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FUZZY TRANSLATIONS AND FUZZY MULTIPLICATIONS OF HYPER BCK-ALGEBRAS

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Abstract. Fuzzy α-translations, (normalized, maximal) fuzzy extensions and fuzzy multiplications of fuzzy hyper BCK-subalgebras in hyper BCK-algebras are discussed. Relations among fuzzy α-translations, (normalized, maximal) fuzzy extensions and fuzzy multiplications are investigated.

1. Introduction

The hyper structure theory (called also multialgebras) was introduced in 1934 by Marty [8] at the 8th congress of Scandinavian Mathematiciens. Around the 40's, several authors worked on hypergroups, especially in France and in the United States, but also in Italy, Russia, and Japan. Over the following decades, many important results appeared, but above all since the 70's onwards the most luxuriant flourishing of hyper structures has been seen. Hyper structures have many applications to several sectors of both pure and applied sciences. In [7], Jun et al. applied the hyper structures to BCK-algebras, and introduced the concept of a hyper BCK-algebra which is a generalization of a BCK-algebra. They also introduced the notion of a (weak, s-weak, strong) hyper BCK-ideal, and gave relations among them. Harizavi [2] studied prime weak hyper BCK-ideals of lower hyper BCK-semilattices. Jun et al. discussed the notion of hyperatoms and scalar elements of hyper BCK-algebras (see [4]). Jun et al. also discussed the fuzzy structures of (implicative) hyper BCK-ideals in hyper BCK-algebras (see [3, 5]).

Fuzzy set theory is established in the paper [9]. In the traditional fuzzy sets, the membership degrees of elements range over the interval $[0, 1]$. The membership degree expresses the degree of belongingness

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of elements to a fuzzy set. The membership degree 1 indicates that an element completely belongs to its corresponding fuzzy set, and the membership degree 0 indicates that an element does not belong to the fuzzy set. The membership degrees on the interval $(0, 1)$ indicate the partial membership to the fuzzy set. Sometimes, the membership degree means the satisfaction degree of elements to some property or constraint corresponding to a fuzzy set (see [1, 10]) In the viewpoint of satisfaction degree, the membership degree 0 is assigned to elements which do not satisfy some property. The elements with membership degree 0 are usually regarded as having the same characteristics in the fuzzy set representation. By the way, among such elements, some have irrelevant characteristics to the property corresponding to a fuzzy set and the others have contrary characteristics to the property. The traditional fuzzy set representation cannot tell apart contrary elements from irrelevant elements.

In this paper, we discuss fuzzy α-translations, (normalized, maximal) fuzzy extensions and fuzzy multiplications of fuzzy hyper BCK-subalgebras in hyper BCK-algebras. We investigate relations among fuzzy α-translations, (normalized, maximal) fuzzy extensions and fuzzy multiplications.

2. Preliminaries

We include some elementary aspects of hyper BCK-algebras that are necessary for this paper, and for more details we refer to [5], [6], and [7].

Let H be a nonempty set endowed with a hyperoperation "$\circ$". For two subsets A and B of H, denote by $A \circ B$ the set $\bigcup\limits_{a\in A, b\in B} a \circ b$. We shall use $x \circ y$ instead of $x \circ \{y\}$, $\{x\} \circ y$, or $\{x\} \circ \{y\}$.

By a *hyper BCK-algebra* we mean a nonempty set H endowed with a hyperoperation "$\circ$" and a constant 0 satisfying the following axioms:

(HK1) $(x \circ z) \circ (y \circ z) \ll x \circ y$,
(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$,
(HK3) $x \circ H \ll \{x\}$,
(HK4) $x \ll y$ and $y \ll x$ imply $x = y$,

for all $x, y, z \in H$, where $x \ll y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A \ll B$ is defined by $\forall a \in A$, $\exists b \in B$ such that $a \ll b$. In such case, we call "$\ll$" the *hyperorder* in H.

