all  $x, y, z \in \mathcal{A}$ . Similarly, it follows that  $C_1 A_2(x, y, z) = 0$  holds for all  $x, y, z \in \mathcal{A}$ . We define a mapping  $\Delta : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$  by

$$\Delta(x, z) = \frac{1}{2} [A_1(x, z) + A_2(x, z)]$$

for all  $x, z \in \mathcal{A}$ . Then we conclude that  $\Delta$  is bi-additive and the inequality

$$\begin{aligned} \|\Delta(x, z) - f(x, z)\| &= \left\| \frac{1}{2} [A_1(x, z) + A_2(x, z)] - f(x, z) \right\| \\ &= \frac{1}{2} \|A_1(x, z) + A_2(x, z) - 2f(x, z)\| \\ &\leq \frac{1}{2} [\|A_1(x, z) - f(x, z)\| + \|A_2(x, z) - f(x, z)\|] \\ &\leq \frac{\theta}{2} \left( \frac{1}{|2-2^p|} + \frac{1}{|2-2^q|} \right) \|x\|^p \|z\|^q \end{aligned}$$

holds for all  $x, z \in \mathcal{A}$ . We now want to prove that  $\Delta$  is unique. Let  $K(p, q, \theta) = \frac{\theta}{2} \left( \frac{1}{|2-2^p|} + \frac{1}{|2-2^q|} \right)$ , where  $p, q < 1$  or  $p, q > 1$ . Assume that there exists another one, denoted by  $\Delta' : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ . Then there exist constants  $\theta_1 > 0$  and  $p_1, q_1 < 1$  or  $\theta_1 > 0$  and  $p_1, q_1 > 1$  such that

$$\|\Delta'(x, z) - f(x, z)\| \leq K(p_1, q_1, \theta_1) \|x\|^{p_1} \|z\|^{q_1}$$

for all  $x, z \in \mathcal{A}$  which yields

$$\begin{aligned} \|\Delta(x, z) - \Delta'(x, z)\| &\leq \|\Delta(x, z) - f(x, z)\| + \|f(x, z) - \Delta'(x, z)\| \\ &\leq K(p, q, \theta) \|x\|^p \|z\|^q + K(p_1, q_1, \theta_1) \|x\|^{p_1} \|z\|^{q_1} \end{aligned}$$

for all  $x, z \in \mathcal{A}$ . Therefore we have

$$\begin{aligned} \|\Delta(x, z) - \Delta'(x, z)\| &= n^{-2\tau} \|\Delta(n^\tau x, n^\tau z) - \Delta'(n^\tau x, n^\tau z)\| \\ &\leq n^{-2\tau} (K(p, q, \theta) \|n^\tau x\|^p \|n^\tau z\|^q \\ &\quad + K(p_1, q_1, \theta_1) \|n^\tau x\|^{p_1} \|n^\tau z\|^{q_1}) \end{aligned}$$

$$= n^{\tau(p+q-2)} K(p, q, \theta) \|x\|^p \|z\|^q \\ + n^{\tau(p_1+q_1-2)} K(p_1, q_1, \theta_1) \|x\|^{p_1} \|z\|^{q_1}$$

for all  $x, z \in \mathcal{A}$ . By letting  $n \rightarrow \infty$ , we get  $\Delta(x, z) = \Delta'(x, z)$  for all  $x, z \in \mathcal{A}$ .

Now, we claim that  $\Delta$  is a bi-derivation. Since  $\Delta$  is bi-additive, we see that  $\Delta(x, z) = 2^{-\tau n} \Delta(2^{\tau n} x, z)$  and  $\Delta(x, z) = 2^{-\tau n} \Delta(z, 2^{\tau n} z)$  holds for all  $x, z \in \mathcal{A}$  and all  $n \in \mathbb{N}$ . First, it follows from (2.1) that

$$\begin{aligned} \|\Delta(x, z) - 2^{-2\tau n} f(2^{\tau n} x, 2^{\tau n} z)\| &= 2^{-2\tau n} \|\Delta(2^{\tau n} x, 2^{\tau n} z) - f(2^{\tau n} x, 2^{\tau n} z)\| \\ &\leq 2^{-2\tau n} K(p, q, \theta) \|2^{\tau n} x\|^p \|2^{\tau n} z\|^q \\ &= 2^{\tau(p+q-2)n} K(p, q, \theta) \|x\|^p \|z\|^q \end{aligned}$$

for all  $x, z \in \mathcal{A}$  and all  $n \in \mathbb{N}$ . Since  $\tau(p+q-2) < 0$ , we have

$$(2.4) \quad \|\Delta(x, z) - 2^{-2\tau n} f(2^{\tau n} x, 2^{\tau n} z)\| \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

