Honam Mathematical J. **33** (2011), No. 3, pp. 407–418
http://dx.doi.org/10.5831/HMJ.2011.33.3.407

# ON BRAID-PLAT RELATIONS IN CONWAY FUNCTION

Ki-Heon Yun

**Abstract.** There are two kinds of closing method for a given braid $\beta \in B_{2n}$, a braid closure $\hat{\beta}$ and a plat closure $\bar{\beta}$. In the article, we find a relation between the Conway potential function $\nabla_{\hat{\beta}}$ of braid closure $\hat{\beta}$ and $\nabla_{\bar{\beta}}$ of plat closure $\bar{\beta}$.

## 1. Introduction

In [FS98] Fintushel and Stern found a relation between the Conway potential function of a link $L \subset S^3$ and Seiberg-Witten invariants of a Fintushel-Stern knot surgery 4-manifold. By using this method, we experienced a big progress in the topology of smooth 4-manifolds. One of major progresses is the construction of exotic smooth structures on lots of smooth 4-manifolds. It was achieved by computing Seiberg-Witten invariants. If we restrict our attention to Fintushel-Stern knot surgery 4-manifolds, then Seiberg-Witten invariants computation is closely related to multi-variable Alexander polynomial of the corresponding knot and link. Therefore lots of examples in 4-manifolds were influenced by some knowledge of knot theory.

One of long standing problems in the Conway potential function is the characterization problem: for a given multi-variable symmetric Laurent polynomial $f$, can we find a link $L \subset S^3$ such that $\nabla_L = f$? For any given symmetric Laurent polynomial $f(t)$ with $f(1) = 1$, there is a knot $K \subset S^3$ which has $\nabla_K(t) \doteq f(t)$. But the problem is not solved yet for multi-variable Laurent polynomial $f$. Torres conditions [Tor53] are well known necessary conditions to the problem. But the Torres conditions are not sufficient [Hil81] [Kid78]. So we want to find more restrictions on

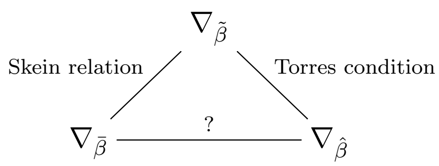
Received August 2, 2011. Accepted August 21, 2011.
2000 Mathematics Subject Classification. 57M25, 57M27.
Key words and phrases. Conway potential function, braid, plat.

symmetric multi-variable Laurent polynomial which has to be satisfied as a multi-variable Conway potential function of a link.

Any knot or link has two kinds well known representatives: a closed braid and a plat. In [BK88] Birman and Kanenobu found a braid plat formula of Jones polynomial. Therefore it is natural to ask a relation between the Conway potential functions $\nabla_{\hat{\beta}}$ and $\nabla_{\bar{\beta}}$ where $\hat{\beta}$ is the braid closure and $\bar{\beta}$ is the plat closure of a fixed admissible braid $\beta \in B_{2n}$.

If we view the problem from the Seiberg-Witten invariants of a Fintushel-Stern knot/link surgery 4-manifold, then the skein relation is related to $\pm 0/1$ log transforms on rim torus and the product formula of Seiberg-Witten invariants along $T^3$ boundary [MMS97] [FS98].

The sewn-up link exteriors studied by Hoste [Hos84] suggests a new link $\tilde{\beta}$ which contains $\hat{\beta}$ as a sublink and $\tilde{\beta}$ can be related to $\bar{\beta}$ by skein relations.




In this article, referring to the notations appeared in Definition 3.8 in Section 3, we get a relation between multi-variable Conway potential functions of these three knots or links which is the following

**Theorem 1.1.** *Let $\beta \in B_{2n}$ be a braid whose plat closure $\bar{\beta}$ is an admissible plat. Then we get*

$$(1.1)\quad \pi_{\bar{\beta}}\Big(\prod_{i=1}^{2n}(t_i - t_i^{-1})\Big)\nabla_{\overline{\beta}}(t_{c_1}, t_{c_2}, \cdots, t_{c_{\bar{r}}}) = (-1)^n \pi_{\bar{\beta}}\big(\nabla_{\widetilde{\beta}}(\pi_{\hat{\beta}}(\{t_1, t_2, \cdots, t_{2n}\}), t_1, t_3, \cdots, t_{2n-1})\big)$$

