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COMPACTNESS OF A SUBSPACE OF THE ZARISKI TOPOLOGY ON SPEC(D)

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Abstract. Let D be an integral domain, Spec(D) the set of prime ideals of D, and X a subspace of the Zariski topology on Spec(D). We show that X is compact if and only if given any ideal I of D with $I \nsubseteq P$ for all $P \in X$, there exists a finitely generated ideal $J \subseteq I$ such that $J \nsubseteq P$ for all $P \in X$. We also prove that if $D = \cap_{P \in X} D_P$ and if $*$ is the star-operation on D induced by X, then X is compact if and only if $*_f$-Max$(D) \subseteq X$. As a corollary, we have that t-Max(D) is compact and that $\mathcal{P}(D) = \{P \in \text{Spec}(D) | P$ is minimal over $(a : b)$ for some $a, b \in D\}$ is compact if and only if t-Max$(D) \subseteq \mathcal{P}(D)$.

1. Introduction

Let D be an integral domain, and let Spec(D) be the set of prime ideals of D. For each subset E of D, let $V(E)$ denote the set of all prime ideals of D which contain E. Then the sets $V(E)$ satisfy the axioms for the closed sets in a topological space. The resulting topology is called the *Zariski topology* on Spec(D). For each $f \in D$, let $X_f = \text{Spec}(D) \setminus V(\{f\})$; then $X_f = \{P \in \text{Spec}(D) | f \notin P\}$ and X_f is an open subset of Spec(D). Obviously, $\{X_f | f \in D\}$ forms a basis of open sets for Spec(D). Note that if X is a nonempty set of prime ideals of D, then the collection $\{X_f \cap X | f \in D\}$ forms a basis of open sets for the subspace topology X of Spec(D). Recall that a topological space T is said to be *compact* if every open covering of T contains a finite subcollection that also covers T.

If Λ is a nonempty set of elements of D such that $\text{Spec}(D) = \cup_{f \in \Lambda} X_f$, then $(\{f | f \in \Lambda\}) = D$, and hence $1 = \sum_{i=1}^n f_i g_i$ for some $f_i \in \Lambda$ and

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$g_i \in D$. Thus $\mathrm{Spec}(D) = X_{f_1} \cup \cdots \cup X_{f_n}$, and this implies that $\mathrm{Spec}(D)$ is a compact topological space (cf. [2, page 12]). Let X be a subspace topology of the Zariski topology on $\mathrm{Spec}(D)$. In this paper, we show that X is compact if and only if, given any ideal I of D with $I \nsubseteq P$ for all $P \in X$, there exists a finitely generated ideal $J \subseteq I$ such that $J \nsubseteq P$ for all $P \in X$. As a corollary, we have that if $*$ is a star-operation on D, then $*_f$-$\mathrm{Max}(D)$ is compact. (Definitions related to star-operations will be reviewed at the end of this section.) We also prove that if $D = \cap_{P \in X} D_P$ and if $*$ is the star-operation on D induced by X, then X is compact if and only if $*_f$-$\mathrm{Max}(D) \subseteq X$. Let $\mathcal{P}(D) = \{P \in \mathrm{Spec}(D) | P$ is minimal over $(a:b)$ for some $a, b \in D\}$ and x be an indeterminate over D. Also, we prove that $\mathcal{P}(D)$ is compact if and only if t-$\mathrm{Max}(D) \subseteq \mathcal{P}(D)$ and that X is compact if and only if $\{P[x] | P \in X\}$ is compact.

Let K be the quotient field of an integral domain D, and let $\mathbf{F}(D)$ be the set of nonzero fractional ideals of D. A star-operation $*$ on D is a mapping $I \mapsto I^*$ from $\mathbf{F}(D)$ into $\mathbf{F}(D)$ such that (i) $(aD)^* = aD$ and $(aI)^* = aI^*$, (ii) $I \subseteq I^*$, and if $I \subseteq J$, then $I^* \subseteq J^*$, and (iii) $(I^*)^* = I^*$ for all $0 \neq a \in K$ and all $I, J \in \mathbf{F}(D)$. An $I \in \mathbf{F}(D)$ is called a $*$-*ideal* if $I^* = I$. For $I \in \mathbf{F}(D)$, let $I^{*_f} = \cup J^*$, where J ranges over all nonzero finitely generated subideals of I. It is well known that $*_f$ is also a star-operation on D. A star-operation is said to be of *finite type* if $* = *_f$. Let $*$-$\mathrm{Max}(D)$ denote the set of $*$-ideals of D maximal among proper integral $*$-ideals of D. It is well known that $*_f$-$\mathrm{Max}(D) \neq \emptyset$ if D is not a field; each proper integral $*_f$-ideal is contained in a maximal $*_f$-ideal; a maximal $*_f$-ideal is a prime ideal; each prime ideal minimal over a $*_f$-ideal is a $*_f$-ideal; and $D = \cap_{P \in *_f\text{-}\mathrm{Max}(D)} D_P$. Let X be a nonempty set of prime ideals of D such that $D = \cap_{P \in X} D_P$. For each $I \in \mathbf{F}(D)$, define $I^* = \cap_{P \in X} ID_P$; then $*$ is a star-operation on D [1, Theorem 1]. We call $*$ the star-operation induced by X.