for all  $x, z \in \mathcal{A}$ . Following the similar argument as the above, we obtain

$$\|\Delta(xy, z) - 2^{-3\tau n} f(2^{2\tau n} xy, 2^{\tau n} z)\| \leq 2^{\tau(2p+q-3)n} K(p, q, \theta) \|x\|^p \|z\|^q$$

for all  $x, y, z \in \mathcal{A}$  and all  $n \in \mathbb{N}$ , and so

$$(2.5) \quad \|\Delta(xy, z) - 2^{-3\tau n} f(2^{2\tau n} xy, 2^{\tau n} z)\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Since  $f$  satisfies (1.5), we get

$$\begin{aligned} &\|2^{-3\tau n} f(2^{2\tau n} xy, 2^{\tau n} z) - 2^{-2\tau n} x f(2^{\tau n} y, 2^{\tau n} z) - f(2^{\tau n} x, 2^{\tau n} z) 2^{-2\tau n} y\| \\ &= 2^{-3\tau n} \|f((2^{\tau n} x)(2^{\tau n} y), 2^{\tau n} z) - 2^{\tau n} x f(2^{\tau n} y, 2^{\tau n} z) - f(2^{\tau n} x, 2^{\tau n} z) 2^{\tau n} y\| \\ &\leq 2^{-3\tau n} \varepsilon \|2^{\tau n} x\|^p \|2^{\tau n} y\|^p \|2^{\tau n} z\|^q \\ &= 2^{\tau(2p+q-3)n} \varepsilon \|x\|^p \|y\|^p \|z\|^q \end{aligned}$$

for all  $x, y, z \in \mathcal{A}$  and all  $n \in \mathbb{N}$ . From reminding of  $\tau(2p+q-3) < 0$ , it follows that

$$(2.6) \quad \|2^{-3\tau n} f(2^{2\tau n} xy, 2^{\tau n} z) - 2^{-2\tau n} x f(2^{\tau n} y, 2^{\tau n} z) - 2^{-2\tau n} y f(2^{\tau n} x, 2^{\tau n} z)\| \rightarrow 0$$

as  $n \rightarrow \infty$ . Using (2.4), (2.5) and (2.6), we see that

$$\begin{aligned} &\|\Delta(xy, z) - x\Delta(y, z) - \Delta(x, z)y\| \\ &\leq \|\Delta(xy, z) - 2^{-3\tau n} f(2^{2\tau n} xy, 2^{\tau n} z)\| \\ &\quad + \|2^{-3\tau n} f(2^{2\tau n} xy, 2^{\tau n} z) - 2^{-2\tau n} x f(2^{\tau n} y, 2^{\tau n} z) - 2^{-\tau n} y f(2^{\tau n} x, 2^{\tau n} z)\| \\ &\quad + \|x\Delta(y, z) - 2^{-2\tau n} x f(2^{\tau n} y, 2^{\tau n} z)\| + \|\Delta(x, z)y - f(2^{\tau n} x, 2^{\tau n} z) 2^{-2\tau n} y\| \\ &\leq \|\Delta(xy, z) - 2^{-3\tau n} f(2^{2\tau n} xy, 2^{\tau n} z)\| \\ &\quad + \|2^{-3\tau n} f(2^{2\tau n} xy, 2^{\tau n} z) - 2^{-2\tau n} x f(2^{\tau n} y, 2^{\tau n} z) - 2^{-\tau n} y f(2^{\tau n} x, 2^{\tau n} z)\| \\ &\quad + \|x\| \|\Delta(y, z) - 2^{-2\tau n} f(2^{\tau n} y, 2^{\tau n} z)\| + \|\Delta(x, z) - f(2^{\tau n} x, 2^{\tau n} z) 2^{-2\tau n}\| \|y\| \end{aligned}$$

and so taking the limit as  $n \rightarrow \infty$  implies that  $\Delta(xy, z) = x\Delta(y, z) + \Delta(x, z)y$  is valid for all  $x, y, z \in \mathcal{A}$ . Since  $f$  also satisfies (1.6), we deduce that  $\Delta(x, yz) = y\Delta(x, z) + \Delta(x, y)z$  holds for all  $x, y, z \in \mathcal{A}$  by applying the same method as above. That is,  $\Delta$  is a bi-derivation as claimed and the proof is complete.  $\square$

**Lemma 2.2.** *Let  $\mathcal{A}$  be a unital Banach algebra. Suppose that  $f : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$  is a mapping satisfying the inequalities (1.3)  $\sim$  (1.6) for some  $\theta, \varepsilon > 0$  and some  $p, q \in \mathbb{R} \setminus \{1\}$ . If  $p, q < 1$  or  $p, q > 1$ , then we have*

$$f(rx, ry) = r^2 f(x, y)$$

for all  $x, y \in \mathcal{A}$  and all  $r \in \mathbb{Q}$ .

