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GENERALIZED DERIVATIONS WITH ANNIHILATOR CONDITIONS IN PRIME RINGS

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Abstract. Let R be a prime ring, H a generalized derivation of R, L a noncentral Lie ideal of R, and $0 \neq a \in R$. Suppose that $au^sH(u)u^t = 0$ for all $u \in L$, where $s, t \geq 0$ are fixed integers. Then $H = 0$ unless R satisfies S_4, the standard identity in four variables.

Throughout this article, R is always a prime ring with extended centended C, right Utumi quotient ring U, and two-sided Martindale quotient ring Q. The definitions and properties of these objects can be found in [3, Chapter 2]. By S_4 we denote the standard identity in four variables.

By a generalized derivation on R one usually means an additive map $H : R \to R$ such that $H(xy) = H(x)y+xd(y)$ for some derivation d of R. Obviously any derivation is a generalized derivation. Another basic example of generalized derivations is the following: $H(x) = ax + xb$ for $a, b \in R$. In [12] Hvala initiated the study of generalized derivations on prime rings. In [16, Theorem 3] Lee proved the following essential result: every generalized derivation H on a dense left ideal of R can be uniquely extended to U and assume the form $H(x) = bx + d(x)$ for some $b \in U$ and a derivation d on U. In recent years, a number of articles discussed generalized derivations in the context of prime and semiprime rings (e.g., [1, 9, 10, 11, 17, 18, 20]).

In [6] Dhara and Sharma proved that, if $a \in R$ such that $au^sd(u)^nu^t = 0$ for all $u \in L$, a noncommutative Lie ideal of R, where d a derivation of R, $s \geq 0, t \geq 0, n \geq 1$ are fixed integers, then either $a = 0$ or $d = 0$ unless $\mathrm{char}R = 2$ and R satisfies S_4. In [5] Dhara and Filippis proved that, if $u^sH(u)u^t = 0$ for all $u \in L$, where L is a noncommutative Lie ideal of R, H is a generalized derivation of R, and $s, t \geq 0$ are fixed integers, then $H(x) = 0$ for all $x \in R$ unless $\mathrm{char}R = 2$ and R satisfies S_4.

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In the present paper we shall extend the result of Dhara and Filippis to the situation when $au^sH(u)u^t = 0$ for all $u \in L$, where $a \in R$, L a noncentral Lie ideal of R, H a generalized derivation of R and $s, t \geq 0$ are fixed integers. More precisely, our main result is the following:

Theorem 1. *Let R be a prime ring, H a generalized derivation of R, L a noncentral Lie ideal of R, and $0 \neq a \in R$. Suppose that $au^sH(u)u^t = 0$ for all $u \in L$, where $s, t \geq 0$ are fixed integers. Then $H = 0$ unless R satisfies S_4, the standard identity in four variables.*

The following example illustrates the necessity of conditions in Theorem 1.

Example 1. Let $R = M_2(F)$, the ring of all 2×2 matrices algebra over a field F (R satisfies S_4). Let $H : R \to R$ such that $H(x) = e_{22}x$ for all $x \in R$. Note that H is a nonzero generalized derivation of R. It is well-known fact that $[x, y]^2 \in F \cdot I_2$ for all $x, y \in R$. Since every element in $[R, R]$ is a single commutator [2, Theorem], we see that $u^2 \in F \cdot I_2$ for all $u \in [R, R]$. So, $e_{11}u^2H(u) = 0$ for all $u \in [R, R]$.

For the proof of the main result we begin with the following simple result.