*and*

$$(1.2)\quad \nabla_{\widetilde{\beta}}(t_{s_1}, t_{s_2}, \cdots, t_{s_r}, \underbrace{1, 1, \cdots, 1}_{n}) = \pi_{\hat{\beta}}\Big(\prod_{i=1}^{n}(t_{2i-1}t_{2i}^{-1} - t_{2i-1}^{-1}t_{2i})\Big)\nabla_{\widehat{\beta}}(t_{s_1}, t_{s_2}, \cdots, t_{s_r}).$$

## 2. Preliminaries

A multi-variable Conway potential function is defined axiomatically by Turaev [Tur86] and Murakami [Mur93].

**Definition 2.1.** [Tur01] A *multi-variable Conway potential function* associates to any ordered oriented link $L = K_1 \cup K_2 \cup \cdots \cup K_n$ a rational function in $n$ variables $t_1, t_2, \cdots, t_n$,

$$\nabla_L(t_1, t_2, \cdots t_n) \in \mathbb{Q}(t_1, t_2, \cdots, t_n),$$

so that the following four axioms are satisfied:

1. $\nabla_L(t_1, t_2, \cdots, t_n)$ is an invariant under ambient isotopies of $L$
2. The function $\nabla_L(t, t, \cdots, t)$ is the one variable Conway function. In particular, it does not depend on the numbering of the components of $L$.
3. If $n \geq 2$, then $\nabla_L(t_1, t_2, \cdots, t_n) \in \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \cdots, t_n^{\pm 1}]$.
4. If the link $L'$ is obtained from $L$ by replacing the $i^{\text{th}}$ component $K_i$ by its $(2,1)$-cable, then
$$\nabla_{L'}(t_1, t_2, \cdots, t_n) = (T + T^{-1})\nabla_L(t_1, \cdots, t_{i-1}, t_i^2, t_{i+1}, \cdots, t_n),$$
where $T = t_i \prod_{j \neq i} t_j^{lk(K_i, K_j)}$.

In the section we give a skein relation which relates the Conway potential functions $\nabla_{\tilde{\beta}}$ and $\nabla_{\bar{\beta}}$. Morton [Mor83] studied a *ring on a band* relation and we generalize it to the multi-variable case.

**Proposition 2.2.** [Mur93] *Let $L_1$ and $L_2$ be two links which are different only in a part as in Figure 1. If we identify the variable $t_d$ of the circle on a knot component with the variable $t_c$ of the knot component. Then we get*

$$(t_c - t_c^{-1})\nabla_{L_1} = \nabla_{L_2}.$$

*Proof.* It comes directly from the first axiom of the Conway potential function which is given by J. Murakami [Mur93]. □

**Proposition 2.3.** *Let $L_3$, $L_4$ and $L_5$ be links as in Figure 1. Then*

$$(t_c - t_c^{-1})\nabla_{L_4} = \nabla_{L_3} - \nabla_{L_5}.$$

*Proof.* Consider $L_3 = L_+$, $L_4 = L_0$ and $L_5 = L_-$. Then we get the result by the Skein relation of a multi-variable Conway potential function. □

