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INTEGRALLY CLOSED MODULES

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Abstract. In this paper, we find some properties on integrally closed modules.

1. Introduction

Throughout the paper, all rings are commutative rings with identity and all modules are unitary. Let M be an R-module and S the set of nonzero divisors of R and R_S the total quotient ring of R. Put $T = T_M = \{t \in S | tm = 0 \text{ for some } m \in M \text{ implies } m = 0\}$. Then we can easily show that T is a multiplicatively closed subset of S and $1 \in T$ and if M is torsion free then $T = S$.

M is called a *multiplication module* if every submodule N of M has the form IM for some ideal I of R.

M. Alkan , B. Sarac and Y.Tyras([1]) introduced the concept of integral closedness for modules, a generalization of the concept of integral closedness for a ring.

In this paper we will find out some properties of integrally closed modules. Specially we prove Theorem 2.12,Theorem 2.13 and Theorem 2.14.

2. Integral Closedness of Modules

Proposition 2.1. *Let M be an R-module , N a submodule of M and R_T the localization of R at T in the usual sense. For $\frac{r}{t} \in R_T$ and $n \in N$, let $\frac{r}{t}n \in M$ if there exists $m \in M$ such that $rn = tm$.Then this is a well defined operation.*

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Proof. Let $\frac{r_1}{t_1} = \frac{r_2}{t_2} \in R_T$ and $n \in N$ such that $r_1 n = t_1 m_1$ and $r_2 n = t_2 m_2$ for some $m_1, m_2 \in M$. Then there exists $s \in T$ such that $sr_1 t_2 = sr_2 t_1$. Hence $st_2 t_1 m_1 = st_2 r_1 n = sr_1 t_2 n = sr_2 t_1 n = st_1 r_2 n = st_1 t_2 m_2$. Since $st_1 t_2 \in T$ and $st_1 t_2(m_1 - m_2) = 0$, $m_1 = m_2$. □

Proposition 2.2. *R is an integrally closed ring if and only if $r_n y^n + \cdots + r_1 y + r_0 = 0$ for some positive integer $n, r_i \in R$ and $y \in R_S$, then $r_n y \in R$.*

Proof. Suppose that R is integrally closed and $r_n y^n + r_{n-1} y^{n-1} + \cdots + r_1 y + r_0 = 0$ for $y \in R_S$ and $r_i \in R$. Then $r_n^{n-1}(r_n y^n + r_{n-1} y^{n-1} + \cdots + r_1 y + r_0) = 0$ and $(r_n y)^n + r_{n-1}(r_n y)^{n-1} + r_{n-2} r_n (r_n y)^{n-2} + \cdots + r_1 r_n^{n-2}(r_n y) + r_0 r_n^{n-1} = 0$. Thus $r_n y \in R_S$ is integral over R and hence $r_n y \in R$. Conversely, Let $y \in R'$, the integral closure of R. Then there are some $r_i \in R$ such that $y^n + r_{n-1} y^{n-1} + \cdots + r_1 y + r_0 = 0$. Put $r_n = 1$. Then by our assumption $r_n y = 1y = y \in R$ and $R' = R$. So R is an integrally closed ring. □

Observations from the above propositons allow us to define the concept of integrally closed modules which are equivalent to the concept of integrally closedness for the rings when they are considered as module over themselves. Alkan([1]) gives the following definition:

Definition 2.3. An R-module M is said to be *integrallyclosed* whenever $y^n m_n + \cdots + y m_1 + m_0 = 0$ for some $y \in R_T$ and $m_i \in M$, then $y m_n \in M$.

Example 2.4. Let $R = Z, M = Q/Z$. Then we know easily that M is an integrally closed R-module.

Note that $S = Z - \{0\}$ and $T = \{s \in S | sm = 0$ for some $m \in M$ implies $m = 0\}$. If $s \in T$ and $s \neq 1$ then there exists $t \in Z$ such that $(s,t) = 1, \frac{t}{s} \in Q - Z$ and $s(\frac{t}{s} + Z) = t + Z = Z$. Then we get $\frac{t}{s} \in Z$ and it is impossible. Hence $T = \{1\}$ and $R_T = \{\frac{z}{1} | z \in Z\}$. Assume that $(\frac{z}{1})^n (q_n + Z) + \cdots + (\frac{z}{1})(q_1 + Z) + (q_0 + Z) = 0$ for some $q_i \in Q, \frac{z}{1} \in R_T$ and $q_i + Z \in Q/Z$. Clearly there exists $zq_n + Z \in Q/Z$ such that $1 \cdot (zq_n + Z) = z \cdot (q_n + Z)$ and so $(\frac{z}{1})(q_n + Z) \in Q/Z$. By Proposition 2.2, $M = Q/Z$ is an integrally closed Z-module.