Note that the condition (HK3) is equivalent to the condition:

$$(\forall x, y \in H)(x \circ y \ll x). \tag{2.1}$$

In any hyper BCK-algebra H, the following hold:

(2.2) $(A \circ B) \circ C = (A \circ C) \circ B,\ A \circ B \ll A,\ 0 \circ A \ll \{0\}.$

(2.3) $0 \circ 0 = \{0\}.$

(2.4) $0 \ll x.$

(2.5) $A \ll A.$

(2.6) $A \subseteq B \ \Rightarrow\ A \ll B.$

(2.7) $0 \circ A = \{0\}.$

(2.8) $A \ll \{0\} \ \Rightarrow\ A = \{0\}.$

(2.9) $x \in x \circ 0.$

(2.10) $x \circ 0 \ll \{y\} \ \Rightarrow\ x \ll y.$

(2.11) $y \ll z \ \Rightarrow\ x \circ z \ll x \circ y.$

(2.12) $x \circ y = \{0\} \ \Rightarrow\ (x \circ z) \circ (y \circ z) = \{0\},\ x \circ z \ll y \circ z.$

for all $x, y, z \in H$ and for all nonempty subsets A, B and C of H.

We now review some fuzzy logic concepts. A *fuzzy set* in a set H is a function $\mu : X \to [0, 1]$. For any $t \in [0, 1]$ and a fuzzy set μ in a nonempty set H, the set

$$U(\mu; t) = \{x \in H \mid \mu(x) \geq t\} \ \text{(resp. } L(\mu; t) = \{x \in H \mid \mu(x) \leq t\})$$

is called an *upper* (resp. *lower*) *level set* of μ.

A fuzzy set μ in a hyper BCK-algebra H is called a *fuzzy hyper BCK-ideal* of H if it satisfies

$$(\forall x, y \in H)(x \ll y \ \Rightarrow\ \mu(y) \leq \mu(x)), \tag{2.13}$$

$$(\forall x, y \in H)(\mu(x) \geq \min\Big\{\inf_{a \in x \circ y} \mu(a), \mu(y)\Big\}). \tag{2.14}$$

3. Fuzzy translations and fuzzy multiplications of fuzzy hyper BCK-subalgebras

In what follows let $H = (H, \circ, 0)$ denote a hyper BCK-algebra, and for any fuzzy set μ of H, we denote $\top := 1 - \sup\{\mu(x) \mid x \in H\}$ unless otherwise specified.

Definition 3.1. [7] Let S be a subset of H. If S is a hyper BCK-algebra with respect to the hyper operation "$\circ$" on H, we say that S is a *hyper subalgebra* of H.

Theorem 3.2. [7] *Let S be a non-empty subset of a H. Then S is a hyper subalgebra of H if and only if $x \circ y \subseteq S$ for all $x, y \in S$.*

Definition 3.3. Let μ be a fuzzy subset of H and let $\alpha \in [0, \top]$. A mapping $\mu_\alpha^T : H \to [0, 1]$ is called a *fuzzy α-translation* of μ if it satisfies:

$$(\forall x \in H)(\mu_\alpha^T(x) = \mu(x) + \alpha).$$

Definition 3.4. A fuzzy set μ in H is called a *fuzzy hyper BCK-subalgebra* of H if it satisfies:

$$(\forall x, y \in H) \left(\inf_{z \in x \circ y} \mu(z) \geq \min\{\mu(x), \mu(y)\} \right). \tag{3.1}$$

Theorem 3.5. *Let μ be a fuzzy hyper BCK-subalgebra of H and $\alpha \in [0, \top]$. Then the fuzzy α-translation μ_α^T of μ is a fuzzy hyper BCK-subalgebra of H.*

Proof. Let $x, y \in H$. Then

$$\begin{aligned} \inf_{z \in x \circ y} \mu_\alpha^T(z) &= \inf_{z \in x \circ y} (\mu(z) + \alpha) = \alpha + \inf_{z \in x \circ y} \mu(z) \\ &\geq \alpha + \min\{\mu(x), \mu(y)\} = \min\{\mu(x) + \alpha, \mu(y) + \alpha\} \\ &= \min\{\mu_\alpha^T(x), \mu_\alpha^T(y)\}. \end{aligned}$$

Hence μ_α^T is a fuzzy hyper BCK-subalgebra of H. □

Theorem 3.6. *Let μ be a fuzzy subset of H such that the fuzzy α-translation μ_α^T of μ is a fuzzy hyper BCK-subalgebra of H for some $\alpha \in [0, \top]$. Then μ is a fuzzy hyper BCK-subalgebra of H.*

