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A NOTE ON THE GENERALIZED BERNSTEIN POLYNOMIALS

A. Bayad, T. Kim, S. H. Lee and D. V. Dolgy

Abstract. We prove two identities for multivariate Bernstein polynomials on simplex, which are considered on a pointwise. In this paper, we study good approximations of Bernstein polynomials for every continuous functions on simplex and the higher dimensional q-analogues of Bernstein polynomials on simplex.

1. Introduction and motivation

Recently many mathematicians study on the theory of multivariate Bernstein polynomials on simplex . This theory has many applications in different areas in mathematics and physics, see [1–17].
Throughout this paper we set $I = [0, 1]$ and $k \in \mathbb{N}$. Taking a k-dimensional simplex Δ_k :

$$\Delta_k = \{\overrightarrow{x} = (x_1, \ldots, x_k) \in I^k : \ x_1 + \ldots + x_k \leq 1\}.$$

As well-known, Bernstein polynomials (1) are the most important and interesting concrete operators on a space of continuous functions(see [15,16]). The purpose of this paper is to study their generalization to k-dimensional simplex.

Definition 1. The n-th degree ordinary Bernstein polynomial $B_{v,n}$: $I \to \mathbb{R}$ is given by

$$B_{v,n}(x) = \binom{n}{v} x^v (1 - x)^{n-v}, \tag{1}$$

$v = 0, 1, \ldots, n$. We extend this to

$$B_{\overrightarrow{v},n} : \Delta_k \to \mathbb{R}$$

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by taking $\vec{v}$ to be a multi-index, $\vec{v}=(v_1,\ldots,v_k)\in\mathbb{N}_0^k$, defining

$$|\vec{v}|:=v_1+\ldots+v_k\in\{0,1,\ldots,n\}$$

and setting

$$B_{\vec{v},n}(\vec{x})=\binom{n}{\vec{v}}\vec{x}^{\vec{v}}(1-|\vec{x}|)^{n-|\vec{v}|}, \tag{2}$$

where

$$\vec{x}=(x_1,\ldots,x_k)\in\Delta_k,\ \vec{x}^{\vec{v}}=\prod_{i=1}^k x_i^{v_i},\ \vec{v}!=v_1!\cdots v_k! \text{ and } \binom{n}{\vec{v}}=\frac{n!}{\vec{v}!(n-|\vec{v}|)!}.$$

For every f defined on Δ_k, we write

$$\mathbb{B}_n(f|\vec{x})=\sum_{|\vec{v}|\le n}f(\vec{v}/n)B_{\vec{v},n}(\vec{x}). \tag{3}$$

Here we have the convergence.

Proposition 1. *If $f:\Delta_k\to\mathbb{R}$ is continuous, then $\mathbb{B}_n(f|.)\to f$ uniformly on Δ_k as $n\to\infty$.*

Proof. The proof of this proposition 1 is quite simple and is based on the partition property of the $B_{\vec{v},n}(\vec{x})$ that

$$\sum_{|\vec{v}|\le n}B_{\vec{v},n}(\vec{x})=1.$$

This proof is similar to that Theorem 1.1.1 in [15, p.5–8]. Since f is uniformly continuous on Δ_k, for given $\epsilon>0$, there exists $\delta>0$ with the property that if $\vec{x}=(x_1,\ldots,x_k)$, $\vec{y}=(y_1,\ldots,y_k)$, and $|x_i-y_i|<\delta$ for all i, then $|f(\vec{x})-f(\vec{y})|<\epsilon$. We define the distance on Δ_k by $d(\vec{x},\vec{y})=\max\{|x_1-y_1|,\ldots,|x_k-y_k|\}$. We suppose that n is sufficiently large that $\frac{1}{4n\delta^2}<\epsilon$. We have

$$|\mathbb{B}_n(f|\vec{x})-f(\vec{x})|\le\sum_{d\left(\frac{\vec{v}}{n},\vec{x}\right)<\delta}|f(\tfrac{\vec{v}}{n})-f(\vec{x})|\,B_{\vec{v},n}(\vec{x})+\sum_{d\left(\frac{\vec{v}}{m},\vec{x}\right)\ge\delta}|f(\tfrac{\vec{v}}{m})-f(\vec{x})|\,B_{\vec{v},n}(\vec{x})$$

where $\vec{v}=(v_1,\ldots,v_k)$, a multi-index. The first sum satisfies

$$\sum_{d\left(\frac{\vec{v}}{n},\vec{x}\right)<\delta}|f(\tfrac{\vec{v}}{n})-f(\vec{x})|\,B_{\vec{v},n}(\vec{x})<\epsilon$$

by uniform continuity of f. Now we study the second sum. Let $M = \max f$. Then

$$\sum_{d\left(\frac{\vec{v}}{m},\vec{x}\right)\geq\delta} |f(\tfrac{\vec{v}}{m}) - f(\vec{x})|\, B_{\vec{v},n}(\vec{x}) \;\leq\; 2M \sum_{d\left(\frac{\vec{v}}{n},x\right)\geq\delta} B_{\vec{v},n}(\vec{x}).$$

