Honam Mathematical J. **33** (2011), No. 3, pp. 381–391
http://dx.doi.org/10.5831/HMJ.2011.33.3.381

REPRESENTATIONS OF $U_{3,6}$ AND $AG(2,3)$

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Abstract. One of the main interesting things of a matroid theory is the representability by a matroid from a matrix over some field F. The representability of uniform matroid $U_{m,n}$ over some field are studied by many authors. In this paper we construct a matrix representing $U_{3,6}$ over the field $GF(4)$. Also we find out matrix of the affine matroid $AG(2,3)$ over the field $GF(4)$.

1. Introduction

Matroid theory came from graph theory. Matriod can be defined in many different ways. Given an $m \times n$ matrix A, we obtain a vector space of n column vectors. The generalization of properties of independent sets of these vectors is the definition of matroid by independence:

Definition 1.1. For a finite set E, let $\mathcal{I}$ be a collection of subsets of E satisfing the following three conditions ;

(a) $\emptyset \in \mathcal{I}$

(b) If $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$.

(c) If I_1 and I_2 are in $\mathcal{I}$ and $|I_1| < |I_2|$, then there is an element $e \in I_2 \setminus I_1$ such that $I_1 \cup e \in \mathcal{I}$.

Then $M = (E, \mathcal{I})$ is called a *matroid* on E and the members of $\mathcal{I}$ are independent sets of M, and $E = E(M)$ is the ground set of M. A subset of E that is not in $\mathcal{I} = \mathcal{I}(M)$ is called dependent. A minimal dependent set is called *circuit.* There are many different ways of defining matroid([4]).

The definition by circuits comes from the cycles of graph. Let E=E(G) be the set of edges of a graph G. The collection $\mathcal{I}$ of all subsets of E(G)

Received July 19, 2011. Accepted September 5, 2011.

2000 Mathematics Subject Classification. 52C40, 60J10.

Key words and phrases. Graphic matroid, Uniform matroid, Affine space, Representation of F.

The first author was partially supported by Chonnam National University.

which does not contain a cycle of G satisfies the conditions of definitions 1.1. Thus (E(G),$\mathcal{I}$) is a matroid which is called cycle matroid and denoted by M(G).

Two matroids M_1 and M_2 are *isomorphic* if there is a bijection ψ from $E(M_1)$ to $E(M_2)$ such that for each $X \in \mathcal{I}(M_1), \psi(X) \in \mathcal{I}(M_2)$.

A matroid that is isomorphic to the cycle matroid of a graph is *graphic*. A rank of a matroid is the number of elements of a maximal independent set. Maximal independent sets are called bases. Let B be a basis of a matroid M and $e \in E(M) - B$. The circuit containing e is called the fundamental circuit with ruspect to B.

Matroids with rank less than 5 can be drawn in Euclidean space $\mathbb{R}^3$. A diagram of a matroid in Euclidean space should be understood as the following; Two identical points, three(four, five) point sets in a line(plane,space) are circuits. This is called the *geometric representation* of a matroid. Sometimes lines can be drawn as arc or circle. A matroid coming from a matrix A will be denoted by M[A].

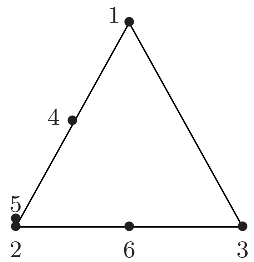
Example 1.2.

$$A = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}.$$

Then the circuits of the matroid M[A] are

$$\{2,5\}, \{1,2,4\}, \{1,4,5\}, \{2,3,6\}, \{3,5,6\}.$$

The geometric representation of M[A] is



Let m and n be non-negative integers such that $m \leq n$. Let E be an n-element set and $\mathcal{I}$ be the collection of subsets of E with cardinality less then or equal to m. Then $(E, \mathcal{I})$ is a matroid which is denoted by

$U_{m,n}$. These matroids are called *uniform matroids*. The representability of $U_{m,n}$ are studied by many authors.([1],[2],[5],[7])

Now, if F is a finite field, then F has exactly p^k-elements for some prime p. When $k > 1$, $GF(p^k)$ can be constructed as follows. Let $h(w)$ be a irreducible polynomial of degree k with coefficients Z_p. Consider the set S of all polynomials in w that have degree at most k-1 and have coefficients in Z_p. Under addition and multiplication both of which are performed modulo $h[w]$, S forms a field and this field is denoted by $GF(p^k)$.