Proof. Assume that μ_α^T is a fuzzy hyper BCK-subalgebra of H for some $\alpha \in [0, \top]$. Let $x, y \in H$, we have

$$\begin{aligned} \alpha + \inf_{z \in x \circ y} \mu(z) &= \inf_{z \in x \circ y} (\mu(z) + \alpha) = \inf_{z \in x \circ y} \mu_\alpha^T(z) \\ &\geq \min\{\mu_\alpha^T(x), \mu_\alpha^T(y)\} = \min\{\mu(x) + \alpha, \mu(y) + \alpha\} \\ &= \min\{\mu(x), \mu(y)\} + \alpha \end{aligned}$$

which implies that $\inf_{z \in x \circ y} \mu(z) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in H$. Hence μ is a fuzzy hyper BCK-subalgebra of H. □

Definition 3.7. Let μ_1 and μ_2 be fuzzy subsets of H. If $\mu_1(x) \leq \mu_2(x)$ for all $x \in H$, then we say that μ_2 is a *fuzzy extension* of μ_1.

Definition 3.8. Let μ_1 and μ_2 be fuzzy subsets of H. Then μ_2 is called a *fuzzy S-extension* of μ_1 if the following assertions are valid:

(i) μ_2 is a fuzzy extension of μ_1.

(ii) If μ_1 is a fuzzy hyper BCK-subalgebra of H, then μ_2 is a fuzzy hyper BCK-subalgebra of H.

Example 3.9. Consider a hyper BCK-algebra $H = \{0, 1, 2\}$ with the following Cayley table:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{0,1\}$
2	$\{2\}$	$\{2\}$	$\{0,1\}$

Define a fuzzy subsets μ_1 and μ_2 of H by

H	0	1	2
μ_1	0.7	0.5	0.3

H	0	1	2
μ_2	0.7	0.5	0.6

Then μ_1 is a fuzzy hyper BCK-subalgebra of H and μ_2 is fuzzy extension of μ_1. Since

$$\inf_{z \in 2 \circ 2} \mu_2(z) \; = \; \mu_2(1) \; = \; 0.5 \; \not\geq \; 0.6 \; = \; \min\{\mu_2(2), \; \mu_2(2)\},$$

μ_2 is not a fuzzy hyper BCK-subalgebra of H. Hence μ_2 is not a fuzzy S-extension of μ_1.

Example 3.10. Consider a hyper BCK-algebra $H = [0, \infty)$, define a hyper operation " $\circ$ " in H by

$$x \circ y \; = \; \begin{cases} [0, x] & if \quad x \leq y, \\ (0, y] & if \quad x > y \neq 0, \\ \{x\} & if \quad y = 0, \end{cases}$$

for all $x, y \in H$. Define a fuzzy subset μ_i of H by

$$\mu_i(x) = 1/(i + x),$$

for all $i \in \mathbb{N}$, for all $x \in H$. Then μ_{i_1} and μ_{i_2} are fuzzy hyper BCK-subalgebras of H for all $i_1, i_2 \in \mathbb{N}$. If $i_1 < i_2$, then

$$\mu_{i_1}(x) \; = \; 1/(i_1 + x) \; > \; 1/(i_2 + x) \; = \; \mu_{i_2}$$

for all $x \in H$. Hence μ_{i_1} is a fuzzy S-extension of μ_{i_2}.

By means of the definition of fuzzy α-translation, we know that $\mu_\alpha^T(x) \geq \mu(x)$ for all $x \in H$. Hence we have the following theorem.

Theorem 3.11. *Let μ be a fuzzy hyper BCK-subalgebra of H and $\alpha \in [0, \top]$. Then the fuzzy α-translation μ_α^T of μ is a fuzzy S-extension of μ.*

Proof. $\mu_\alpha^T(x) = \mu(x) + \alpha \geq \mu(x)$ for all $x \in H$. Thus μ_α^T is a fuzzy extension of μ. Assume that μ is a fuzzy hyper BCK-subalgebra of H and $\alpha \in [0, \top]$. By Theorem 3.5, μ_α^T is a fuzzy hyper BCK-subalgebra of H. Hence μ_α^T is a fuzzy S-extension of μ. □

The converse of Theorem 3.11 is not true in general as seen in the following example.

Example 3.12. Consider μ_{i_1} and μ_{i_2} in Example 3.10. μ_{i_1} is a fuzzy S-extension of μ_{i_2}. But μ_{i_1} is not a fuzzy α-translation $(\mu_{i_2})_\alpha^T$ of μ_{i_2} for all $\alpha \in [0, \top]$, since $\mu_{i_1}(5) - \mu_{i_2}(5) \neq \mu_{i_1}(6) - \mu_{i_2}(6)$.

