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A NOTE ON THE BRÜCK CONJECTURE

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Abstract. In 1996, Brück studied the relation between f and f' if an entire function f shares one value a CM with its first derivative f' and posed the famous Brück conjecture. In this work, we generalize the value a in the Brück conjecture to a small function α. Meanwhile, we prove that the Brück conjecture holds for a class of meromorphic functions.

1. Introduction and main results

To state our main result, we need the following concepts and definitions.

Definition 1. The order $\rho(f)$ and the super order $\sigma_2(f)$ of a meromorphic function f are defined by

$$\rho(f) = \overline{\lim_{r\to\infty}}\frac{\log T(r,f)}{\log r},\ \sigma_2(f) = \overline{\lim_{r\to\infty}}\frac{\log\ \log T(r,f)}{\log r}$$

respectively.

Let f, g and α be three meromorphic functions in the complex plane $\mathbb{C}$. We say f and g share α CM provided that $f-\alpha$ and $g-\alpha$ have the same zeros with the same multiplicities. If $f-\alpha$ and $g-\alpha$ have the same zeros, then we say f and g share α IM and denote it by $f(z)=\alpha(z)\Leftrightarrow g(z)=\alpha(z)$. It is assumed that the reader is familiar with the standard symbols and fundamental results of Nevanlinna theory, as found in [6, 13].

The subject on sharing values between entire functions and their derivatives was first studied by Rubel and Yang [11]. They proved a result in 1977 that if a non-constant entire function f and its first derivative f' share two distinct finite numbers $a,\ b$ CM, then $f=f'$. Since then, shared value problems have been studied by many authors and a number of profound results have been obtained(see, e.g., [5, 10]).

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In 1996, Brück [1] studied the relation between f and f' if an entire function f shares one value CM with its first derivative f'. Meanwhile, he posed the following famous conjecture.

Conjecture. *Let f be a non-constant entire function such that the hyper-order $\sigma_2(f)$ of f is not a positive integer and $\sigma_2(f) < \infty$. If f and f' share a finite value a CM, then*

$$\frac{f'-a}{f-a} = c,$$

where c is nonzero constant.

In fact, the conjecture for the case $a = 0$ had been proved by R. Brück in [1]. From the differential equations

$$\frac{f'-a}{f-a} = e^{z^n}, \ \frac{f'-a}{f-a} = e^{e^z}, \tag{1.1}$$

we see that when the hyper-order of $\sigma_2(f)$ is a positive integer or infinite, the conjecture does not hold.

The conjecture for the case that f is of finite order had been proved by Gundersen and Yang [5] ; the case that f is of infinite order with $\sigma_2(f) < \frac{1}{2}$ had been proved by Chen and Shon [3]. But the case $\sigma_2(f) \geq \frac{1}{2}$ is still open.

Under some additional assumptions, there are many results related to the Brück conjecture, see, e.g., [12, Theorem 1] and [16, Theorem 1.1].

In fact, to prove the conjecture is a hard work. Some have considered adding a deficient value condition to help solve the problem, see [15, Theorem 1]. Based on this, it is interesting to ask whether the Brück conjecture holds or not if the function f is replaced by n-th powers f^n, since $f^n (n \geq 2)$ obviously satisfies the following deficient value condition,

$$\Theta(0, f^n) = 1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}(r, \frac{1}{f^n})}{T(r, f^n)} \geq 1 - \overline{\lim_{r\to\infty}} \frac{1}{n} \frac{N(r, \frac{1}{f^n})}{T(r, f^n)} \geq 1 - \frac{1}{n} = \frac{n-1}{n}.$$

From (1.1), we see that the conjecture does not hold when $n = 1$. Thus, we only need to discuss the problem when $n \geq 2$. Recently, Yang and Zhang [14] proved that the Brück conjecture holds for the function f^n, and the order restriction on f is not needed if n is relatively large. Actually, they proved the following result.

Theorem A. *Let f be a non-constant entire function, let $n \geq 7$ be an integer, and let $F = f^n$. If F and F' share 1 CM, then $F = F'$, and f assumes the form*

$$f(z) = ce^{\frac{1}{n}z},$$

where c is a non-zero constant.

It is natural to ask whether Theorem A still holds or not if the value 1 is replaced by a small function of f. In this work, we further study the problem and prove that the Brück conjecture holds for a special class of meromorphic functions. In fact, we obtain the following result.