Lemma 1. *Let R be a prime ring with extended centroid C and $a, b, c \in R$ with $a \neq 0$. If $a[x_1, x_2]^s(b[x_1, x_2] + [x_1, x_2]c)[x_1, x_2]^t = 0$ for all $x_1, x_2 \in R$, then either R satisfies a nontrivial generalized polynomial identity (GPI) or $b + c = 0$.*

Proof. Suppose that $b \notin C$. We have that

$$a[X_1, X_2]^s(b[X_1, X_2] + [X_1, X_2]c)[X_1, X_2]^t$$

is a nonzero GPI for R as it has a nonzero monomial $a(X_1X_2)^sb(X_1X_2)^{t+1}$. Similarly, if $c \notin C$, we also know that R is a nontrivial GPI ring. Now we assume that $b, c \in C$. Then

$$a[x_1, x_2]^{s+t+1}(b + c) = 0$$

for all $x_1, x_2 \in R$. If $b + c \neq 0$, it is obvious that $a[X_1, X_2]^{s+t+1}(b + c)$ is a nonzero GPI for R. This proves the lemma. □

The following result is crucial to the proof of our main result.

Lemma 2. *Let $R = M_m(F)$, the ring of $m \times m$ matrices algebra over a field F with $m > 2$ and $0 \neq a \in R$ and $b, c \in R$ such that*

$$a[x_1, x_2]^s(b[x_1, x_2] + [x_1, x_2]c)[x_1, x_2]^t = 0$$

for all $x_1, x_2 \in R$, where $s, t \geq 0$ are fixed integers. Then $b + c = 0$.

Proof. Let φ be an inner F-automorphism of R. Since

$$a^\varphi[x_1, x_2]^s(b^\varphi[x_1, x_2] + [x_1, x_2]c^\varphi)[x_1, x_2]^t = 0$$

for all $x_1, x_2 \in R$, we may replace a, b, and c by a^φ, b^φ, and c^φ respectively, to prove that $b^\varphi + c^\varphi = 0$. Let $a = \sum_{i,j=1}^{m} a_{ij}e_{ij}$ where $a_{ij} \in F$. Multiplying

a by some suitable e_{kj} from the left-hand side, we may assume $a = e_{kk} + \sum_{s=k+1}^{m} a_{ks}e_{ks}$. Let φ_i be the inner F-automorphism of R defined by $x^{\varphi_i} = (1+a_{ki}e_{ki})x(1-a_{ki}e_{ki})$ for $k+1 \le i \le m$. Then $a^{\varphi_{k+1}} = e_{kk} + \sum_{s=k+2}^{m} a_{ks}e_{ks}$, $a^{\varphi_{k+1}\varphi_{k+2}} = e_{kk} + \sum_{s=k+3}^{m} a_{ks}e_{ks}$, $\ldots$, $a^{\varphi_{k+1}\varphi_{k+2}\cdots\varphi_m} = e_{kk}$. Replacing a, b, c by $a^{\varphi_{k+1}\varphi_{k+2}\cdots\varphi_m}$, $b^{\varphi_{k+1}\varphi_{k+2}\cdots\varphi_m}$, $c^{\varphi_{k+1}\varphi_{k+2}\cdots\varphi_m}$, respectively, we may write $a = e_{kk}$.

Now putting $x_1 = e_{ki}$, $x_2 = e_{ik}$ for $i \ne k$, we have

$$\begin{aligned} 0 &= e_{kk}[e_{ki}, e_{ik}]^s(b[e_{ki}, e_{ik}] + [e_{ki}, e_{ik}]c)[e_{ki}, e_{ik}]^t \\ &= e_{kk}(e_{kk} + (-1)^s e_{ii})(b(e_{kk} - e_{ii}) + (e_{kk} - e_{ii})c)(e_{kk} + (-1)^t e_{ii}) \\ &= e_{kk}(b+c)e_{kk} + (-1)^t e_{kk}(-b+c)e_{ii} \end{aligned}$$

implying $(b+c)_{kk} = 0$. Next putting $x_1 = e_{ki}$, $x_2 = e_{ik} + e_{ij}$ for $1 \le i, j \le m$ such that k, i, j are mutually different, we have

$$\begin{aligned} 0 &= e_{kk}[e_{ki}, e_{ik} + e_{ij}]^s(b[e_{ki}, e_{ik} + e_{ij}] + [e_{ki}, e_{ik} + e_{ij}]c)[e_{ki}, e_{ik} + e_{ij}]^t \\ &= (e_{kk} + e_{kj})b(e_{kk} + e_{kj} + (-1)^{t+1}e_{ii}) + (e_{kk} + e_{kj})c(e_{kk} + e_{kj} + (-1)^t e_{ii}). \end{aligned}$$

Right multiplying by e_{kk}, we get that $(e_{kk} + e_{kj})(b+c)e_{kk} = 0$, this implies $(b+c)_{kk} + (b+c)_{jk} = 0$. Since $(b+c)_{kk} = 0$, we obtain that $(b+c)_{jk} = 0$ for all $1 \le j \le m$.