Theorem 3.13. *Let μ be a fuzzy hyper BCK-subalgebra of H. If μ_1 and μ_2 are fuzzy S-extensions of μ, then $\nu := \mu_1 \cap \mu_2$ is a fuzzy S-extension of μ.*

Proof. Assume that μ_1 and μ_2 are fuzzy S-extensions of μ. Then $\mu_1(x) \geq \mu(x)$ and $\mu_2(x) \geq \mu(x)$ for all $x \in H$. Then

$$\nu(x) \;=\; (\mu_1 \cap \mu_2)(x) \;=\; \min\{\mu_1(x), \mu_2(x)\} \;\geq\; \mu(x)$$

for all $x \in H$. Hence ν is a fuzzy extension of μ. Let μ be a fuzzy hyper BCK-subalgebra of H. Then μ_1 and μ_2 are fuzzy hyper BCK-subalgebras of H. Then

$$\begin{aligned}
\inf_{z \in x \circ y} \nu(z) &= \inf_{z \in x \circ y} (\mu_1 \cap \mu_2)(z) \;=\; \inf_{z \in x \circ y} (\min\{\mu_1(z), \mu_2(z)\}) \\
&= \min\left\{ \inf_{z \in x \circ y} \mu_1(z), \inf_{z \in x \circ y} \mu_2(z) \right\} \\
&\geq \min\{\min\{\mu_1(x), \mu_1(y)\}, \min\{\mu_2(x), \mu_2(y)\}\} \\
&= \min\{\min\{\mu_1(x), \mu_2(x)\}, \min\{\mu_1(y), \mu_2(y)\}\} \\
&= \min\{(\mu_1 \cap \mu_2)(x), (\mu_1 \cap \mu_2)(y)\} \;=\; \min\{\nu(x), \nu(y)\}
\end{aligned}$$

for all $x, y \in H$. Thus $\inf\limits_{z \in x \circ y} \nu(z) \geq \min\{\nu(x), \nu(y)\}$ for all $x, y \in H$. Hence ν is a fuzzy hyper BCK-subalgebra of H. Therefore ν is a fuzzy S-extension of ν. □

Example 3.14. Consider a hyper BCK-algebra $H = \{0, 1, 2, 3\}$ with the following Cayley table:

$\circ$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
2	$\{2\}$	$\{2\}$	$\{0,2\}$	$\{0,1,2\}$
3	$\{3\}$	$\{3\}$	$\{3\}$	$\{0,3\}$

Define a fuzzy subsets μ, μ_1 and μ_2 of H by

H	0	1	2	3
μ	0.9	0.2	0.1	0.1

H	0	1	2	3
μ_1	0.95	0.3	0.3	0.6

H	0	1	2	3
μ_2	0.95	0.3	0.7	0.3

Then μ is a fuzzy hyper BCK-subalgebra of H, μ_1 and μ_2 are fuzzy S-extensions of μ. But

$$\nu := \mu_1 \cup \mu_2 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0.95 & 0.3 & 0.7 & 0.6 \end{pmatrix}$$

is not a fuzzy S-extension of μ, since $\inf_{z \in 2 \circ 3} \nu(1) = 0.3 \ngeq 0.6 = \min\{\nu(2), \nu(3)\}$.

For a fuzzy subset μ of H, $\alpha \in [0, \top]$ and $t \in [0, 1]$ with $t \geq \alpha$, let

$$U_\alpha(\mu; t) := \{x \in H \mid \mu(x) \geq t - \alpha\}.$$

Theorem 3.15. *If μ is a fuzzy hyper BCK-subalgebra of H, then $U_\alpha(\mu; t)$ is a hyper subalgebra of H for all $t \in \mathrm{Im}(\mu)$ with $t \geq \alpha$, $\alpha \in [0, \top]$.*

Proof. Let $x, y \in U_\alpha(\mu; t)$. Then $\mu(x) \geq t - \alpha$ and $\mu(y) \geq t - \alpha$. Then $\inf_{z \in x \circ y} \mu(z) \geq \min\{\mu(x), \mu(y)\} \geq t - \alpha$. Then $\mu(z) \geq t - \alpha$ for all $z \in x \circ y$. i.e., $z \in U_\alpha(\mu; t)$ for all $z \in x \circ y$. Then $x \circ y \subseteq U_\alpha(\mu; t)$. Hence $U_\alpha(\mu; t)$ is a hyper subalgebra of H. □

In Theorem 3.15, if we do not give a condition that μ is a fuzzy hyper BCK-subalgebra of H, then $U_\alpha(\mu; t)$ is not a hyper subalgebra of H as seen in the following example.