**Proposition 2.4.** *Let $L_3$, $L_4$ and $L_6$ be links as in Figure 1 and we identify $t_c$ and $t_d$. Then*

$$\nabla_{L_3} = (t_c - t_c^{-1})\nabla_{L_4} - (t_c - t_c^{-1})\nabla_{L_6}.$$

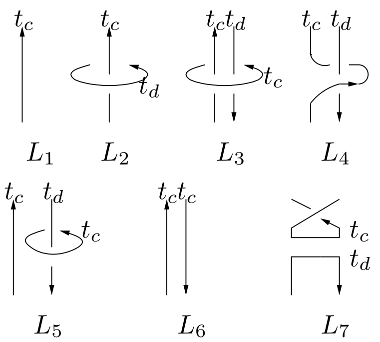



FIGURE 1

*Proof.* If we apply Proposition 2.2 with an identification $t_c = t_d$, then

$$\nabla_{L_5} = -(t_c - t_c^{-1})\nabla_{L_6}$$

and we get from Proposition 2.3

$$\nabla_{L_3} = (t_c - t_c^{-1})\nabla_{L_4} - (t_c - t_c^{-1})\nabla_{L_6}.$$

□

**Theorem 2.5.** *If we identify the variable $t_d$ with the variable $t_c$ in Figure 1, then we get*

$$(t_c - t_c^{-1})^2\nabla_{L_7} = -\nabla_{L_3}.$$

*Proof.* Let us put $t_c = t_d$ and consider $L_6 = L_+$, $L_7 = L_0$ and $L_4 = L_-$, then we have

$$\begin{aligned}(t_c - t_c^{-1})^2\nabla_{L_7} &= (t_c - t_c^{-1})\{\nabla_{L_6} - \nabla_{L_4}\}\\ &= -(t_c - t_c^{-1})\{\nabla_{L_4} - \nabla_{L_6}\} = -\nabla_{L_3}\end{aligned}$$

from Proposition 2.4. □

## 3. Braid closure and plat closure

In this section, we study a relation between $\nabla_{\bar{\beta}}$ and $\nabla_{\hat{\beta}}$ for a given braid $\beta \in B_{2n}$.

Let us consider the *braid permutation* $\rho : B_n \to S_n$ defined by $\rho(\sigma_i) = (i\ i+1)$ for the standard Artin braid $\sigma_i$ and $\rho(\sigma_i\sigma_j) = \rho(\sigma_j) \circ \rho(\sigma_i)$.

**Definition 3.1.** 1. A *pure braid* is a braid $\beta \in B_n$ with braid permutation $\rho(\beta) = id_n \in S_n$. The group of pure braid of $n$-strands is denoted by $PB_n$.

2. Let $\beta \in B_n$ for some positive integer $n$. A *braid-plat of type* $L_\beta(2k, n-2k)$ is the link obtained from $\beta$ by closing first $2k$ strands as a plat and the remaining $n-2k$ strands as a closed braid.
3. An *admissible braid-plat* is an oriented braid-plat $L_\beta(2k, n-2k)$ such that strands $1, 3, \cdots, 2k-1$ are oriented upward and strands $2, 4, \cdots, 2k$ are oriented downward at the top and at the bottom.
4. A *preferred plat* is an admissible braid-plat $L_\beta(2n, 0)$ with a restriction such that $\rho(\beta)(2i) = 2i$ for $i = 1, 2, \cdots, n$ and there is a sequence of integers $0 = n_0 < n_1 < \cdots < n_r = n$ such that the plat closing of the strands $\{2n_{i-1}+1, 2n_{i-1}+2, \cdots, 2n_i\}$ gives the link component $L_i$ of $\bar{\beta} = L_1 \cup L_2 \cup \cdots \cup L_r$ for each $i = 1, \cdots, r$.

**Remark 3.2.** [Bir74] [BK88]

1. A plat $\bar{\beta}$ of $\beta \in B_{2n}$ is a braid-plat of type $L_\beta(2n, 0)$ and a closed braid $\hat{\beta}$ of $\beta \in B_n$ is a braid-plat of type $L_\beta(0, n)$.
2. Any braid $\beta \in B_n$ may be altered to $\alpha \in B_n$, with $L_\alpha(2k, n-2k) = L_\beta(2k, n-2k)$ as a link and $L_\alpha(2k, n-2k)$ is an admissible braid-plat by adding appropriate half-twists at the top and at the bottom.
3. Any admissible braid-plat of type $L_\beta(2n, 0)$ may be altered to a preferred plat by adding appropriate $(\sigma_{2i}\sigma_{2i-1}\sigma_{2i+1}\sigma_{2i})^{\pm 1}$ or $(\sigma_{2i}^{-1}\sigma_{2i-1}^{-1}\sigma_{2i+1}\sigma_{2i})^{\pm 1}$ at the top.
4. Since any link is represented as $2n$-plat for some $n$, it may be considered as an admissible braid-plat or a preferred plat. But it may not be possible to represent a knot/link as a plat of a pure braid.