Let ψ be the F-automorphism of R defined by $x^{\psi} = (1 + e_{ik})x(1 - e_{ik})$ for all $x \in R$, where $i \ne k$. Then

$$e_{kk}^{\psi}[x_1, x_2]^s(b^{\psi}[x_1, x_2] + [x_1, x_2]c^{\psi})[x_1, x_2]^t = 0$$

for all $x_1, x_2 \in R$. Since $e_{kk}^{\psi} = e_{kk} + e_{ik}$, left multiplying by e_{kk}, we get

$$e_{kk}[x_1, x_2]^s(b^{\psi}[x_1, x_2] + [x_1, x_2]c^{\psi})[x_1, x_2]^t = 0$$

for all $x_1, x_2 \in R$. As above we can obtain that $(b+c)_{jk}^{\psi} = 0$ for all j.

On the other hand, we have

$$\begin{aligned} (b+c)^{\psi} &= b + c + e_{ik}(b+c) - (b+c)e_{ik} - e_{ik}(b+c)e_{ik} \\ &= b + c + \sum_q (b+c)_{kq}e_{iq} - \sum_p (b+c)_{pi}e_{pk} - (b+c)_{ki}e_{ik}. \end{aligned}$$

This implies that $(b+c)_{jk}^{\psi} = (b+c)_{jk} - (b+c)_{ji}$ for $j \ne i$ and

$$(b+c)_{ik}^{\psi} = (b+c)_{ik} + (b+c)_{kk} - (b+c)_{ii} - (b+c)_{ki}.$$

Since $(b+c)_{jk} = (b+c)_{jk}^{\psi} = 0$ for all j, we see that $(b+c)_{ji} = 0$ for all i, j, that is, $b + c = 0$. This proves the result. □

Applying the above two results we can obtain the following:

Lemma 3. *Let R be a prime ring with extended centroid C and $a, b, c \in R$ with $a \ne 0$. If $a[x_1, x_2]^s(b[x_1, x_2] + [x_1, x_2]c)[x_1, x_2]^t = 0$ for all $x_1, x_2 \in R$, then $b + c = 0$ unless $\dim_C RC \le 4$.*

Proof. We assume that $\dim_C RC > 4$. Our goal is to show $b + c = 0$. By assumption R satisfies generalized polynomial identity

$$f(x_1, x_2) = a[x_1, x_2]^s(b[x_1, x_2] + [x_1, x_2]c)[x_1, x_2]^t.$$

Suppose on the contrary that $b+c \neq 0$. In view of Lemma 1 we see that R is a nonzero GPI ring. Since R and U satisfy same generalized polynomial identity (see [4]), U satisfies $f(x_1, x_2)$. In case C is infinite, we have $f(x_1, x_2) = 0$ for all $x_1, x_2 \in U \otimes_C \overline{C}$, where $\overline{C}$ is the algebraic closure of C. Since both U and $U \otimes_C \overline{C}$ are prime and centrally closed [7], we may replace R by U or $U \otimes_C \overline{C}$ according to C finite or infinite. Thus we may assume that R is centrally closed over C (i.e., $RC = R$) which is either finite or algebraically closed and $f(x_1, x_2) = 0$ for all $x_1, x_2 \in R$. By Martindale's theorem [19], R is a primitive ring having nonzero socle H with C as the associated division ring. By Jacobson's density theorem [13, p. 75], R is isomorphic to a dense ring of linear transformations of a vector space V over C, and H consists of the linear transformations in R of finite rank.