Example 3.16. Consider a hyper BCK-algebra $H = \{0, 1, 2\}$ with the following Cayley table:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0, 1\}$	$\{0, 1\}$
2	$\{2\}$	$\{1, 2\}$	$\{0, 1, 2\}$

Define a fuzzy subset μ of H by

$$\mu : H \to [0, 1], \ x \mapsto \begin{cases} 0.7 & \text{if } x = 0, \\ 0.3 & \text{if } x = 1, \\ 0.5 & \text{if } x = 2. \end{cases}$$

Then μ is not a fuzzy hyper BCK-subalgebra of H since

$$\inf_{z\in 2\circ 2}\mu(z)=\mu(1)=0.3\ngeq 0.5=\mu(2)=\min\{\mu(2),\mu(2)\}.$$

For $\alpha = 0.2$ and $t = 0.7$, we obtain $U_\alpha(\mu;t)=\{0,2\}$ which is not a hyper subalgebra of H since $2\circ 2=\{0,1,2\}\nsubseteq\{0,2\}=U_\alpha(\mu;t)$.

Theorem 3.17. *If $U_\alpha(\mu;t)$ is a hyper subalgebra of H for all $t\in\mathrm{Im}(\mu)$ with $t\geq\alpha$, $\alpha\in[0,\top]$, then μ is a fuzzy hyper BCK-subalgebra of H,*

Proof. Assume that $U_\alpha(\mu;t)$ is a hyper subalgebra of H, for all $t\in\mathrm{Im}(\mu)$ with $t\geq\alpha$, $\alpha\in[0,\top]$. Then $U_0(\mu;t)$ is a hyper subalgebra of H, for all $t\in\mathrm{Im}(\mu)$ with $t\geq 0$. Assume that there exists $a,b\in H$ such that

$$\inf_{c\in a\circ b}\mu(c)\ <\ \min\{\mu(a),\mu(b)\}.$$

Let $\beta:=\min\{\mu(a),\mu(b)\}$. Then $\mu(a)\geq\beta$ and $\mu(b)\geq\beta$. Then $\mu(a)\geq\beta-0$ and $\mu(b)\geq\beta-0$, so $a,b\in U_0(\mu;\beta)$. $\mu(w)<\beta$ for some $w\in a\circ b$, since $\inf_{c\in a\circ b}\mu(c)<\beta$. Then $\mu(w)<\beta-0$ for some $w\in a\circ b$. i.e., $w\notin U_0(\mu;\beta)$ for some $w\in a\circ b$. Then $a\circ b\nsubseteq U_0(\mu;\beta)$, which is contradiction. So

$$\inf_{z\in x\circ y}\mu(z)\ \geq\ \min\{\mu(x),\mu(y)\}$$

for all $x,y\in H$. Hence μ is a fuzzy hyper BCK-subalgebra of H. □

Theorem 3.18. *Let μ be a fuzzy subset of H and $\alpha\in[0,\top]$. Then the fuzzy α-translation μ_α^T of μ is a fuzzy hyper BCK-subalgebra of H if and only if $U_\alpha(\mu;t)$ is a hyper subalgebra of H for all $t\in\mathrm{Im}(\mu)$ with $t\geq\alpha$.*

Proof. Let $x,y\in U_\alpha(\mu;t)$. Then $\mu(x)\ \geq\ t-\alpha$ and $\mu(y)\ \geq\ t-\alpha$. Then

$$\begin{aligned}\alpha+\inf_{z\in x\circ y}\mu(z)\ &=\ \inf_{z\in x\circ y}(\mu(z)+\alpha)\ =\ \inf_{z\in x\circ y}\mu_\alpha^T(z)\\ &\geq\ \min\{\mu_\alpha^T(x),\mu_\alpha^T(y)\}\ =\ \min\{\mu(x)+\alpha,\mu(y)+\alpha\}\\ &=\ \alpha+\min\{\mu(x),\mu(y)\}\ \geq\ \alpha+t-\alpha\ =\ t.\end{aligned}$$

Thus $\inf_{z\in x\circ y}\mu(z)\ \geq\ t-\alpha$. Then $\mu(z)\ \geq\ t-\alpha$ for all $z\in x\circ y$. i.e., $z\in U_\alpha(\mu;t)$ for all $z\in x\circ y$. Then $x\circ y\subseteq U_\alpha(\mu;t)$. Hence $U_\alpha(\mu;t)$ is a hyper subalgebra of H.