Example 1.3. Since $h(w) = w^2 + w + 1$ is irreducible over Z_2, the multiplication tables for $GF(4) = \{0, 1, w, w + 1\}$ are the following.

$+$	0	1	w	$w+1$
0	0	1	w	$w+1$
1	1	0	$w+1$	w
w	w	$w+1$	0	1
$w+1$	$w+1$	w	1	0

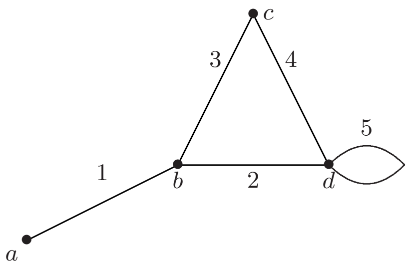
$\times$	0	1	w	$w+1$
0	0	0	0	0
1	0	1	w	$w+1$
w	w	$w+1$	0	1
$w+1$	$w+1$	w	1	0

Let $V = V(n+1, GF(q))$ be an (n+1)-dimensional vector space over $GF(q)$. The *projective space* of V will be denoted by $PG(n, q)$. The *affine space* $AG(n, q)$ is obtained from $PG(n, q)$ by deleting from the later all the points of a hyperplane. $PG(n, q)$ and $AG(n, q)$ can be considered as matroids. Let G be a graph. Form a directed graph $D(G)$ from G by arbitrarily assigning a direction to each edge. Let $A_{D(G)} = [a_{ij}]$ denote the incidence matrix of $D(G)$. That is, $A_{D(G)}$ is the matrix $[a_{ij}]$ where rows and columns are indexed by the vertices and arcs, respectively of $D(G)$ where

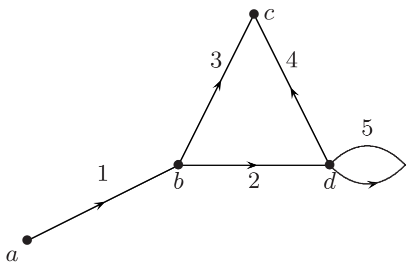
$$a_{ij} = \begin{cases} 1 & \text{if vertex } i \text{ is the tail of non-loop arc } j, \\ -1 & \text{if vertex } i \text{ is the head of non-loop arc } j, \\ 0 & \text{if otherwise.} \end{cases}$$

A matroid M is *F-represeutable* if M is isomorphic to M[A] for a matrix A over F.

Example 1.4. If G and $D(G)$ are the following,



G



$D(G)$

then

$$A_{D(G)} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \left[\begin{array}{rrrrr} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 \end{array}\right] \end{array}.$$

We have the following property:

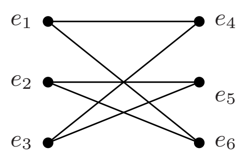
Proposition 1.5. If G is a graph, then $M(G)$ is representable over every field([6]).

2. Constructing representations for matroids

Two matrices A_1, A_2 are equivalent if $M[A_1]$ and $M[A_2]$ are isomorphic. By the properties of independence of vectors, it is easy to see that if M is a rank r matroid representation by A, then A is equivalent to a standard representation $[I_r \mid D]$, where I_r is the $r \times r$ identity matrix. Let D^* be the matrix obtained from D by replacing each non-zero entry of D by 1, then D^* is called the B-*fundamental circuit incidence matrix* of M and $[I_r \mid D^*]$ is called a *partial representation* for M. Now let M be a rank r matroid and B be the basis $\{e_1, \cdots, e_r\}$ for M. Let $X = \{e_{r+1}, e_{r+2}, \cdots, e_n\}$ be the B-fundamental circuit incidence matrix of M. Then $X = D^*$. Let $G(D^*)$ denote the associated simple bipartite graph. That is $V(G(D^*)) = \{e_1, \cdots, e_r\} \cup \{e_{r+1}, \cdots, e_n\}$ and two vertices e_i and e_j are adjacent if and only if row e_i and column e_j is 1.