Proposition 2.5. *Let M, M' be R-modules and $\phi : M \to M'$ an R-isomorphism. If M is an integrally closed module then so is M'.*

Proof. Suppose that $y^n m'_n + \cdots + y m'_1 + m'_0 = 0$ for some $y = \frac{r}{t} \in R_T, m'_i \in M'$. Then there exists $m_i \in M$ such that $\phi(m_i) = m'_i$. Then $(\frac{r}{t})^n \phi(m_n) + \cdots + (\frac{r}{t})\phi(m_1) + \phi(m_0) = 0$. Since $\frac{1}{t^n} \cdot \phi(r^n m_n + r^{n-1} t m_{n-1} +$

$\cdots + rt^{n-1}m_1 + t^n m_0) = 0$ and ϕ is an isomorphism, $r^n m_n + r^{n-1}tm_{n-1} + \cdots + rt^{n-1}m_1 + t^n m_0 = 0$. Again, $(\frac{r}{t})^n m_n + \cdots + (\frac{r}{t})m_1 + m_0 = 0$. Since M is integrally closed, $\frac{r}{t}m_n \in M$ and hence there exists $m \in M$ such that $rm_n = tm$. Therefore $r\phi(m_n) = t\phi(m)$ and $\frac{r}{t}\phi(m_n) \in \phi(M) = M'$ i.e., $ym'_n = \frac{r}{t}m'_n \in M'$ □

Proposition 2.6. *Let $\{M_i\}_{i\in\Lambda}$ be a collection of integrally closed submodules of an R-module M. Then $\cap M_i (i \in \Lambda)$ is an integrally closed submodule of M.*

Proof. Let $K = \cap M_i (i \in \Lambda)$ and let $(\frac{r}{t})^n k_n + \cdots + (\frac{r}{t})k_1 + k_0 = 0$ for some $\frac{r}{t} \in R_T$ and $k_i \in K$. Then for any $\lambda \in \Lambda, k_i \in M_\lambda$ and since M_λ is integrally closed, $(\frac{r}{t})k_n \in M_\lambda$. Hence $(\frac{r}{t})k_n \in K$. □

Proposition 2.7. *If M_1 and M_2 are integrally closed R-modules,then so is $M = M_1 \oplus M_2$.*

Proof. Let $(\frac{r}{t})^n m_n + (\frac{r}{t})^{n-1}m_{n-1} + \cdots + (\frac{r}{t})m_1 + m_0 = 0$ for some $t \in T, r \in R$ and $m_i \in M$. Put $m_i = m_1^i + m_2^i, m_1^i \in M_1$ and $m_2^i \in M_2$. Then $r^n(m_1^n + m_2^n) + tr^{n-1}(m_1^{n-1} + m_2^{n-1}) + \cdots + t^{n-1}r(m_1^1 + m_2^1) + t^n(m_1^0 + m_2^0) = 0$. Hence $r^n m_1^n + tr^{n-1}m_1^{n-1} + \cdots + t^{n-1}rm_1^1 + t^n m_1^0 = -\{r^n m_2^n + tr^{n-1}m_2^{n-1} + \cdots + t^n m_2^0\} \in M_1 \cap M_2 = \{0\}$. Therefore $r^n m_1^n + tr^{n-1}m_1^{n-1} + \cdots + t^n m_1^0 = 0$ and so, $(\frac{r}{t})^n m_1^n + (\frac{r}{t})^{n-1}m_1^{n-1} + \cdots + (\frac{r}{t})m_1^1 + m_1^0 = 0$. Since M_1 is an integrally closed R-module, $(\frac{r}{t})m_1^n \in M_1$. Similarly $(\frac{r}{t})m_2^n \in M_2$ and $(\frac{r}{t})(m_1^n + m_2^n) = (\frac{r}{t})m_n \in M$. i.e., M is integrally closed. □

The converse of the above proposition is true if M_i is torsion free module over a domain

Proposition 2.8. *Let M_1 and M_2 be torsion free modules over a domain R. If $M = M_1 \oplus M_2$ is integrally closed, then M_1 and M_2 are integrally closed.*