Let $\beta \in PB_{2n}$ and let $\bar{\beta}$ be the plat of $\beta$ with orientation defined by oriented upward at each odd-th strand and oriented downward at each even-th strand. Then $\bar{\beta}$ is a link with $n$ -components. Let $\hat{\beta}$ be the oriented $2n$-components braid closure of $\beta$ which is obtained by closing the $i$-th strand at the bottom and the $i$-th strand at the top. It has an orientation defined by each odd-th strand is oriented upward and each even-th strand is oriented downward.

Let $\tilde{\beta}$ be the $3n$-components link as in Figure 2 with ordering such that $t_i$, $1 \le i \le 2n$, is the color of the link component coming from the $i$-th strand and $t_{2n+i}$, $1 \le i \le n$, is the color of the link component which binds the $(2i-1)$-th strand and $(2i)$-th strand.

One of well-known constraints in the Conway potential function is the following Torres conditions.

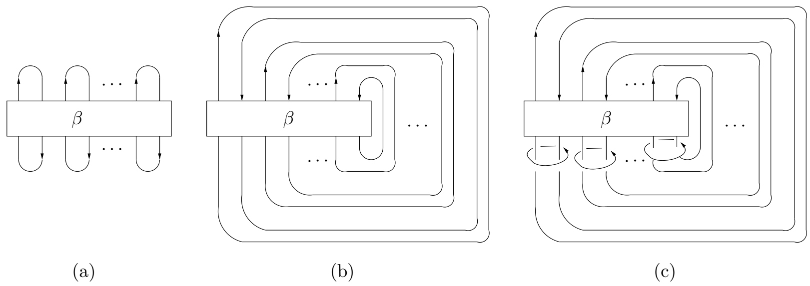



FIGURE 2. (a) $\bar{\beta}$ (b) $\hat{\beta}$ (c) $\tilde{\beta}$

**Proposition 3.3.** [Tor53] *Let $L = K_1 \cup K_2 \cup \cdots \cup K_n$ be an $n$ component oriented link and $t_i$ be the color corresponding to the preferred meridian of link component $K_i$.*

1. *Let $L' = K_2 \cup K_3 \cup \cdots \cup K_n$ be a sublink of $L$ obtained by dropping the link component $K_1$, then we get*

$$\nabla_L(1, t_2, t_3, \cdots, t_n) = (\prod_{i=2}^{n} t_i^{lk(K_1,K_i)} - \prod_{i=2}^{n} t_i^{-lk(K_1,K_i)})\nabla_{L'}(t_2, t_3, \cdots, t_n)$$

2. $\nabla_L(1, \cdots, 1, t_i, 1, \cdots, 1) = \nabla_{K_i}(t_i) \prod_{k=1,k\neq i}^{n} (t_i^{lk(K_i,K_k)} - t_i^{-lk(K_i,K_k)})$

**Proposition 3.4.** *For a pure braid $\beta \in PB_{2n}$ with orientation as in Figure 2, we get the following relations:*

$$(3.1)\quad \pi_{\bar{\beta}}(\prod_{i=1}^{2n}(t_i - t_i^{-1}))\nabla_{\bar{\beta}}(t_1, t_3, \cdots, t_{2n-1}) = (-1)^n \nabla_{\tilde{\beta}}(t_1, t_1, t_3, t_3, \cdots, t_{2n-1}, t_{2n-1}, t_1, t_3, \cdots, t_{2n-1})$$

*and*

$$(3.2)\quad \nabla_{\tilde{\beta}}(t_1, t_2, \cdots, t_{2n}, \underbrace{1, \cdots, 1}_{n}) = (\prod_{i=1}^{n}(t_{2i-1}t_{2i}^{-1} - t_{2i-1}^{-1}t_{2i}))\nabla_{\hat{\beta}}(t_1, t_2, \cdots, t_{2n})$$

*where $\pi_{\bar{\beta}} : \{1, 2, \cdots, 2n\} \to \{1, 3, \cdots, 2n-1\}$ is a map defined by $\pi_{\bar{\beta}}(2i-1) = 2i-1$ and $\pi_{\bar{\beta}}(2i) = 2i-1$.*