We see that

$$\sum_{d\left(\frac{\vec{v}}{n},\vec{x}\right)\geq\delta} B_{\vec{v},n}(\vec{x}) < n\epsilon.$$

Thus $|\mathbb{B}_n(f|\,\vec{x}) - f(\vec{x})| < \epsilon + 2Mn\epsilon$, and we are done. □

The generating functions of the k-dimensional Bernstein polynomials are as follows:

Proposition 2. *For $n, v \in \mathbb{Z}_+$, we have*

$$\sum_{n\geq|\vec{v}|} B_{\vec{v},n}(\vec{x})\frac{t^n}{n!} = \frac{(t\,\vec{x})^{\vec{v}}}{\vec{v}!}e^{t(1-|\vec{x}|)}. \tag{4}$$

Proof. Writing

$$\sum_{n\geq|\vec{v}|} B_{\vec{v},n}(\vec{x})\frac{t^n}{n!} = \sum_{n\geq|\vec{v}|}\binom{n}{\vec{v}}\,\vec{x}^{\,\vec{v}}\,(1-|\,\vec{x}\,|)^{n-|\vec{v}|}\frac{t^n}{n!} \tag{5}$$

$$= \sum_{n\geq|\vec{v}|}\frac{(t\,\vec{x})^{\vec{v}}}{\vec{v}!}\frac{(1-|\,\vec{x}\,|)^{n-|\vec{v}|}t^{n-|\vec{v}|}}{(n-|\,\vec{v}\,|)!} \tag{6}$$

$$= \frac{(t\,\vec{x})^{\vec{v}}}{\vec{v}!}\sum_{m\geq 0}\frac{(1-|\,\vec{x}\,|)^m t^m}{m!}. \tag{7}$$

This yields the equality

$$\sum_{n\geq|\vec{v}|} B_{\vec{v},n}(\vec{x})\frac{t^n}{n!} = \frac{(t\,\vec{x})^{\vec{v}}}{\vec{v}!}e^{t(1-|\vec{x}|)}. \tag{8}$$

□

2. Main results and proofs

This section contains the main results of this paper. The first main result can be state as follows.

Theorem 1. *For $n \in \mathbb{N}, m \in \mathbb{N}_0$ and $\vec{v} \in \mathbb{N}_0^k$ such that $m \leq min\Big(|\ \vec{v}\ |, n\Big)$, we have the following identity*

$$\sum_{\substack{\vec{u} \leq \vec{v} \\ |\vec{u}| \leq m}} \frac{\vec{u}!(m - |\ \vec{u}\ |)!}{m!} B_{\vec{u},m}(\vec{x}) B_{\vec{v}-\vec{u},n-m}(\vec{x}) = B_{\vec{v},n}(\vec{x}), \tag{9}$$

where $\vec{u} \leq \vec{v}$ means that $0 \leq u_i \leq v_i$ for all $i = 1, \cdots, k$.

The formula (9) can be viewed as a pointwise recurrence or orthogonality formula for the Bernstein polynomials in k-dimensional simplex.

Remark 1. For $m = 1$, we obtain from Theorem 1 the following recurrence formula

$$(1 - |\ \vec{x}\ |) B_{\vec{v},n-1}(\vec{x}) + \sum_{\substack{|\vec{u}|=1 \\ \vec{u} \leq \vec{v}}} \vec{x}^{\vec{u}} B_{\vec{v}-\vec{u},n-1}(\vec{x}) = B_{\vec{v},n}(\vec{x}) \tag{10}$$

Proof. We prove this theorem by induction on n and m. For $n = 0, 1$ the statement is trivial. Let $n \geq 2$ and let us take $m = 1$. Then the sum is given by

$$\begin{aligned}
&\sum_{\substack{\vec{u} \leq \vec{v} \\ |\vec{u}| \leq m}} \frac{\vec{u}!(m - |\ \vec{u}\ |)!}{m!} B_{\vec{u},m}(\vec{x}) B_{\vec{v}-\vec{u},n-m}(\vec{x}) \\
&= B_{\vec{0},1}(\vec{x}) B_{\vec{v},n-1}(\vec{x}) + \sum_{\substack{\vec{u} \leq \vec{v} \\ |\vec{u}|=1}} B_{\vec{u},1}(\vec{x}) B_{\vec{v}-\vec{u},n-1}(\vec{x}) \\
&= (1 - |\ \vec{x}\ |) B_{\vec{v},n-1}(\vec{x}) + \sum_{\substack{|\vec{u}|=1 \\ \vec{u} \leq \vec{v}}} \vec{x}^{\vec{u}} B_{\vec{v}-\vec{u},n-1}(\vec{x}).
\end{aligned}$$