*Proof.* Let  $e$  be a unit element of  $\mathcal{A}$  and  $r \in \mathbb{Q} \setminus \{0\}$  arbitrarily. Put  $\tau = 1$  if  $p, q < 1$  and  $\tau = -1$  if  $p, q > 1$ . Then we see that  $\tau(p-1) < 0$  and  $\tau(q-1) < 0$ . By Theorem 2.1, there exists a unique bi-derivation  $\Delta : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$  satisfying the inequality (2.1). Recall that  $\Delta$  is bi-additive, and hence it is easy to see that  $\Delta(rx, y) = r\Delta(x, y)$  and  $\Delta(x, ry) = r\Delta(x, y)$  for all  $x, y \in \mathcal{A}$ . Then we obtain that  $f(rx, y) = rf(x, y)$  for all  $x, y \in \mathcal{A}$ . For,

$$\begin{aligned} & \|\Delta((2^{\tau n}e)(rx), y) - r2^{\tau n}ef(x, y) - f(2^{\tau n}e, y)rx\| \\ & \leq r\|\Delta(2^{\tau n}ex, y) - f(2^{\tau n}ex, y)\| \\ & \quad + r\|f(2^{\tau n}ex, y) - 2^{\tau n}ef(x, y) - f(2^{\tau n}e, y)x\| \end{aligned}$$

for all  $x, y \in \mathcal{A}$  and all  $n \in \mathbb{N}$ . Now the inequalities (1.5) and (2.1) yield that

$$\begin{aligned} & \|\Delta((2^{\tau n}e)(rx), y) - r2^{\tau n}ef(x, y) - f(2^{\tau n}e, y)rx\| \\ & \leq rK(p, q, \theta)\|2^{\tau n}ex\|^p\|y\|^q + r\varepsilon\|2^{\tau n}e\|^p\|x\|^p\|y\|^q \\ (2.7) \quad & = 2^{\tau pn}r(K(p, q, \theta) + \varepsilon)\|x\|^p\|y\|^q \end{aligned}$$

for all  $x, y \in \mathcal{A}$  and all  $n \in \mathbb{N}$ .

It follows from (2.1) and (2.7) that

$$\begin{aligned} & \|f((2^{\tau n}e)(rx), y) - r2^{\tau n}ef(x, y) - f(2^{\tau n}e, y)rx\| \\ & \leq \|f((2^{\tau n}e)(rx), y) - \Delta((2^{\tau n}e)(rx), y)\| \\ & \quad + \|\Delta((2^{\tau n}e)(rx), y) - r2^{\tau n}ef(x, y) - f(2^{\tau n}e, y)rx\| \\ & \leq K(p, q, \theta)\|(2^{\tau n}e)(rx)\|^p\|y\|^q + 2^{\tau pn}r(K(p, q, \theta) + \varepsilon)\|x\|^p\|y\|^q \\ & = 2^{\tau pn}K(p, q, \theta)(|r|^p + r)\|x\|^p\|y\|^q + 2^{\tau pn}r\varepsilon\|x\|^p\|y\|^q \end{aligned}$$

for all  $x, y \in \mathcal{A}$  and all  $n \in \mathbb{N}$ . That is, we have

$$\begin{aligned} & \|f((2^{\tau n}e)(rx), y) - r2^{\tau n}ef(x, y) - f(2^{\tau n}e, y)rx\| \\ (2.8) \quad & \leq 2^{\tau pn}K(p, q, \theta)(|r|^p + r)\|x\|^p\|y\|^q + 2^{\tau pn}r\varepsilon\|x\|^p\|y\|^q \end{aligned}$$

for all  $x, y \in \mathcal{A}$  and all  $n \in \mathbb{N}$ . From (1.5) and (2.8), we obtain

$$\begin{aligned} & \|2^{\tau n}\{f(rx, y) - rf(x, y)\}\| \\ & = \|2^{\tau n}e\{f(rx, y) - rf(x, y)\}\| \end{aligned}$$

$$\begin{aligned}
&\leq \|2^{\tau n}ef(rx, y) + f(2^{\tau n}e, y)rx - f((2^{\tau n}e)(rx), y)\| \\
&\quad + \|f((2^{\tau n}e)(rx), y) - r2^{\tau n}ef(x, y) - f(2^{\tau n}e, y)rx\| \\
&\leq \varepsilon \|2^{\tau n}e\|^p \|rx\|^p \|y\|^q + 2^{\tau pn}K(p, q, \theta)(|r|^p + r)\|x\|^p \|y\|^q + 2^{\tau pn}r\varepsilon \|x\|^p \|y\|^q \\
&= 2^{\tau pn}(|r|^p + r)(K(p, q, \theta) + \varepsilon)\|x\|^p \|y\|^q
\end{aligned}$$