Theorem 1.1. *Let f be a transcendental meromorphic function with finitely many poles, let $n \geq 2$ be an integer, and let $\alpha = Pe^Q$ $(\neq \alpha')$ be an entire function such that the order of α is less than that of f, where P, Q are two polynomials. If f^n and $(f^n)'$ share α CM, then*

$$f = Ae^{\frac{1}{n}z},$$

where A is a non-zero constant.

As a matter of fact, we deduce a more general result.

Theorem 1.2. *Let f be a transcendental meromorphic function with finitely many poles, and let $\alpha = Pe^Q$ $(\neq \alpha')$ be an entire function such that the order of α is less than that of f, where P, Q are two polynomials. If f has only multiple zeros and f and f' share α CM, then $f = f'$ and α reduces to a polynomial.*

Remark. If Q is a constant, then Theorem 1.2 still holds even without the assumption that the order of α is less than that of f. Using a result of [9], this is easily deduced in a manner similar to our proof of Theorem 1.2.

2. Some lemmas

In order to prove our theorems, we need the following lemmas.

Lemma 2.1 ([8]). *Let $\{f_n\}$ be a family of functions meromorphic (analytic) on the unit disc $\triangle$. If $a_n \to a$, $|a| < 1$, and $f_n^\sharp(a_n) \to \infty$, then there exist*
(a) *a subsequence of f_n (which we still write as $\{f_n\}$);*
(b) *points $z_n \to z_0$ $|z_0| < 1$;*
(c) *positive numbers $\rho_n \to 0$*
such that $f_n(z_n + \rho_n\xi) = g_n(\xi) \to g(\xi)$ locally uniformly, where g is a non-constant meromorphic (entire) function on $\mathbb{C}$, such that

$$\rho_n \leq \frac{M}{f_n^\sharp(a_n)},$$

where M is a constant which is independent of n.

Here, as usual, $g^\sharp(\xi) = \frac{|g'(\xi)|}{1+|g(\xi)|^2}$ is the spherical derivative.

Lemma 2.2 ([7]). *Let f be a meromorphic function of infinite order on $\mathbb{C}$. Then there exist points $z_n \to \infty$, such that for every $N > 0$, $f^\sharp(z_n) > |z_n|^N$ if n is sufficiently large.*

Lemma 2.3 ([2]). *Let f and α be meromorphic functions of finite order such that both of f and α have finitely many poles, f and α have no common poles and the order of α is less than the order of f. If f and f' share α CM, then $f' - \alpha = c(f - \alpha)$ for some non-zero constant c.*

3. Proof of Theorem 1.2

We prove the main theorem with the method of J. Grahl and C. Meng [4, Theorem 1.1]. For the convenience of the reader, we present our proof in all detail.

We consider the function $F = \frac{f}{\alpha}$.

Case 1: F has finite order.

Hence $f = F\alpha$ has finite order as well.

Set

$$\mu = \frac{f' - \alpha}{f - \alpha}. \tag{3.1}$$

By Lemma 2.3, we obtain that μ is a constant. If $\mu = 1$, we have $f = f'$. Now, we suppose that $\mu \neq 1$. Rewriting (3.1) as

$$f' = \mu f + \alpha(1 - \mu). \tag{3.2}$$

Now, we consider into two subcases.

Subcase 1.1. f has finitely many zeros.

It follows from (3.2) that f is an entire function. Since f is of finite order, then we set $f = P_1 e^{Q_1}$, where P_1 and Q_1 are two polynomials. Putting the form of f into (3.2) yields

$$[P_1' + P_1 Q' - \mu P_1]e^{Q_1} = \alpha(1 - \mu) = Pe^{Q}(1 - \mu). \tag{3.3}$$

It is obvious that $\deg Q_1 = \deg Q$. It is obvious that $\rho(f) = \rho(\alpha) = \deg Q$, a contradiction.

Subcase 1.2. f has infinitely many zeros.

Assume that z_0 is a zero of f. Noting that f has only multiple zeros, we have $f'(z_0) = 0$. Substituting z_0 into (3.2) yields

$$0 = \alpha(z_0)(1 - \mu) = P(z_0)e^{Q(z_0)}(1 - \mu),$$

which implies that $P(z_0) = 0$. Thus, the zeros of f are also zeros of P. While P is a polynomial and has only finitely many zeros, so f has only finitely many zeros, which contradicts the assumption of Subcase 1.2.

Case 2. F has infinite order.