By using the following fact $|\ \vec{u}\ | = 1$ if and only if one index of $\vec{u}$ is 1 and all the others are zero, after simple manipulation, we obtain the relation

$$(1-|\vec{x}|)B_{\vec{v},n-1}(\vec{x})+\sum_{\substack{|\vec{u}|=1\\ \vec{u}\leq\vec{v}}}\vec{x}^{\vec{u}}\,B_{\vec{v}-\vec{u},n-1}(\vec{x})=B_{\vec{v},n}(\vec{x}).$$

Then the Theorem 1 is valid for any n and $m=1$. Now we suppose the theorem holds up to $n\geq m\geq 1$. We can write for $n+1$ and $m=1$ the following

$$B_{\vec{v},n+1}(\vec{x})=\sum_{\substack{\vec{u}\leq\vec{v}\\ |\vec{u}|\leq 1}}\frac{\vec{u}!(1-|\vec{u}|)!}{1!}B_{\vec{u},1}(\vec{x})B_{\vec{v}-\vec{u},n}(\vec{x})$$

Then we get

$$B_{\vec{v},n+1}(\vec{x})=\sum_{\substack{\vec{u}+\leq\vec{v}\\ |\vec{u}|\leq 1}}B_{\vec{u},1}(\vec{x})B_{\vec{v}-\vec{u},n}(\vec{x}), \tag{11}$$

by using the recurrence hypothesis, from (11), we obtain

$$\begin{aligned}
B_{\vec{v},n+1}(\vec{x}) &= \sum_{\substack{\vec{u}\leq\vec{v}\\ |\vec{u}|\leq 1}}B_{\vec{u},1}(\vec{x})\sum_{\substack{\vec{u'}\leq\vec{v}-\vec{u}\\ |\vec{u'}|\leq m}}\frac{\vec{u'}!(m-|\vec{u'}|)!}{m!}B_{\vec{u'},m}(\vec{x})B_{\vec{v}-\vec{u}-\vec{u'},n-m}(\vec{x})\\
&= \sum_{\substack{\vec{u}+\vec{u'}\leq\vec{v}\\ |\vec{u}|\leq 1,|\vec{u'}|\leq m}}\frac{\vec{u'}!(m-|\vec{u'}|)!}{m!}B_{\vec{u},1}(\vec{x})B_{\vec{u'},m}(\vec{x})B_{\vec{v}-\vec{u}-\vec{u'},n-m}(\vec{x}).
\end{aligned}$$

Setting $\vec{w}=\vec{u}+\vec{u'}$. From the relation (2), we deduce the identity

$$B_{\vec{v},n+1}(\vec{x})=\sum_{\substack{\vec{w}\leq\vec{v}\\ |\vec{w}|\leq m+1}}\frac{\vec{w}!(m+1-|\vec{w}|)!}{(m+1)!}B_{\vec{w},m+1}(\vec{x})B_{\vec{v}-\vec{w},n-m}(\vec{x}).$$

This completes the proof of the theorem. □

For every $j,m,1\leq j\leq k,m\geq 1$, we define the affine transformations $T_{j,m}$ by

$$T_{j,m}(\vec{x})=(x_1,\cdots,x_{j-1},m-|\vec{x}|,x_{j+1},\cdots,x_k). \tag{12}$$

and for every $\sigma\in\mathcal{S}_k$ permutation of the set $\{1,\cdots,k\}$ we put

$$\sigma(\vec{x})=(x_{\sigma(1)},\cdots,(x_{\sigma(k)}). \tag{13}$$

We state now the second main result of this paper.

Theorem 2. *For $n \in \mathbb{N}$ and $\vec{v} \in \mathbb{N}_0^k$, we have the following identities*

$$B_{\vec{v},n}(T_{j,1}(\vec{x})) = B_{T_{j,n}(\vec{v}),n}(\vec{x}), \tag{14}$$

and

$$B_{\vec{v},n}(\sigma(\vec{x})) = B_{\sigma^{-1}(\vec{v}),n}(\vec{x}). \tag{15}$$

The relation (14) is a multivariate symmetry formula for the Bernstein polynomilas.