Conversely assume that there exists $a,b\in H$ such that

$$\inf_{c\in a\circ b}\mu_\alpha^T(c)\ <\ \beta\ \leq\ \min\{\mu_\alpha^T(a),\mu_\alpha^T(b)\}.$$

Then $\mu_\alpha^T(a) \geq \beta$ and $\mu_\alpha^T(b) \geq \beta$. Then $\mu(a) \geq \beta - \alpha$ and $\mu(b) \geq \beta - \alpha$, so $a, b \in U_\alpha(\mu; \beta)$. $\mu_\alpha^T(w) < \beta$ for some $w \in a \circ b$, since $\inf\limits_{c \in a \circ b} \mu_\alpha^T(c) < \beta$. Then $\mu(w) < \beta - \alpha$ for some $w \in a \circ b$. i.e., $w \notin U_\alpha(\mu; \beta)$ for some $w \in a \circ b$. Then $a \circ b \nsubseteq U_\alpha(\mu; \beta)$, which is contradiction. So

$$\inf_{z \in x \circ y} \mu_\alpha^T(z) \ \geq \ \min\{\mu_\alpha^T(x), \mu_\alpha^T(y)\}$$

for all $x, y \in H$. Hence μ_α^T is a fuzzy hyper BCK-subalgebra of H. □

Theorem 3.19. *Let μ be a fuzzy hyper BCK-subalgebra of H and let $\alpha, \beta \in [0, \top]$. If $\alpha \geq \beta$, then the fuzzy α-translation μ_α^T of μ is a fuzzy S-extension of the fuzzy β-translation μ_β^T of μ.*

Proof. Straightforward. □

For every fuzzy hyper BCK-subalgebra μ of H and $\beta \in [0, \top]$, the fuzzy β-translation μ_β^T of μ is a fuzzy hyper BCK-subalgebra of H. If ν is a fuzzy S-extension of μ_β^T, then there exists $\alpha \in [0, \top]$ such that $\alpha \geq \beta$ and $\nu(x) \geq \mu_\alpha^T(x)$ for all $x \in H$. Thus we have the following theorem.

Theorem 3.20. *Let μ be a fuzzy hyper BCK-subalgebra of H and $\beta \in [0, \top]$. For every fuzzy S-extension ν of the fuzzy β-translation μ_β^T of μ, there exists $\alpha \in [0, \top]$ such that $\alpha \geq \beta$ and ν is a fuzzy S-extension of the fuzzy α-translation μ_α^T of μ.*

The following example illustrates Theorem 3.20.

Example 3.21. Consider a hyper BCK-algebra $H = \{0, 1, 2\}$ with the following Cayley table:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0, 1\}$	$\{0, 1\}$
2	$\{2\}$	$\{2\}$	$\{0, 1\}$

Define a fuzzy subset μ of H by

H	0	1	2
μ	0.7	0.5	0.3

Then μ is a fuzzy hyper BCK-subalgebra of H, and $\top = 0.3$. If we take $\beta = 0.1$, then the fuzzy β-translation μ_β^T of μ is given by

H	0	1	2
μ_β^T	0.8	0.6	0.4

Let ν be a fuzzy subset of H defined by

H	0	1	2
ν	0.93	0.77	0.56

Then ν is clearly a fuzzy hyper BCK-subalgebra of H which is fuzzy extension of μ_β^T, and hence ν is a fuzzy S-extension of the fuzzy β-translation μ_β^T of μ. But ν is not a fuzzy α-translation of μ for all $\alpha \in [0, \top]$. Take $\alpha = 0.2$. Then $\alpha = 0.2 > 0.1 = \beta$, and the fuzzy α-translation μ_α^T of μ is given as follows:

H	0	1	2
μ_α^T	0.9	0.7	0.5

Note that $\nu(x) \geq \mu_\alpha^T(x)$ for all $x \in H$, and hence ν is a fuzzy S-extension of the fuzzy α-translation μ_α^T of μ.