*Proof.* Equation (3.1) is an application of Theorem 2.5 because

$$\begin{aligned}
&\pi_{\bar\beta}(\prod_{i=1}^{2n}(t_i - t_i^{-1}))\nabla_{\bar\beta}(t_1, t_3, \cdots, t_{2n-1}) \\
&\quad = (\prod_{i=1}^{n}(t_{2i-1} - t_{2i-1}^{-1})^2)\nabla_{\bar\beta}(t_1, t_3, \cdots, t_{2n-1}) \\
&\quad = (-1)^n\nabla_{\tilde\beta}(t_1, t_1, t_3, t_3, \cdots, t_{2n-1}, t_{2n-1}, t_1, t_3, \cdots, t_{2n-1}).
\end{aligned}$$

Equation (3.2) is an application of Torres conditions: Observe that for $1 \leq j \leq 2n$ and $1 \leq i \leq n$,

$$lk(K_j, K_{2n+i}) = \begin{cases} 1 & \text{, if } j = 2i-1 \\ -1 & \text{, if } j = 2i \\ 0 & \text{, otherwise} \end{cases}$$

and $\hat\beta$ is a sublink of $\tilde\beta$. □

Let us consider a preferred plat case from now on.

**Proposition 3.5.** *Let $\beta \in B_{2n}$ be a braid whose plat closure $\bar\beta$ is a knot and a preferred plat. Then we get*

$$(t - t^{-1})^{2n}\nabla_{\bar\beta}(t) = (-1)^n\nabla_{\tilde\beta}(\underbrace{t, t, \cdots, t}_{n+1}, \underbrace{t, \cdots, t}_{n})$$

*and*

$$\begin{aligned}
&\nabla_{\tilde\beta}(t_1, t_2, t_4 \cdots, t_{2n}, \underbrace{1, 1, \cdots, 1}_{n}) \\
&\qquad = (\prod_{i=1}^{n}(t_1 t_{2i}^{-1} - t_1^{-1}t_{2i}))\nabla_{\hat\beta}(t_1, t_2, t_4 \cdots, t_{2n}).
\end{aligned}$$

*Proof.* If we close $\beta$ as a closed braid, the number of link components is reduced to the corresponding $\rho(\beta)$, so we need a projection map

$$\pi_{\hat\beta} : \{1, 2, 3, 4, \cdots, 2n\} \to \{1, 2, 4, \cdots, 2n\}$$

such that

$$\pi_{\hat\beta}(i) = \begin{cases} 1, & \text{if } i = 1, 3, \cdots, 2n-1 \\ i, & \text{if } i = 2, 4, \cdots, 2n. \end{cases}$$

Hence we need to apply $\pi_{\hat\beta}$ on both sides of (3.1) and (3.2) of Proposition 3.4, which gives the desired result. □

Note that, for a given any knot $K$, we can always find a braid $\beta \in B_{2n}$ for some positive integer $n$ such that $K = \bar\beta$ and $\bar\beta$ is a preferred plat.

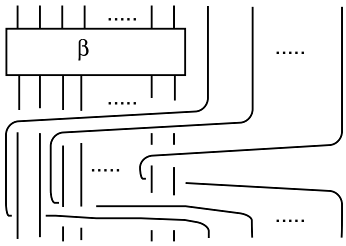



FIGURE 3

**Proposition 3.6.** *Suppose that a braid $\beta \in B_{2n}$ has a plat closure $\bar{\beta}$ which is a preferred plat and a knot. Then the braid closure $\hat{\beta}$ of $\beta$ has $(2n+1)$ components as follows:*

$$\tilde{\beta} = K_1 \cup K_2 \cup K_4 \cup \cdots \cup K_{2n} \cup K_{2n+1} \cup \cdots \cup K_{3n}.$$