Example 2.1.

$$X = D^* = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \end{array} \begin{array}{c} \begin{array}{ccc} e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$$



$G(D^*)$

Theorem 2.2. Let $r \times n$ matrix $[I_r \mid D]$ be an F-representation for the matroid M. Let $\{b_1, \cdots, b_m\}$ be a basis of the cycle matroid of $G(D^*)$. Then $m = n - w(G(D^*))$, where $w(G(D^*))$ is the number of connected components of $G(D^*)$. Moreover, if $(\theta_1, \theta_2, \cdots, \theta_m) \in (F - \{0\})^m$, then M has an F-representation $[I_r \mid D_1]$ that is equivalent to $[I_r \mid D]$ such that, for each $i \in \{1, 2, \cdots, m\}$, the entry corresponding to b_i is θ_i.

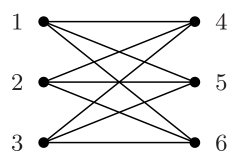
Proof. If G is a connected graph, then the rank of $M(G)$ is the number of edges of a spanning tree T of G. Because $|V(T)| = |E(T)| + 1$, we can see that $m = n - w(G(D^*))$.

Let G be a forest of $G(D^*)$. For each $x \in E(G)$, let $\theta(x)$ be a non-zero element of F. We shall show, by induction on m, that we can obtain a matrix $[I_r \mid D_1]$ which is equivalent to $[I_r \mid D]$ such that for each $x \in E(G)$, the entry in D_1 corresponding to x is $\theta(x)$. It is trivially true for $|E(G)| = 0$. Assume it is true for $|E(G)| < m$ and let $|E(G)| = m \geq 1$. As G is a forest with at least one edge, it has a vertex v of degree one. Let y be the edge of G with vertex v. By induction hypothesis we obtain a matrix $[I_r \mid D_1]$ which is equivalent to $[I_r \mid D]$ such that for each $x \in E(G|y)$, the entry of D_1 corresponding to x is $\theta(x)$. The vertex v of $G(D^*)$ corresponds to a row or a column of D_1. If v corresponds to a row, none of the entries of D_1 corresponding to edge of $G_1 \setminus y$ is in this row. We can multiply this row by an appropriate non-zero scalar t (if the y entry is a, $t = \theta(y) \diagup a$) to make the y-entry equal to $\theta(y)$ without changing any of the $(G|y)$-entries. The multiplication may alter the entry in row v of I_r. But multiplying the corresponding column by t^{-1} will fix this without affecting any other entries. In the other case, by multiplying non-zero scalar to the column, the y-entry can be equal to $\theta(y)$ without affecting any of the $(G|y)$-entries. It follows the result by induction. □

3. Representations

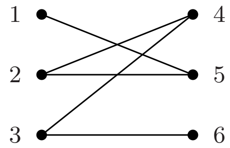
Let $E = \{1, 2, 3, 4, 5, 6\}$ and let $\mathcal{I} = \{I \subset E||I| \leq 3\}$. Then $U_{3,6} = (E, \mathcal{I})$, as was defined in the introduction. By the definition of $U_{3,6}$, any of the three columns of the partial representation $[I_3|D^*]$ is independent. Thus all the entries of D^* should be 1. That is

$$D^* = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \end{array}$$



$G(D^*)$

By Theorem 2.2, taking a basis,



the entries of $[I_3|D_1]$ corresponding to the five entries can be filled with any non-zero elements of F. If we fill out 1's to the entries the matrix $[I_3|D_1^*]$ which we are looking for is the form of

$$\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & a & 1 & b \\ 0 & 1 & 0 & 1 & 1 & c \\ 0 & 0 & 1 & 1 & d & 1 \end{array}\right] \end{array}.$$

Because any of the three columns of this matrix are independent, we have the following equations,

$$\begin{array}{c} 1\ 4\ 5 \\ \left|\begin{array}{ccc} 1 & a & 1 \\ 0 & 1 & 1 \\ 0 & 1 & d \end{array}\right| \end{array} = d - 1 \neq 0, \qquad \begin{array}{c} 1\ 4\ 6 \\ \left|\begin{array}{ccc} 1 & a & b \\ 0 & 1 & c \\ 0 & 1 & 1 \end{array}\right| \end{array} = 1 - c \neq 0,$$