The most well-known star-operations are the v-, t-, and d-operations. The v-operation is defined by $I_v = (I^{-1})^{-1}$, where $I^{-1} = \{a \in K | aI \subseteq D\}$, the t-operation is defined by $t = v_f$, and the d-operation is the identity function on $\mathbf{F}(D)$, i.e., $I_d = I$ for all $I \in \mathbf{F}(D)$. It is well known that $I = I_d \subseteq I^{*_f} \subseteq I_t \subseteq I_v$ for all $I \in \mathbf{F}(D)$. For more on star-operations, see [3, §32 and §34].

2. Main Results

Throughout D is an integral domain. Let $\mathcal{P}(D) = \{P \in \mathrm{Spec}(D) | P$ is minimal over $(a:b)$ for some $a, b \in D\}$. In [6, Lemma 3.1], Papick showed that $\mathcal{P}(D)$ is compact if and only if, given any ideal I of D with $I \nsubseteq P$ for all $P \in \mathcal{P}(D)$, there exists a finitely generated ideal $J \subseteq I$ such that $J \nsubseteq P$ for all $P \in \mathcal{P}(D)$.

We first generalize Papick's result to an arbitrary nonempty set of prime ideals of D, which is the main result of this paper.

Theorem 1. *Let $\emptyset \neq X \subseteq Spec(D)$. Then X is compact if and only if, given any ideal I of D with $I \nsubseteq P$ for all $P \in X$, there exists a finitely generated ideal $J \subseteq I$ such that $J \nsubseteq P$ for all $P \in X$.*

Proof. ($\Rightarrow$) For each $P \in X$, choose $a_p \in I \setminus P$, and note that

$$X = \cup_{P \in X}(X_{a_p} \cap X),$$

where $X_{a_p} = \{P' \in \mathrm{Spec}(D) | a_p \notin P'\}$. Note also that each $X_{a_p} \cap X$ is an open set in X; so by assumption, there are $a_{p_1}, \ldots, a_{p_k}$ such that

$$X = (X_{a_{p_1}} \cap X) \cup \cdots \cup (X_{a_{p_k}} \cap X).$$

Let $J = (a_{p_1}, \ldots, a_{p_k})$; then obviously, $J \subseteq I$ and $J \nsubseteq P$ for all $P \in X$.

($\Leftarrow$) It suffices to show that any open cover of X consisting of basic open sets has a finite subcover. Suppose that $X = \cup_{\alpha \in \Lambda}(X_{a_\alpha} \cap X)$, where $\{X_{a_\alpha} | \alpha \in \Lambda\}$ is a family of basic open sets for $\mathrm{Spec}(D)$, and let $I = (\{a_\alpha | \alpha \in \Lambda\})$. Then $I \nsubseteq P$ for all $P \in X$, and hence there are some $\alpha_1, \ldots, \alpha_s \in \Lambda$ such that $J = (a_{\alpha_1}, \ldots, a_{\alpha_s}) \nsubseteq P$ for all $P \in X$ by assumption. Thus $X = (X_{a_{\alpha_1}} \cap X) \cup \cdots \cup (X_{a_{\alpha_s}} \cap X)$. □

Corollary 2. *Let X be a nonempty set of prime ideals of D such that (i) there are no containment relations among distinct members of X and (ii) each prime ideal of D contained in $\cup_{P \in X} P$ is contained in some $P \in X$. Then X is compact. In particular, the set of maximal ideals of D is compact.*