If $\dim_C V < \infty$, then $R \cong M_m(C)$ for some $m > 2$. By Lemma 2 we have $b + c = 0$, a contradiction. Now we assume that $\dim_C V = \infty$. It is clear that there exist $h_1, h_2 \in H$ such that $h_1 a \neq 0$ and $bh_2 + ch_2 \neq 0$. Left multiplying $f(x_1, x_2)$ by h_1 we may assume that $a \in H$. By Litoff's theorem [8], there exists idempotent $e \in H$ such that $h_2, a, bh_2, ch_2 \in eRe$ and $eRe \cong M_k(C)$ with $k > 2$. Hence

$$eae[ex_1e, ex_2e]^s(ebe[ex_1e, ex_2e] + [ex_1e, ex_2e]ece)[ex_1e, ex_2e]^t = 0$$

for all $x_1, x_2 \in R$. Since $eae = a \neq 0$, by Lemma 2 we have $ebe + ece = 0$. Thus

$$bh_2 + ch_2 = e(bh_2)e + e(ch_2)e = ebeh_2 + eceh_2 = (ebe + ece)h_2 = 0,$$

a contradiction. □

We are ready to give:

Proof of Theorem 1. We assume that R does not satisfy S_4. Our aim is to show $H = 0$. By a theorem of Lanski and Montgomery [15, Theorem 13] we have $0 \neq [I, R] \subseteq L$, where I is a nonzero ideal of R. Hence we may assume without loss of generality that $L = [I, I]$. Since I and U satisfy the same differential identities [4], we have

$$a[x_1, x_2]^s H([x_1, x_2])[x_1, x_2]^t = 0$$

for all $x_1, x_2 \in U$. By [16, Theorem 3] we may assume that $H(x) = bx + d(x)$ for all $x \in U$, where $b \in U$ and d is a derivation of U. So

$$a[x_1, x_2]^s(b[x_1, x_2] + d([x_1, x_2]))[x_1, x_2]^t = 0$$

for all $x_1, x_2 \in U$. Assume first that d is Q-inner, i.e., there exists $p \in U$ such that $d(x) = [p, x]$ for all $x \in U$. Thus

$$a[x_1, x_2]^s((b + p)[x_1, x_2] - [x_1, x_2]p)[x_1, x_2]^t = 0$$

for all $x_1, x_2 \in U$. By Lemma 3 we have $b = (b + p) - p = 0$ and so

$$a[x_1, x_2]^s d([x_1, x_2])[x_1, x_2]^t = 0$$

for all $x_1, x_2 \in U$. It follows from [6, Theorem 1] that $d = 0$ and so $H = 0$ as desired.

Suppose that d is not Q-inner. Then

$$a[x_1, x_2]^s (b[x_1, x_2] + [d(x_1), x_2] + [x_1, d(x_2)])[x_1, x_2]^t = 0$$

for all $x_1, x_2 \in U$. In view of the powerful Kharchenko's theorem [14] we have

$$a[x_1, x_2]^s (b[x_1, x_2] + [x_3, x_2] + [x_1, x_4])[x_1, x_2]^t = 0$$

for all $x_1, x_2, x_3, x_4 \in U$. Thus, U satisfies its blended component

$$a[x_1, x_2]^s ([x_3, x_2] + [x_1, x_4])[x_1, x_2]^t.$$

In particular, we have

$$a[x_1, x_2]^s d([x_1, x_2])[x_1, x_2]^t = a[x_1, x_2]^s ([d(x_1), x_2] + [x_1, d(x_2)])[x_1, x_2]^t = 0$$

for all $x_1, x_2 \in U$. It follows from [6, Theorem 1] that $d = 0$, a contradiction. $\square$

We conclude this paper with the following.

Conjecture. Let R be a prime ring, $0 \neq a \in R$, H a generalized derivation and L a noncentral Lie ideal of R. Suppose that $au^s H(u)^n u^t = 0$ for all $u \in L$, where $s, t \geq 0$ and $n \geq 1$ are fixed integers. Then $H = 0$ unless R satisfies S_4, the standard identity in four variables.

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