Proof. Suppose that $(\frac{r}{t})^n m_1^n + \cdots (\frac{r}{t})m_1^1 + m_1^0 = 0$ for $m_1^i \in M_1, r \in R$ and $t \in T$ (Note that $T = S$ since M_i is torsion free). Then $(\frac{r}{t})^n(m_1^n + 0_2) + \cdots + (\frac{r}{t})(m_1^1 + 0_2) + (m_1^0 + 0_2) = 0$ where 0_2 is a zero element in M_2. Since $M_1 \oplus M_2$ is integrally closed, $(\frac{r}{t})(m_1^n + 0_2) \in M_1 \oplus M_2$ and hence $r(m_1^n + 0_2) = t(m'_1 + m'_2)$ for some $m'_1 \in M_1$ and $m'_2 \in M_2$. So $rm_1^n - tm'_1 = tm'_2 \in M_1 \cap M_2 = \{0\}$ and $tm'_2 = 0$. Since M_2 is torsion free, $m'_2 = 0$ and hence $(\frac{r}{t})m_1^n = m'_1 + m'_2 = m'_1 \in M_1$. Thus M_1 is integrally closed. Similarly we know that M_2 is integrally closed. □

Corollary 2.9. *Let $\{M_i\}_{i\in\Lambda}$ be a finite collection of torsion free modules over a domain R, then the direct sum $M = \oplus_{i\in\Lambda} M_i$ is a integrally closed module if and only if each of M_i is an integrally closed module.*

Corollary 2.10. *Let $0 \to M_1 \to M \to M_2 \to 0$ be a split exact sequence of torsion free modules over a domain R, then M is an integrally closed module if and only if M_1 and M_2 are integrally closed modules.*

Theorem 2.11. *If R is an integral domain and M a faithful multiplication R-module , then M is finitely generated and torsion free*

Proof. It follows from Theorem 3.1 of [4] and Lemma 4.1 of [3] □

Theorem 2.12. *If M is a faithful multiplication module over an integrally closed domain R and if $x \in R_S$ such that $xM \subseteq M$, then $x \in R$.*

Proof. By Theorem 2.11, M is finitely generated and torsion free. Let $x = \frac{r}{t}$ and $M = Rm_1 + \cdots + Rm_s$. Suppose that $\frac{r}{t}M \subseteq M$. Then for each $i(i = 1, \cdots, s)$, there exists $a_{ij} \in R$ such that $rm_i = a_{i1}m_1 + \cdots + a_{is}m_s$. Hence,

$(r - a_{11})m_1 - a_{12}m_2 - \cdots - a_{1s}m_s = 0$

$\vdots$

$-a_{s1}m_1 - a_{s2}m_2 - \cdots + (r - a_{ss})m_s = 0$

Let $b_{ij} = \delta_{ij} \cdot r - a_{ij}$, a matrix $B = (b_{ij})$ and $d = detB$ where, δ_{ij} is a kronecker symbol. Let B_{ij} be a cofactor of b_{ij} in a matrix B. Now $0 = \Sigma_i B_{ij} \cdot 0 = \Sigma_i B_{ij}(\Sigma_k b_{ik} m_k) = \Sigma_i B_{ij}(b_{ij} m_j + \Sigma_{k\neq j} b_{ik} m_k) = \Sigma_i B_{ij} b_{ij} m_j + \Sigma B_{ij} b_{i1} m_1 + \cdots + \Sigma_i B_{ij} b_{ij-1} m_{j-1} + \Sigma_i B_{ij} b_{ij+1} m_{ij+1} + \cdots + \Sigma_i B_{ij} b_{is} m_s = dm_j$ because $d = detB = \Sigma_i B_{ij} b_{ij}$ and $0 = \Sigma_i B_{ij} b_{ik}$ if $j \neq k$.

On the other hand, $d = r^s + \alpha_{s-1} r^{s-1} + \cdots + \alpha_1 r + \alpha_0$ for some $\alpha_i \in R$ and we know that $dm_i = 0$ for $i = 1, \cdots, s$. Since M is torsion free, $d = 0$ and so $0 = (\frac{r}{t})^s + \alpha_{s-1}(\frac{r}{t})^{s-1} + \cdots + \alpha_1(\frac{r}{t}) + \alpha_0$ and $x = \frac{r}{t}$ is integral over R and hence $x \in R$ because R is integrally closed.