Proof. Let I be an ideal of D such that $I \nsubseteq P$ for all $P \in X$. Let $S = D \setminus \cup_{P \in X} P$, and note that $\{PD_S | P \in X\}$ is the set of maximal ideals of D_S [3, Proposition 4.8]. Since $I \nsubseteq P$, we have $ID_S \nsubseteq PD_S$ for all $P \in X$; so $ID_S = D_S$. Hence $1 = \sum_{i=1}^n f_i \frac{d_i}{s_i}$ for some $f_i \in I$, $d_i \in D$, and $s_i \in S$. So if we set $s = s_1 \cdots s_n$, then $s \in I \setminus \cup_{P \in X} P$, and hence $sD \subseteq I$ with $sD \nsubseteq P$ for all $P \in X$. Thus X is compact by Theorem 1. □

Corollary 3. *If $*$ is a star-operation on D, then $*_f$-Max(D) is compact. In particular, t-Max(D) is compact.*

Proof. Let I be an ideal of D with $I \nsubseteq P$ for all $P \in *_f\text{-Max}(D)$. Then $I^{*_f} = D$, and since $*_f$ is of finite type, there exists a finitely generated ideal $J \subseteq I$ with $J^{*_f} = D$. Again, since $*_f$ is of finite type, $J \nsubseteq P$ for each $P \in *_f\text{-Max}(D)$. Thus $*_f\text{-Max}(D)$ is compact by Theorem 1. □

Tang proved that, for a finitely generated ideal I of D, $I \subseteq P$ for some $P \in \mathcal{P}(D)$ if and only if $I_v \subsetneq D$ [7, Theorem E]. This was improved by Zafrullah [8, Theorem 1] as follows: *Let $\{M_i\}_{i\in\Lambda}$ be a set of prime ideals of D such that $D = \cap D_{M_i}$. If I is an ideal of D with $I_v \subsetneq D$, then I_v, and hence I, is contained in at least one M_i.*

An integral domain D is a *P-domain* if D_P is a valuation domain for each $P \in \mathcal{P}(D)$, while D is a *Prüfer v-multiplication domain* (PvMD) if each nonzero finitely generated ideal I of D is t-invertible, i.e., $(II^{-1})_t = D$. It is well known that D is a PvMD if and only if D_P is a valuation domain for each $P \in t\text{-Max}(D)$; so a PvMD is a P-domain. Let D be a P-domain that is not a PvMD [5, Example 2.1]. Then there is a maximal t-ideal Q of D such that $Q \notin \mathcal{P}(D)$; in particular, $Q \nsubseteq P$ for each $P \in \mathcal{P}(D)$. Thus I being finitely generated is necessary for [7, Theorem E] and Zafrullah's result does not hold for an ideal I with $I_t \subsetneq D$. The next result shows that if $\{M_i\}$ is compact, then each ideal I of D with $I_t \subsetneq D$ is contained in at least one M_i.

Corollary 4. *Let $X \subseteq Spec(D)$ such that $D = \cap_{P\in X} D_P$, and let $*$ be the star-operation on D induced by X. Then X is compact if and only if $*_f$-Max$(D) \subseteq X$. In this case, each $P \in t$-Max(D) is contained in at least one $Q \in X$.*

Proof. Suppose that X is compact, and let Q be a maximal $*_f$-ideal of D. Assume to the contrary that $Q \nsubseteq P$ for all $P \in X$. By Theorem 1, there exists a finitely generated ideal $J \subseteq Q$ such that $J \nsubseteq P$ for all $P \in X$. Then $J^* = \cap_{P\in X} JD_P = \cap_{P\in X} D_P = D$, and hence $D = J^* \subseteq Q^{*_f} = Q \subsetneq D$, a contradiction. So $Q \subseteq P$ for some $P \in X$. Note that if $P' \in X$, then $P' \subseteq (P')^{*_f} \subseteq (P')^* = P'$; so $(P')^{*_f} = (P')^* = P'$. Thus $Q = P$. Conversely, assume that $*_f\text{-Max}(D) \subseteq X$, and let I be an ideal of D such that $I \nsubseteq P$ for all $P \in X$. Then by assumption, $I^{*_f} = D$, and since $*_f$ is of finite type, there exists a finitely generated ideal $J \subseteq I$ such that $J^{*_f} = D$. As we note, each $P \in X$ is a $*_f$-ideal, and hence $J \nsubseteq P$ for all $P \in X$. Thus X is compact by Theorem 1.