$$\begin{array}{c} 1\ 5\ 6 \\ \left|\begin{array}{ccc} 1 & 1 & b \\ 0 & 1 & c \\ 0 & d & 1 \end{array}\right| \end{array} = 1 - cd \neq 0, \qquad \begin{array}{c} 2\ 4\ 5 \\ \left|\begin{array}{ccc} 0 & a & 1 \\ 1 & 1 & 1 \\ 0 & 1 & d \end{array}\right| \end{array} = 1 - ad \neq 0,$$

$$\begin{array}{c} 2\ 4\ 6 \\ \left|\begin{array}{ccc} 0 & a & b \\ 1 & 1 & c \\ 0 & 1 & 1 \end{array}\right| \end{array} = b - a \neq 0, \qquad \begin{array}{c} 2\ 5\ 6 \\ \left|\begin{array}{ccc} 0 & 1 & b \\ 1 & 1 & c \\ 0 & d & 1 \end{array}\right| \end{array} = bd - 1 \neq 0,$$

$$\begin{array}{c} 3\ 4\ 5 \\ \left|\begin{array}{ccc} 0 & a & 1 \\ 0 & 1 & 1 \\ 1 & 1 & d \end{array}\right| \end{array} = a - 1 \neq 0, \qquad \begin{array}{c} 3\ 4\ 6 \\ \left|\begin{array}{ccc} 0 & a & b \\ 0 & 1 & c \\ 0 & 1 & 1 \end{array}\right| \end{array} = ac - b \neq 0,$$

$$\begin{array}{c} 3\ 5\ 6 \\ \left|\begin{array}{ccc} 0 & 1 & b \\ 0 & 1 & c \\ 1 & d & 1 \end{array}\right| \end{array} = c - b \neq 0$$

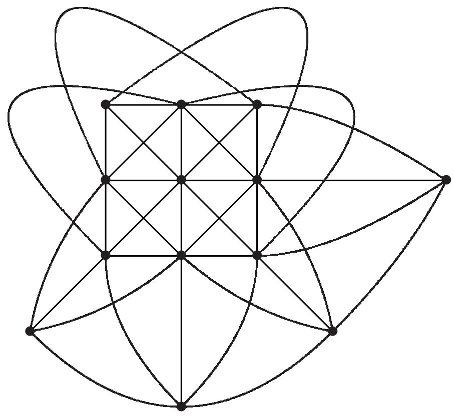
In the field $GF(4)$, if we take $b = 1$ and $a = c = d = w + 1$, then $ac = cd = ad = (w+1)(w+1) = w$, $bd = 1(w+1) = w+1$. Therefore a matrix which we want to find is

$$D_1 = \begin{vmatrix} w+1 & 1 & 1 \\ 1 & 1 & w+1 \\ 1 & w+1 & 1 \end{vmatrix}$$

Thus we have the following property;

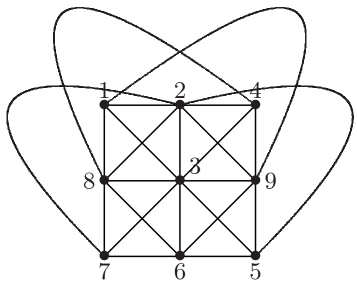
Theorem 3.1. $U_{3,6}$ is F-representable if and only if $|F| \geq 4$.

Now if we look at the projective space $PG(2,3)$ from the vector space $V(3, \mathbb{Z}_3)$, we can see that there are 13 points and there exist four hyperplanes passing through each point in $PG(2,3)$. Also each hyperplane contains four points. Thus a geometric representation of $PG(2,3)$ is the following:



$PG(2,3)$

Outer four points in the diagram are in a hyperplane of $PG(2,3)$. By deleting these points, we have the following geometric representation of $AG(2,3)$.