for all  $x, y \in \mathcal{A}$  and all  $n \in \mathbb{N}$ . This means that

$$(2.9) \quad \|f(rx, y) - rf(x, y)\| \leq 2^{\tau(p-1)n}(|r|^p + r)(K(p, q, \theta) + \varepsilon)\|x\|^p \|y\|^q$$

for all  $x, y \in \mathcal{A}$  and all  $n \in \mathbb{N}$ . Since  $\tau(p-1) < 0$  and  $r$  was arbitrary, if we take  $n \rightarrow \infty$  in (2.9), then we arrive at

$$f(rx, y) = rf(x, y)$$

for all  $x, y \in \mathcal{A}$  and all  $r \in \mathbb{Q} \setminus \{0\}$ . By the similar process, we also obtain that

$$f(x, ry) = rf(x, y)$$

for all  $x, y \in \mathcal{A}$  and all  $r \in \mathbb{Q} \setminus \{0\}$ . Consequently, we find that

$$f(rx, ry) = r^2f(x, y)$$

for all  $x, y \in \mathcal{A}$  and all  $r \in \mathbb{Q} \setminus \{0\}$ . It is obvious that  $f(0x, 0y) = f(0, 0) = 0 = 0f(x, y)$  for all  $x, y \in \mathcal{A}$ . This completes the proof.  $\square$

Our main result is as follows:

**Theorem 2.3.** *Let  $\mathcal{A}$  be a unital Banach algebra. Suppose that  $f : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$  is a mapping satisfying the inequalities (1.3)  $\sim$  (1.6) for some  $\theta, \varepsilon > 0$  and some  $p, q \in \mathbb{R} \setminus \{1\}$ . If  $p, q < 1$  or  $p, q > 1$ , then  $f : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$  is a bi-derivation.*

*Proof.* Let  $\Delta$  be a unique bi-derivation as in Theorem 2.1. Put  $\tau = 1$  if  $p, q < 1$  and  $\tau = -1$  if  $p, q > 1$ . Since  $f(2^{\tau n}x, 2^{\tau n}y) = 2^{2\tau n}f(x, y)$  for all  $x, y \in \mathcal{A}$  and all  $n \in \mathbb{N}$  by Lemma 2.2, it follows from (2.1) that

$$\begin{aligned}
\|f(x, y) - \Delta(x, y)\| &= \|2^{-2\tau n}f(2^{\tau n}x, 2^{\tau n}y) - 2^{-\tau n}\Delta(2^{\tau n}x, 2^{\tau n}y)\| \\
&\leq 2^{-2\tau n}K(p, q, \theta)\|2^{\tau n}x\|^p \|2^{\tau n}y\|^q \\
&= 2^{\tau(p+q-2)n}K(p, q, \theta)\|x\|^p \|y\|^q
\end{aligned}$$

for all  $x, y \in \mathcal{A}$  and all  $n \in \mathbb{N}$ . Namely,

$$(2.10) \quad \|f(x, y) - \Delta(x, y)\| \leq 2^{\tau(p+q-2)n}K(p, q, \theta)\|x\|^p \|y\|^q$$

for all  $x \in \mathcal{A}$  and all  $n \in \mathbb{N}$ . Since  $\tau(p+q-2) < 0$ , if we let  $n \rightarrow \infty$  in (2.10), then we conclude that  $f(x, y) = \Delta(x, y)$  for all  $x, y \in \mathcal{A}$  which implies that  $f$  is a bi-derivation. The proof is complete  $\square$

M. Brešar [4, Theorem 3.5] proved that if  $\Delta$  is a symmetric bi-derivation on noncommutative 2-torsion free prime rings, then we have  $\Delta = 0$ . The following is the Brešar's result for approximate bi-derivations.

**Corollary 2.4.** *Let  $\mathcal{A}$  be a unital Banach algebra which is prime. Suppose that  $f : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$  is a symmetric mapping satisfying (1.3) and (1.5) for some  $\theta, \varepsilon > 0$  and some  $p, q \in \mathbb{R} \setminus \{1\}$ . If  $p, q < 1$  or  $p, q > 1$ , then we have  $f = 0$ .*

*Proof.* Applying Theorem 2.3, we see that  $f$  is a symmetric bi-derivation. Hence we have  $f = 0$  by Brešar's result [4, Theorem 3.5] which completes the proof.  $\square$

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