For the "in this case" part, note that $I \subseteq I^{*_f} \subseteq I_t$ for all $I \in \mathbf{F}(D)$. Hence each $Q' \in t\text{-Max}(D)$ is a $*_f$-ideal, and thus $Q' \subseteq P' \in *_f\text{-Max}(D) \subseteq X$. □

The result of Corollary 4 shows that if there are no containment relations among distinct members of X, then X is compact if and only if $X = *_f$-Max(D).

Corollary 5. 1. *If X is a set of prime t-ideals of D with $D = \cap_{P\in X} D_P$, then X is compact if and only if t-Max$(D) \subseteq X$.*
2. *$\mathcal{P}(D)$ is compact if and only if t-Max$(D) \subseteq \mathcal{P}(D)$.*

Proof. (1) If X is compact, then $*_f$-Max$(D) \subseteq X$ by Corollary 4, and since each ideal in X is a t-ideal, each maximal $*_f$-ideal of D is a t-ideal, and hence $*_f$-Max$(D) = t$-Max(D). Thus t-Max$(D) \subseteq X$. Conversely, assume t-Max$(D) \subseteq X$, and let I be an ideal of D such that $I \nsubseteq P$ for all $P \in X$. Then by assumption, $I_t = D$, and thus there is a finitely generated ideal $J \subseteq I$ with $J_v = D$. Since each ideal in X is a t-ideal, $J \nsubseteq P$ for all $P \in X$. Thus X is compact by Theorem 1.

(2) Note that each prime ideal in $\mathcal{P}(D)$ is a t-ideal and $D = \cap_{P\in\mathcal{P}(D)} D_P$. Thus the result follows directly from (1). □

Let D be a P-domain. If t-Max$(D) \subseteq \mathcal{P}(D)$, then D_P is a valuation domain for each $P \in t$-Max(D), and hence D is a PvMD. Thus if D is not a PvMD (see [5, Example 2.1]), then $\mathcal{P}(D)$ is not compact by Corollary 5(2).

Corollary 6. ([6, Proposition 3.2]) *Let x be an indeterminate over D. Then $\mathcal{P}(D)$ is compact if and only if $\mathcal{P}(D[x])$ is compact.*

Proof. Note that if Q is a maximal t-ideal of $D[x]$, then either $Q \cap D = (0)$ or $Q = P[x]$ for some $P \in t$-Max(D) [4, Proposition 1.1]. Also, note that $(a : b)D[x] = (aD[x] : bD[x])$; so t-Max$(D) \subseteq \mathcal{P}(D)$ if and only if t-Max$(D[x]) \subseteq \mathcal{P}(D[x])$. Thus the result follows from Corollary 5. □

Corollary 7. *Let $\emptyset \neq X \subseteq$ Spec(D) and x an indeterminate over D.*
1. *X is compact if and only if $\{P[x] | P \in X\}$ is compact.*
2. *If $*$ is a star-operation on D, then $\{P[x] | P \in *_f\text{-Max}(D)\}$ is compact.*

Proof. (1) Let $T = \{P[x] | P \in X\}$. Suppose that X is compact, and let $T = \cup_{f\in\Lambda}(X_f \cap T)$, where $\Lambda \subseteq D[x]$ and $X_f = \{Q \in \text{Spec}(D[x]) | f \notin Q\}$. Put $\Gamma = \{a \in D | a \text{ is a coefficient of a polynomial } f \in \Lambda\}$. Obviously, $X = \cup_{a\in\Gamma}(X_a \cap X)$, where $X_a = \{P \in \text{Spec}(D) | a \notin P\}$, and since X is compact, there exist some $a_1, \ldots, a_k \in \Gamma$ such that $X = (X_{a_1} \cap X) \cup \cdots \cup (X_{a_k} \cap X)$. Hence if $f_i \in \Lambda$ is such that a_i is a coefficient of f_i, then $T = (X_{f_1} \cap T) \cup \cdots \cup (X_{f_k} \cap T)$.

Conversely, suppose that T is compact, and let $X = \cup_{a \in I}(X_a \cap X)$, where $I \subseteq D$ and $X_a = \{P \in \text{Spec}(D) | a \notin P\}$. Clearly, $T = \cup_{a \in I}(X_a \cap T)$, where $X_a = \{Q \in \text{Spec}(D[x]) | a \notin Q\}$, and since T is compact, we have $T = (X_{a_1} \cap T) \cup \cdots \cup (X_{f_n} \cap T)$ for some $a_1, \ldots, a_n \in I$. Thus $X = (X_{a_1} \cap X) \cup \cdots \cup (X_{a_n} \cap X)$.

(2) This follows from (1) and Corollary 3. □

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