$AG(2,3)$

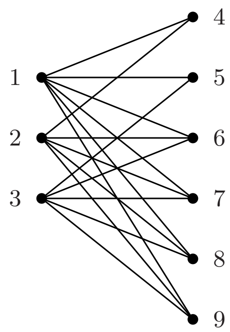
If we take a base $\{1,2,3\}$ in $AG(2,3)$, we have the following partial representation of $AG(2,3)$,

$$[I_3|D^*] = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & & 4 & 5 & 6 & 7 & 8 & 9 \end{matrix} \\ \left[\begin{array}{ccc|cccccc} 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \end{array}\right] \end{array}$$

From

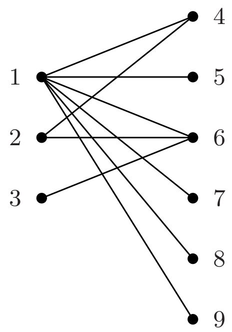
$$D^* = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{matrix} 4 & 5 & 6 & 7 & 8 & 9 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

the bipartite graph $G(D^*)$ is the following:



$G(D^*)$

Let's take a maximal tree



Then, by Theorem 2.2, a matrix $[I_3 \mid D_1]$ of $AG(2,3)$ is the following form:

$$\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array}$$
$$\left[\begin{array}{ccccccccc} 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & b & c & d \\ 0 & 1 & 1 & 0 & a & 1 & e & f & g \end{array}\right]$$

Because the determinants of the circuit columns are zero, we have the following equations:

$$\overset{\begin{array}{ccc}5&6&7\end{array}}{\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & b \\ a & 1 & e \end{vmatrix}} = e - a - b = 0, \quad \overset{\begin{array}{ccc}3&4&7\end{array}}{\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & b \\ a & 1 & e \end{vmatrix}} = b - 1 = 0,$$

$$\overset{\begin{array}{ccc}4&5&9\end{array}}{\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & d \\ 0 & a & g \end{vmatrix}} = a - ad - g = 0, \quad \overset{\begin{array}{ccc}2&7&9\end{array}}{\begin{vmatrix} 0 & 1 & 1 \\ 1 & b & d \\ 0 & e & g \end{vmatrix}} = e - g = 0,$$

$$\overset{\begin{array}{ccc}1&6&9\end{array}}{\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & d \\ 0 & 0 & g \end{vmatrix}} = g - d = 0, \quad \overset{\begin{array}{ccc}1&7&8\end{array}}{\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & c \\ 0 & e & f \end{vmatrix}} = f - ce = 0,$$

$$\overset{\begin{array}{ccc}4&6&8\end{array}}{\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & c \\ 0 & 1 & f \end{vmatrix}} = f + 1 - c = 0, \quad \overset{\begin{array}{ccc}3&8&9\end{array}}{\begin{vmatrix} 0 & 1 & 1 \\ 0 & c & d \\ 1 & f & g \end{vmatrix}} = d - c = 0.$$

By the equations, we have $c = d = g = e$.

If we denote these same number by x, we have the following equation:

$f - x^2 = 0$, $f + 1 - x = 0$

From these equations, we have $x^2 - x + 1 = 0$.

Because
$(w+1)^2-(w+1)+1=w-(w+1)+1=0$,
$x^2-x+1=0$ has a solution $w+1$ in $GF(4)$. If we take $f=a=w$, the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & w+1 & w+1 \\ 0 & 0 & 1 & 0 & w & 1 & w+1 & w & w+1 \end{bmatrix}$$

is a representations of $AG(2,3)$ over $GF(4)$.
Furthermore we proved the following theorem

Theorem 3.2. $AG(2,3)$ is F-representable if and only if F has a root of the equation $x^2-x+1=0$.

References

[1] Bixby, R,E. *On Reid's characterization of the tenary matroids*, J.Comb. Theory Ser. B 174-204,1979.
[2] Geelen,J.F, Gerards, A.M.H and A.kapoor, *The Excluded minorns for $GF(4)$-representable matroids*, J.Combin. Theory Ser. B, 247-299, 2000.
[3] Geelen,J. Maxhew, Dillon *Inequivalent representations of matroids having no $U_{3,6}$-minor*, J.Combin Theory Ser. B, no.1 55-67, 2004.
[4] Kung, P.S, *Twelve view of matroid theory, combinatorial and computational mathematics*, (Pohang, 2000) 56-96, World Sci.pull., River edge, NJ, 2001.
[5] Kahn.J and Seymour P., *On forbidden minors for $GF(3)$*, Proc. Amer.Math. Soc, 102 437-440, 1988.
[6] J.Oxley, *Matroid theory*, Oxford university press, Inc, New york, 1992.
[7] Seymour,P.D, *matroid representation over $GF(3)$*, J.Combin Theory set, B26,159-173, 1979.

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