By Lemma 2.2, there exist $w_n \to \infty$, such that for every $N > 0$, if n is sufficiently large

$$F^{\sharp}(w_n) > |w_n|^N. \tag{3.4}$$

Observing that f has only finitely many poles and α has only finitely many zeros, then there exists a $r > 0$ such that $f(z)$ is analytic and $\alpha(z) \neq 0$ in $|z| \geq r$. Let $D = \{z : |z| \geq r\}$, then $F(z)$ is analytic in D. In view of $w_n \to \infty$ as $n \to \infty$, without loss of generality we may assume $|w_n| \geq r + 1$ for all n. Define $D_1 = \{z : |z| < 1\}$ and

$$F_n(z) = F(w_n + z).$$

Then all $F_n(z)$ are analytic in D_1 and $F_n^\sharp(0) = F^\sharp(w_n) \to \infty$ as $n \to \infty$. It follows from Marty's criterion that $(F_n)_n$ is not normal at $z = 0$.

Therefore, we can apply Lemma 2.1. Choosing an appropriate subsequence of $(F_n)_n$ if necessary, we may assume that there exist sequence $(z_n)_n$ and $(\rho_n)_n$ with $|z_n| < r < 1$ and $\rho_n \to 0$ such that

$$g_n(\zeta) = F_n(z_n + \rho_n\zeta) = \frac{f(w_n + z_n + \rho_n\zeta)}{\alpha(w_n + z_n + \rho_n\zeta)} \to g(\zeta) \tag{3.5}$$

locally uniformly in $\mathbb{C}$ with a non-constant entire function g whose zeros have multiplicity at least 2 and

$$\rho_n \leq \frac{M}{F_n^\sharp(0)} = \frac{M}{F^\sharp(w_n)} \tag{3.6}$$

for a positive number M. From (3.4) and (3.6), we deduce that, for every $N > 0$, if n is sufficiently large,

$$\rho_n \leq M|w_n|^{-N}. \tag{3.7}$$

Differentiating (3.5) yields

$$\begin{aligned} g_n'(\zeta) &= \rho_n \frac{f'(w_n + z_n + \rho_n\zeta)}{\alpha(w_n + z_n + \rho_n\zeta)} - \rho_n \frac{f(w_n + z_n + \rho_n\zeta)\alpha'(w_n + z_n + \rho_n\zeta)}{\alpha(w_n + z_n + \rho_n\zeta)^2} \\ &= \rho_n \frac{f'(w_n + z_n + \rho_n\zeta)}{\alpha(w_n + z_n + \rho_n\zeta)} - \rho_n \frac{g_n(\zeta)\alpha'(w_n + z_n + \rho_n\zeta)}{\alpha(w_n + z_n + \rho_n\zeta)} \to g'(\zeta). \end{aligned}$$

Noting (3.7), we deduce

$$\rho_n \frac{g_n(\zeta)\alpha'(w_n + z_n + \rho_n\zeta)}{\alpha(w_n + z_n + \rho_n\zeta)} = \rho_n \frac{g_n(\zeta)(P' + PQ')(w_n + z_n + \rho_n\zeta)}{P(w_n + z_n + \rho_n\zeta)} \to 0. \tag{3.8}$$

Thus, we have

$$\rho_n \frac{f'(w_n + z_n + \rho_n\zeta)}{\alpha(w_n + z_n + \rho_n\zeta)} \to g'(\zeta). \tag{3.9}$$

Next, we prove that $g(\zeta) = 1 \Rightarrow g'(\zeta) = 0$.

Suppose that $g(\zeta_0) = 1$, then by Hurwitz's theorem, there exists a sequence $\{\zeta_n\}$, $\zeta_n \to \zeta_0$, such that (for n sufficiently large)

$$g_n(\zeta_n) = \frac{f(w_n + z_n + \rho_n\zeta_n)}{\alpha(w_n + z_n + \rho_n\zeta_n)} = 1.$$

Thus $f_n(w_n + z_n + \rho_n\zeta_n) = \alpha(w_n + z_n + \rho_n\zeta_n)$. The assumption $f(z) = \alpha(z) \Rightarrow f'(z) = \alpha(z)$ implies that $f_n'(w_n + z_n + \rho_n\zeta_n) = \alpha(w_n + z_n + \rho_n\zeta_n)$.