*Furthermore, if $t_i$ is a color of the $i$-th strand, then we also get*

$$\begin{aligned}
&\nabla_{\tilde{\beta}}(1, t_2, t_4, \cdots, t_{2n}, t_{2n+1}, \cdots, t_{3n}) \\
&= (-1)^n (\prod_{i=1}^{n}(t_{2i} - t_{2i}^{-1}))((\prod_{i=1}^{n} t_{2i}^{lk(K_1,K_{2i})})(\prod_{j=1}^{n} t_{2n+j}) \\
&-(\prod_{i=1}^{n} t_{2i}^{-lk(K_1,K_{2i})})(\prod_{j=1}^{n} t_{2n+j}^{-1}))\nabla_{K_2\cup K_4\cup\cdots\cup K_{2n}}(t_2, t_4, \cdots, t_{2n})
\end{aligned}$$

*and*

$$\begin{aligned}
&\nabla_{\tilde{\beta}}(t_1, \underbrace{1, \cdots, 1}_{n}, t_1, \cdots, t_1) \\
&= (\prod_{i=1}^{n}(t_1^{lk(K_1,K_{2i})} t_{2n+i}^{-1} - t_1^{-lk(K_1,K_{2i})} t_{2n+i}))(t_1 - t_1^{-1})^n \nabla_{K_1}(t_1).
\end{aligned}$$

*Proof.* By Torres conditions, we get

$$\begin{aligned}
&\nabla_{\tilde{\beta}}(1, t_2, t_4, \cdots, t_{2n}, t_{2n+1}, \cdots, t_{3n}) \\
&= ((\prod_{i=1}^{n} t_{2i}^{lk(K_1,K_{2i})})(\prod_{j=1}^{n} t_{2n+j}) - (\prod_{i=1}^{n} t_{2i}^{-lk(K_1,K_{2i})})(\prod_{j=1}^{n} t_{2n+j}^{-1})) \\
&\times \nabla_{K_2\cup K_4\cup\cdots\cup K_{2n}\cup K_{2n+1}\cup\cdots\cup K_{3n}}(t_2, t_4, \cdots, t_{2n}, t_{2n+1}, \cdots, t_{3n})
\end{aligned}$$

and, by Proposition 2.2, we also have

$$\nabla_{K_2\cup K_4\cup\cdots\cup K_{2n}\cup K_{2n+1}\cup\cdots\cup K_{3n}}(t_2,t_4,\cdots,t_{2n},t_{2n+1},\cdots,t_{3n})$$
$$=(-1)^n(\prod_{i=1}^{n}(t_{2i}-t_{2i}^{-1}))\nabla_{K_2\cup K_4\cup\cdots\cup K_{2n}}(t_2,t_4,\cdots,t_{2n}).$$

One can prove the second equation in a similar way. □

Now we will consider a general preferred plat case. Let $\beta\in B_{2n}$ be a braid whose plat closure $\bar{\beta}=K_1\cup K_2\cup\cdots\cup K_m$ is an $m$-component link and a preferred plat. Then $\rho(\beta)=\prod_{i=1}^{m}(\prod_{j=1}^{r_i}\rho_{ij})$ is a product of disjoint cycles in $S_{2n}$ where

$$\rho_{ij}\in S_{\{2n_{i-1}+1,2n_{i-1}+2,\cdots,2n_i\}}.$$

Let $s_{ij}$ be the smallest element of $\rho_{ij}$ with an order relation defined by

$$s_{ij}<s_{i'j'} \text{ if } i<i' \text{ or } (i=i' \text{ and } j<j').$$

Let

$$\pi_{\hat{\beta}}:\{1,2,\cdots,2n\}\to\{s_{11},s_{12},\cdots,s_{1r_1},\cdots,s_{m1},\cdots,s_{mr_m}\}$$

be a map which is defined

$$\pi_{\hat{\beta}}|_{\{2n_{i-1}+1,2n_{i-1}+2,\cdots,2n_i\}}:\{2n_{i-1}+1,2n_{i-1}+2,\cdots,2n_i\}\to\{s_{i1},s_{i2},\cdots,s_{ir_i}\}$$

as in Proposition 3.5 above. Then we get

**Theorem 3.7.**

$$(\prod_{i=0}^{n-1}(t_{2n_i+1}-t_{2n_i+1}^{-1})^{2(n_{i+1}-n_i)})\nabla_{\overline{\beta}}(t_1,t_{n_1+1},\cdots,t_{n_{m-1}+1})$$
$$=(-1)^n\nabla_{\tilde{\beta}}(\underbrace{t_1,\cdots,t_1}_{n_1+1},\cdots,\underbrace{t_{n_{m-1}+1},\cdots,t_{n_{m-1}+1}}_{(n_m-n_{m-1})+1},\underbrace{t_1,\cdots,t_1}_{n_1},\cdots,\underbrace{t_{n_{m-1}+1},\cdots,t_{n_{m-1}+1}}_{n_m-n_{m-1}})$$