Remark 2. Taking $j = 1$ and $\sigma = (12)$ we get, from the Theorem 2, the symmetries relations

$$B_{\vec{v},n}(1 - |\vec{x}|, x_2, \cdots, x_k) = B_{(n-|\vec{v}|, v_2, \cdots, v_k),n}(\vec{x}), \tag{16}$$

and

$$B_{\vec{v},n}(x_2, x_1, x_3, \cdots, x_k) = B_{(v_2, v_1, x_3, \cdots, v_k),n}(\vec{x}). \tag{17}$$

Proof. By using the equalities (2) and (12) we have

$$\begin{aligned} B_{\vec{v},n}(T_{j,1}(\vec{x})) &= B_{\vec{v},n}(x_1, \cdots, x_{j-1}, \cdots, 1 - |\vec{x}|, x_{j+1}, \cdots, x_k) \\ &= \binom{n}{\vec{v}} x_1^{v_1} \cdots x_{j-1}^{v_{j-1}} (1 - |\vec{x}|)^{v_j} x_{j+1} \cdots x_k^{v_k} x_j^{1-|\vec{v}|} \\ &= \binom{n}{\vec{v}} x_1^{v_1} \cdots x_{j-1}^{v_{j-1}} x_j^{n-|\vec{v}|} x_{j+1} \cdots x_k^{v_k} (1 - |\vec{x}|)^{v_j}. \end{aligned}$$

This implies, by using the relation (12), the identity

$$B_{\vec{v},n}(T_{j,1}(\vec{x})) = B_{T_{j,n}(\vec{v}),n}(\vec{x}).$$

The equality (15) of the Theorem 2 can be obtained in a similar way of (14). □

3. q-extension of Bernstein polynomials on simplex

When one talks of q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, then we always assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we usually assume that $|1 - q|_p < 1$. Here, the symbol $|\cdot|_p$ stands for the p-adic absolute value on $\mathbb{C}_p$ with $|p|_p \leq 1/p$. For each x, the q-basic numbers are defined by

$$[x]_q = \frac{1 - q^x}{1 - q}.$$

We extend this by

$$[\vec{x}]_q = ([x_1]_q, \ldots, [x_k]_q)$$

and the q-extension of Bernstein polynomials on Δ_k is defined by

$$B_{\vec{v},n}(\vec{x}|\, q) = \binom{n}{\vec{v}} [\vec{x}^{\,\vec{v}}]_q [1 - |\,\vec{x}\,|]_q^{n-|\vec{v}|}. \tag{18}$$

Here again we have the q-extensions of Theorem 1 and Theorem 2.

Theorem 3. *For $n \in \mathbb{N}, m \in \mathbb{N}_0$ and $\vec{v} \in \mathbb{N}_0^k$ such that $m \leq min\Big(|\,\vec{v}\,|, n\Big)$. Then we have the following identity*

$$\sum_{\substack{\vec{u} \leq \vec{v} \\ |\vec{u}| \leq m}} \frac{\vec{u}!(m - |\,\vec{u}\,|)!}{m!} B_{\vec{u},m}(\vec{x}\,|q) B_{\vec{v}-\vec{u},n-m}(\vec{x}\,|q) = B_{\vec{v},n}(\vec{x}\,|q),$$

where $\vec{u} \leq \vec{v}$ means that $0 \leq u_i \leq v_i$ for all $i = 1, \cdots, k$.

Theorem 4. *For $n \in \mathbb{N}$ and $\vec{v} \in \mathbb{N}_0^k$, we have the following identities*

$$B_{\vec{v},n}(T_{j,1}(\vec{x})|q) = B_{T_{j,n}(\vec{v}),n}(\vec{x}\,|q), \tag{19}$$

and

$$B_{\vec{v},n}(\sigma(\vec{x})|q) = B_{\sigma^{-1}(\vec{v}),n}(\vec{x}\,|q). \tag{20}$$

The proofs of these theorems are quite similar to those of Theorems 1 and 2. Then we omit them.

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A. Bayad
Département de Mathématiques, Université d'Evry Val d'Essonne,
Bd. F. Mitterrand, 91025 Evry Cedex, France.
E-mail: abayad@maths.univ-evry.fr

T. Kim
Division of General Education, Kwangwoon University,
Seoul 139-701, Korea.
E-mail: tkkim@kw.ac.kr

S. H. Lee
Division of General Education, Kwangwoon University,

Seoul 139-701, Korea.
E-mail: dgekw2011@gmail.com

D. V. Dolgy
Institute of Mathematics and Computer Sciences , Far Eastern National University,
Vladivostok 690060, Russia.
E-mail: $d\_dol@mail.ru$