Then, by (3.9) we derive that

$$g'(\zeta_0) = \lim_{n\to\infty} \rho_n \frac{f'(w_n + z_n + \rho_n\zeta_n)}{\alpha(w_n + z_n + \rho_n\zeta_n)} = \lim_{n\to\infty} \rho_n = 0,$$

which indicates that $g(\zeta) = 1 \Rightarrow g'(\zeta) = 0$.

In the following, we prove $g(\zeta) \neq 1$.

Suppose ξ_0 is a zero of $g-1$ with multiplicity $m(\geq 2)$, then $g^{(m)}(\xi_0) \neq 0$. Thus there exists a positive number δ, such that

$$g(\zeta) \neq 1,\ g'(\zeta) \neq 0,\ g^{(m)}(\zeta) \neq 0 \tag{3.10}$$

on $D_\delta^o = \{z : 0 < |\zeta - \xi_0| < \delta\}$.

Noting that $g \not\equiv 1$, by Rouché theorem there exist $\xi_{n,j} (j = 1, 2, \ldots, m)$ on $D_{\delta/2} = \{\xi : |\zeta - \xi_0| < \delta/2\}$ such that

$$g_n(\xi_{n,j}) = \frac{f(w_n + z_n + \rho_n \xi_{n,j})}{\alpha(w_n + z_n + \rho_n \xi_{n,j})} = 1\ (j = 1, \ldots, m).$$

That is $f(w_n + z_n + \rho_n\xi_{n,j}) = \alpha(w_n + z_n + \rho_n\xi_{n,j})$ $(j = 1, \ldots, m)$. The fact that $f(z) = \alpha(z) \Rightarrow f'(z) = \alpha(z)$ yields $f'(w_n + z_n + \rho_n\xi_{n,j}) = \alpha(w_n + z_n + \rho_n\xi_{n,j})$.

Let

$$H(\zeta) = f(w_n + z_n + \rho_n\zeta) - \alpha(w_n + z_n + \rho_n\zeta).$$

Thus, $H(\xi_{n,j}) = 0$ $(j = 1, \ldots, m)$. We have

$$\begin{aligned} H'(\xi_{n,j}) &= \rho_n[f'(w_n + z_n + \rho_n\xi_{n,j}) - \alpha'(w_n + z_n + \rho_n\xi_{n,j})] \\ &= \rho_n[\alpha(w_n + z_n + \rho_n\xi_{n,j}) - \alpha'(w_n + z_n + \rho_n\xi_{n,j})] \\ &= \rho_n e^{Q(w_n+z_n+\rho_n\xi_{n,j})}[P - P' - PQ']|_{\zeta=w_n+z_n+\rho_n\xi_{n,j}}. \end{aligned}$$

Since $\alpha \not\equiv \alpha'$, so $P - P' - PQ' \not\equiv 0$. Thus, for n large enough, we have $[P - P' - PQ']|_{\zeta=w_n+z_n+\rho_n\xi_{n,j}} \neq 0$. Therefore $H'(\xi_{n,j}) \neq 0$, which means that $\xi_{n,j}$ is a simple zero of $H(\zeta)$ $(j = 1, \ldots, m)$. That is $\xi_{n,j} \neq \xi_{n,i} (1 \leq i \neq j \leq m)$.

By (3.9), we have

$$\begin{aligned} K_n(\zeta) &= \rho_n \frac{f_n'(w_n + z_n + \rho_n\zeta) - \alpha(w_n + z_n + \rho_n\zeta)}{\alpha(w_n + z_n + \rho_n\zeta)} \\ &= \rho_n[\frac{f'(w_n + z_n + \rho_n\zeta)}{\alpha(w_n + z_n + \rho_n\zeta)} - 1] \to g'(\zeta), \end{aligned} \tag{3.11}$$

and $K_n(\xi_{n,j}) = 0$, $(j = 1, \ldots, m)$. From (3.10) we have

$$\lim_{n\to\infty} \xi_{n,j} = \xi_0\ (j = 1, 2, \ldots, m).$$

Noting that (3.11) and the fact that $K_n(\zeta)$ has m zeros $\xi_{n,j}(j = 1, 2, \ldots, m)$ in $D_{\delta/2}$, we obtain from the Hurwitz's theorem that ξ_0 is a zero of $g'(\zeta)$ with multiplicity m, and thus $g^{(m)}(\xi_0) = 0$. This is a contradiction. Hence $g(\zeta) \neq 1$.

With the Nevanlinna's second fundamental theorem, it is not difficult to deduce a contradiction. Thus, Case 2 cannot occur.

This completes the proof of Theorem 1.2.

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