*and*

$$\nabla_{\tilde{\beta}}(\pi_{\hat{\beta}}(\{t_1,t_2,\cdots,t_{2n}\}),\underbrace{1,1,\cdots,1}_{n})$$
$$=\pi_{\hat{\beta}}(\prod_{i=1}^{n}(t_{2i-1}t_{2i}^{-1}-t_{2i-1}^{-1}t_{2i}))\nabla_{\hat{\beta}}(t_{s_{11}},\cdots,t_{s_{1r_1}},\cdots,t_{s_{m1}},\cdots,t_{s_{mr_m}}).$$

*Proof.* It is the most general preferred plat case and we can get it by applying Proposition 3.5 and Proposition 3.6 inductively. □

Now we will consider an admissible plat case.

**Definition 3.8.** Let us define the following two maps:

(i)

$$\pi_{\bar{\beta}} : \{1, 2, \cdots, 2n\} \to \{1, 3, 5, \cdots, 2n-1\} \to \{1, 3, 5, \cdots, 2n-1\}/\sim$$

where $(2i-1) \sim (2j-1)$ if there is a closing such that

$$\{\text{ Top color, Bottom color}\} = \{2i-1, 2j-1\}.$$

Let $\bar{r} = |\{1, 3, 5, \cdots, 2n-1\}/\sim|$ and let

$$\{c_1, c_2, \cdots, c_{\bar{r}}\} = \{1, 3, 5, \cdots, 2n-1\}/\sim$$

with $c_i < c_j$ if $i < j$.

(ii) Let $\rho(\beta) = \rho_1\rho_2\cdots\rho_r \in S_{2n}$ be a product of disjoint cycles in $S_{2n}$ and $s_i$ be the smallest element of $\rho_i$. Then we will define

$$\pi_{\hat{\beta}} : \{1, 2, 3, \cdots, 2n\} \to \{s_1, s_2, \cdots, s_r\}$$

such that $\pi_{\hat{\beta}}(i) = s_j$ if $i$ is in the cycle of $s_j$.

By using notations above, we will get the following relations.

**Theorem 3.9.** *Let $\beta \in B_{2n}$ be a braid whose plat closure $\overline{\beta}$ is an admissible plat with color restriction. Then*

$$\begin{aligned}(3.3)\quad & \pi_{\bar{\beta}}(\prod_{i=1}^{2n}(t_i - t_i^{-1}))\nabla_{\overline{\beta}}(t_{c_1}, t_{c_2}, \cdots, t_{c_{\bar{r}}}) \\ &= (-1)^n \pi_{\bar{\beta}}(\nabla_{\tilde{\beta}}(\pi_{\hat{\beta}}(\{t_1, t_2, \cdots, t_{2n}\}), t_1, t_3, \cdots, t_{2n-1}))\end{aligned}$$

*and*

$$\begin{aligned}(3.4)\quad & \nabla_{\tilde{\beta}}(t_{s_1}, t_{s_2}, \cdots, t_{s_r}, \underbrace{1, 1, \cdots, 1}_{n}) \\ &= \pi_{\hat{\beta}}(\prod_{i=1}^{n}(t_{2i-1}t_{2i}^{-1} - t_{2i-1}^{-1}t_{2i}))\nabla_{\hat{\beta}}(t_{s_1}, t_{s_2}, \cdots, t_{s_r}).\end{aligned}$$

*Proof.* Equation (3.3) is a result of Skein relation and Equation (3.4) is a result of Torres conditions of a multi-variable Conway polynomial. □