□

Theorem 2.13. *Every injective module over an integral domain is integrally closed*

Proof. Let M be an injective module over an integral domain R.For any $m \in M$ and any $0 \neq s$ in R, consider a short exact sequence of R-modules

$0 \longrightarrow Rs \longrightarrow R$

where $Rs \longrightarrow R$ is an inclusion. Now we define $f : Rs \longrightarrow M$ by $f(rs) = rm$. Then f is a well defined R-module homomorphism. Since M is injective, there exists an R-module homomorphism $g : R \longrightarrow M$ such that $g|_{Rs} = f$. Hence $m = f(s) = g(s) = g(s \cdot 1) = sg(1) \in sM$ and so $M = sM$ for every nonzero element s in R. Now, let $y^n m_n + \cdots + ym_1 + m_0 = 0$ for some $y \in R_T$ and $m_i \in M$. Since $y \in R_T$, $y = \frac{r}{t}$ for some $r \in R$ and $t \in T = T_M$. If $r = 0$ then clearly $ym_n = \frac{0}{t}m_n \in M$ since $0_R m_n = t0_M$. If $r \neq 0$, then by above observation $rM = M$. So $yM = \frac{r}{t}M = \frac{1}{t}(rM) = \frac{1}{t}M$ and hence $M = t(yM) = y(tM) = yM$ since $t \in T = T_M$. Therefore $ym_n \in M$ and M is integrally closed. □

Compare the following theorem with Proposition 13.29 of [5].

Theorem 2.14. *Let R be an integral domain and M a faithful multiplication R-module. Then the following statements are equivalent;*

(1) *M is integrally closed.*
(2) *$M_{\mathcal{P}}$ is integrally closed for each prime ideal $\mathcal{P}$ of R.*
(3) *$M_{\mathcal{M}}$ is integrally closed for each maximal ideal $\mathcal{M}$ of R.*

Proof. (1)⇒ (2)

Let $y^n \frac{m_n}{t_n} + \cdots + y\frac{m_1}{t_1} + \frac{m_0}{t_0} = 0$ for some positive integer $n, y(= \frac{r}{s}) \in R_T$ and $\frac{m_i}{t_i} \in M_{\mathcal{P}}$. Put $\breve{t_i} = t_0 \cdots t_{i-1}t_{i+1} \cdot t_n$, $t = \prod_{i=0}^{n} t_i$. Then $t \notin \mathcal{P}$

$y^n(\breve{t_n}m_n) + \cdots + y(\breve{t_1}m_1) + (\breve{t_0}m_0) = 0$. Since M is integrally closed, $y\breve{t_n}m_n \in M$ and $r\breve{t_n}m_n = sm'$ for some $m' \in M$. Thus $t_n r\breve{t_n}m_n = t_n sm'$, $rtm_n = t_n sm'$. So $r \cdot \frac{m_n}{t_n} = s \cdot \frac{m'}{t}$. Hence $y\frac{m_n}{t_n} \in M_{\mathcal{P}}$.

(2)⇒ (3) is clear.

(3)⇒ (1). □

By theorem 2.11, M is torsion free and hence $M \subseteq M_{\mathcal{M}}$ for each maximal ideal $\mathcal{M}$ of R. Thus $M \subseteq K = \bigcap_{\mathcal{M}} M_{\mathcal{M}}$,where the intersection runs over all maximal ideals $\mathcal{M}$ of R. Suppose that $K \neq M$.Then there exists $\frac{m}{r} \in K - M$. Now we put $\mathcal{A} = \{k \in R | km = rn \text{ for some } n \in M\}$. Then clearly $\mathcal{A}$ is a proper ideal of R. So there exists a maximal ideal $\mathcal{N}$ containing $\mathcal{A}$. Since $\frac{m}{r} \in M_{\mathcal{N}}$, there exist $m' \in M$ and $r' \notin \mathcal{N}$ such that $\frac{m}{r} = \frac{m'}{r'}$. Hence there exists $s \notin \mathcal{N}$ such that $s(r'm - rm') = 0$. Since M is torsion free, $r'm = rm'$ and $r' \in \mathcal{A} \subseteq \mathcal{N}$. We get a contradiction and $K = M$. For any maximal ideal $\mathcal{M}$ of R, $M_{\mathcal{M}}$ is an R- submodule of M_0. Therefore M is integrally closed by Proposition 2.6. Note that we regard both $M_{\mathcal{P}}$ and $M_{\mathcal{M}}$ as an R-modules in this theorem.

References

[1] M.Alkan,B.Sarac,Y.Tiras., *Dedekind Modules*, Comm.in Algebra **vol 33** (2005), 1617-1626.
[2] M.M.Ali., *Invertibility of Multiplication Modules II*, New Zealand J. of Math. **vol 39** (2009), 45-64.
[3] Z.E.Bast and P.F.Smith, *Multiplication Modules*, Comm.in Algebra **vol 16(4)** (1988), 755-779.
[4] Y.H.Cho and Y.H.Kim., *On Multiplication Modules(V)*, Honam Math.J **vol 30(2)** (2008), 363-368.
[5] R.Y.Sharp., *Steps in Commutative Algebra*, Cambridge University Press (1990).

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