A fuzzy S-extension ν of a fuzzy hyper BCK-subalgebra μ of H is said to be *normalized* if there exists $x_0 \in H$ such that $\nu(x_0) = 1$. Let μ be a fuzzy hyper BCK-subalgebra of H. A fuzzy subset ν of H is called a *maximal fuzzy S-extension* of μ if it satisfies:

(i) ν is a fuzzy S-extension of μ,
(ii) there does not exist any fuzzy hyper BCK-subalgebra of H which is a fuzzy extension of ν.

Example 3.22. Consider a hyper BCK-algebra $H = [0, \infty)$, define a hyper operation "$\circ$" in H by

$$x \circ y = \begin{cases} [0, x] & if \quad x \leq y, \\ (0, y] & if \quad x > y \neq 0, \\ \{x\} & if \quad y = 0, \end{cases}$$

for all $x, y \in H$. Then H is a hyper BCK-algebra. Let μ and ν be fuzzy subsets of H which are defined by $\mu(x) = \frac{2}{3}$ and $\nu(x) = 1$ for all $x \in H$. Clearly μ and ν are fuzzy hyper BCK-subalgebras of H. It is easy to verify that ν is a maximal fuzzy S-extension of μ.

Proposition 3.23. *If a fuzzy subset ν of H is a normalized fuzzy S-extension of a fuzzy hyper BCK-subalgebra μ of H, then $\nu(0) = 1$.*

Proof. Assume that μ is a fuzzy hyper BCK-subalgebra of H. Since ν is a fuzzy S-extension of μ, ν is a fuzzy hyper BCK-subalgebra of H.

Then

$$\nu(0) \geq \inf_{z \in x \circ x} \mu(z) \geq \min\{\mu(x), \mu(x)\} = \mu(x)$$

for all $x \in H$. Since ν is a normalized fuzzy S-extension of μ, $\nu(0) = 1$. □

Theorem 3.24. *Let μ be a fuzzy hyper BCK-subalgebra of H. Then every maximal fuzzy S-extension of μ is normalized.*

Proof. This follows from the definitions of the maximal and normalized fuzzy S-extensions. □

Definition 3.25. Let μ be a fuzzy subset of H and $\gamma \in [0,1]$. A *fuzzy γ-multiplication* of μ, denoted by μ_γ^m, is defined to be a mapping

$$\mu_\gamma^m : H \to [0,1],\ x \mapsto \mu(x) \cdot \gamma.$$

For any fuzzy subset μ of H, a fuzzy 0-multiplication μ_0^m of μ is clearly a fuzzy hyper BCK-subalgebra of H.

Theorem 3.26. *If μ is a fuzzy hyper BCK-subalgebra of H, then the fuzzy γ-multiplication of μ is a fuzzy hyper BCK-subalgebra of H for all $\gamma \in [0,1]$.*

Proof.

$$\begin{aligned}
\inf_{z \in x \circ y} \mu_\gamma^m(z) &= \inf_{z \in x \circ y} (\mu(z) \cdot \gamma) = \gamma \cdot \inf_{z \in x \circ y} \mu(z) \\
&\geq \gamma \cdot \min\{\mu(x),\ \mu(y)\} = \min\{\mu(x) \cdot \gamma,\ \mu(y) \cdot \gamma\} \\
&= \min\{\mu_\gamma^m(x),\ \mu_\gamma^m(y)\}
\end{aligned}$$

for all $\gamma \in [0,1]$ for all $x, y \in H$. Hence μ_γ^m is a fuzzy hyper BCK-subalgebra of H. □

Theorem 3.27. *For any fuzzy subset μ of H, the following are equivalent:*

(i) *μ is a fuzzy hyper BCK-subalgebra of H.*
(ii) $(\forall \gamma \in (0,1])$ (μ_γ^m *is a fuzzy hyper BCK-subalgebra of H*).

Proof. Necessity follows from Theorem 3.26. Let $\gamma \in (0,1]$ be such that μ_γ^m is a fuzzy hyper BCK-subalgebra of H. Then

$$\begin{aligned}
\gamma \cdot \inf_{z \in x \circ y} \mu(z) &= \inf_{z \in x \circ y} (\mu(z) \cdot \gamma) = \inf_{z \in x \circ y} \mu_\gamma^m(z) \\
&\geq \min\{\mu_\gamma^m(x),\ \mu_\gamma^m(y)\} = \min\{\mu(x) \cdot \gamma,\ \mu(y) \cdot \gamma\} \\
&= \gamma \cdot \min\{\mu(x),\ \mu(y)\}
\end{aligned}$$

for all $x, y \in H$, and so $\inf_{z \in x \circ y} \mu(z) \geq \min\{\mu(x),\ \mu(y)\}$ for all $x, y \in H$ since $\gamma \in (0,1]$. Hence μ is a fuzzy hyper BCK-subalgebra of H. □