**Remark 3.10.** Note that, under the same condition as in Theorem 3.9 above, the Conway potential function of $\tilde{\beta}$ is given by

$$\nabla_{\tilde{\beta}}(\pi_{\hat{\beta}}(\{t_1, t_2^{-1}, \cdots, t_{2n-1}, t_{2n}^{-1}\}), t_{2n+1}, \cdots, t_{3n}) \doteq \pi_{\hat{\beta}}(\frac{det(I - \bar{B}_{\beta'}(t_1^2, t_2^2, \cdots, t_{3n}^2))}{1 - t_1^2 t_2^2 \cdots t_{3n}^2})$$

where

$$\beta' = \beta(\prod_{i=1}^{n}(\prod_{j=0}^{2n-(2i+1)} \sigma_{2n-j}))(\sigma_{2i}\sigma_{2i-1}^2\sigma_{2i})(\prod_{i=1}^{n}(\prod_{j=0}^{2n-(2i+1)} \sigma_{2n-j}))^{-1}.$$

The reason is following: The link $\tilde{\beta}$ can be considered as in Figure 3. Let $\hat{\beta}'$ be the same as $\tilde{\beta}$ as an unoriented link but all strands are oriented upward. Then we can compute the (non-normalized) Conway potential function by using a Morton's result in [Mor98]. Now, by the orientation change formula for Conway function, we get $\nabla_{\tilde{\beta}}$ as above.

## References

[Bir74] J. S. Birman, *Braids, links, and mapping class groups*, Princeton University Press, Princeton, N.J., 1974, Annals of Mathematics Studies, No. 82.

[BK88] J. S. Birman and Taizo Kanenobu, *Jones' braid-plat formula and a new surgery triple*, Proc. Amer. Math. Soc. **102** (1988), no. 3, 687–695. MR **89c:**57003

[FS98] R. Fintushel and R. Stern, *Knots, links, and 4-manifolds*, Inventiones Mathematicae **134** (1998), 363–400.

[Hil81] J. A. Hillman, *The Torres conditions are insufficient*, Math. Proc. Cambridge Philos. Soc. **89** (1981), no. 1, 19–22.

[Hos84] J. Hoste, *Sewn-up r-link exteriors*, Pacific J. Math. **112** (1984), no. 2, 347–382.

[Kid78] M. E. Kidwell, *Alexander polynomials of links of small order*, Illinois J. Math. **22** (1978), no. 3, 459–475.

[Lev65] J. Levine, *A characterization of knot polynomials*, Topology **4** (1965), 135–141.

[Lev67] ______, *A method for generating link polynomials*, Amer. J. Math. **89** (1967), 69–84.

[MMS97] J. W. Morgan, Tomasz S. Mrowka, and Zoltán Szabó, *Product formulas along $T^3$ for Seiberg-Witten invariants*, Math. Res. Lett. **4** (1997), no. 6, 915–929.

[Mor83] H. R. Morton, *Fibred knots with a given Alexander polynomial*, Knots, braids and singularities (Plans-sur-Bex, 1982), Monogr. Enseign. Math., vol. 31, Enseignement Math., Geneva, 1983, pp. 205–222.

[Mor98] ______, *The multivariable Alexander polynomial for a closed braid*, Low Dimensional Topology, Contemporary Mathematics, vol. 233, American mathematical Society, 1998, pp. 167–172.

[Mur93] J. Murakami, *A state model for the multivariable Alexander polynomial*, Pacific J. Math. **157** (1993), no. 1, 109–135.

[Sei35] H. Seifert, *Über das Geschlecht von Knoten*, Math. Ann. **110**(1935), no. 1, 571–592.

[Tor53] G. Torres, *On the Alexander polynomial*, Ann. of Math. (2) **57** (1953), 57–89.

[Tur86] V.G Turaev, *Reidermeister torsion in knot theory*, Russian math. Surveys **41** (1986), no. 1, 119–182.

[Tur01] ______, *Introduction to combinatorial torsions*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2001, Notes taken by Felix Schlenk.

Ki-Heon Yun
Department of Mathematics, Sungshin Women's University,
Seoul 136-742, Korea.
E-mail: kyun@sungshin.ac.kr