Theorem 3.28. *Let μ be a fuzzy subset of H, $\alpha \in [0, \top]$ and $\gamma \in (0, 1]$. Then every fuzzy α-translation μ_α^T of μ is a fuzzy S-extension of the fuzzy γ-multiplication μ_γ^m of μ.*

Proof. For every $x \in H$, we have

$$\mu_\alpha^T(x) = \mu(x) + \alpha \geq \mu(x) \geq \mu(x) \cdot \gamma = \mu_\gamma^m(x),$$

and so μ_α^T is a fuzzy extension of μ_γ^m. Assume that μ_γ^m is a fuzzy hyper BCK-subalgebra of H. Then μ is a fuzzy hyper BCK-subalgebra of H by Theorem 3.27. It follows from Theorem 3.5 that μ_α^T is a fuzzy hyper BCK-subalgebra of H for all $\alpha \in [0, \top]$. Hence every fuzzy α-translation μ_α^T is a fuzzy S-extension of the fuzzy γ-multiplication μ_γ^m. $\square$

The following example shows that Theorem 3.28 is not valid for $\gamma = 0$.

Example 3.29. Consider a hyper BCK-algebra $H := \mathbb{N} \cup \{0\}$. Define a fuzzy set $\mu : H \to [0, 1]$ by

$$\mu(x) := \begin{cases} 0 & \text{if } x = 0, \\ \frac{1}{2} \cdot \left(1 - \frac{1}{x}\right) & \text{if } x \neq 0. \end{cases}$$

Taking $\gamma = 0$, we see that

$$\inf_{z \in x \circ y} \mu_0^m(z) = 0 = \min\{\mu_0^m(x),\ \mu_0^m(y)\}$$

for all $x, y \in H$, that is, μ_0^m is a fuzzy hyper BCK-subalgebra of H. But if we take $x = 2$ and $y = 4$, then

$$\begin{aligned} \inf_{z \in 2 \circ 4} \mu_\alpha^T(z) &= \mu_\alpha^T(0) = \mu(0) + \alpha = \alpha \\ &< \tfrac{1}{4} + \alpha = \min\{\mu(2) + \alpha,\ \mu(4) + \alpha\} \\ &= \min\{\mu_\alpha^T(2),\ \mu_\alpha^T(4)\} \end{aligned}$$

for all $\alpha \in [0, \frac{1}{2}]$. Hence μ_α^T is not a fuzzy S-extension of μ_0^m for all $\alpha \in [0, \frac{1}{2}]$.

The following example illustrates Theorem 3.28.

Example 3.30. Consider a hyper BCK-algebra $H := \mathbb{N} \cup \{0\}$. Define a fuzzy set $\mu : H \to [0, 1]$ by

$$\mu(x) := \begin{cases} \frac{1}{2} & \text{if } x = 0, \\ \frac{1}{2x} & \text{if } x \neq 0. \end{cases}$$

Clearly μ is a fuzzy hyper BCK-subalgebra of H. If we take $\gamma = \frac{1}{2}$, then the fuzzy γ-multiplication $\mu_{\frac{1}{2}}^m$ of μ is given by

$$\mu_{\frac{1}{2}}^m(x) := \begin{cases} \frac{1}{4} & \text{if } x = 0, \\ \frac{1}{4x} & \text{if } x \neq 0. \end{cases}$$

Clearly $\mu^m_{\frac{1}{2}}$ is a fuzzy hyper BCK-subalgebra of H. Also, for any $\alpha \in [0, 0.5]$, the fuzzy α-translation μ^T_α of μ is given by

$$\mu^T_\alpha(x) := \begin{cases} \frac{1}{2} + \alpha & \text{if } x = 0, \\ \frac{1}{2x} + \alpha & \text{if } x \neq 0. \end{cases}$$

Then μ^T_α is a fuzzy extension of $\mu^m_{\frac{1}{2}}$ and μ^T_α is always a fuzzy hyper BCK-subalgebra of H for all $\alpha \in [0, 0.5]$. Hence μ^T_α is a fuzzy S-extension of $\mu^m_{\frac{1}{2}}$ for all $\alpha \in [0, 0.5]$.

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