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DETERMINANTS OF THE LAPLACIANS ON THE n-DIMENSIONAL UNIT SPHERE $\mathbf{S}^n$ $(n = 8,\, 9)$

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Abstract. During the last three decades, the problem of evaluation of the determinants of the Laplacians on Riemann manifolds has received considerable attention by many authors. The functional determinant for the n-dimensional sphere $\mathbf{S}^n$ with the standard metric has been computed in several ways. Here we aim at computing the determinants of the Laplacians on $\mathbf{S}^n$ $(n = 8,\, 9)$ by mainly using ceratin known closed-form evaluations of series involving Zeta function.

1. Introduction and preliminaries

During the last three decades, the problem of evaluation of the determinants of the Laplacians on Riemann manifolds has received considerable attention by many authors including (among others) D'Hoker and Phong [9, 10], Sarnak [14] and Voros [20], who computed the determinants of the Laplacians on compact Riemann surfaces of constant curvature in terms of special values of the Selberg Zeta function. Although the first interest in the determinants of the Laplacians arose mainly for Riemann surfaces, it is also interesting and potentially useful to compute these determinants for classical Riemannian manifolds of higher dimensions, such as spheres. Here, we are particularly concerned with the evaluation of the functional determinant for the n-dimensional sphere $\mathbf{S}^n$ $(n = 8,\, 9)$ with the standard metric.

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For this purpose we need the following definitions. Let $\{\lambda_n\}$ be a sequence such that

$$0 = \lambda_0 < \lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_n \leqq \cdots\,;\ \lambda_n \uparrow \infty \ (n \to \infty); \tag{1.1}$$

henceforth we consider only such nonnegative increasing sequences diverging to infinity. Then we can show that

$$Z(s) := \sum_{n=1}^{\infty} \frac{1}{\lambda_n^s}, \tag{1.2}$$

which is known to converge absolutely in a half-plane $\Re(s) > \sigma$ for some $\sigma \in \mathbb{R}$.

Definition 1 (*cf.* Osgood *et al.* [12]). The determinant of the Laplacian Δ on the compact manifold M is defined to be

$$\det{}'\Delta := \prod_{\lambda_k \neq 0} \lambda_k, \tag{1.3}$$

where $\{\lambda_k\}$ is the sequence of eigenvalues of the Laplacian Δ on M. The sequence $\{\lambda_k\}$ is known to satisfy the condition as in (1.1), but the product in (1.3) is always divergent; so, in order for the expression (1.3) to make sense, some sort of regularization procedure must be used. It is easily seen that, formally, $e^{-Z'(0)}$ is the product of nonzero eigenvalues of Δ. This product does not converge, but $Z(s)$ can be continued analytically to a neighborhood of $s = 0$. Therefore, we can give a *meaningful* definition:

$$\det{}'\Delta := e^{-Z'(0)}, \tag{1.4}$$

which is called the *Functional Determinant of the Laplacian* Δ on M.

Definition 2. The *order* μ of the sequence $\{\lambda_k\}$ is defined by

$$\mu := \inf \left\{ \alpha > 0 \,\middle|\, \sum_{k=1}^{\infty} \frac{1}{\lambda_k^{\alpha}} < \infty \right\}. \tag{1.5}$$

The analogous and shifted analogous Weierstrass canonical products $E(\lambda)$ and $E(\lambda, a)$ of the sequence $\{\lambda_k\}$ are defined, respectively, by

$$E(\lambda) := \prod_{k=1}^{\infty} \left\{ \left(1 - \frac{\lambda}{\lambda_k}\right) \exp\left(\frac{\lambda}{\lambda_k} + \frac{\lambda^2}{2\lambda_k^2} + \cdots + \frac{\lambda^{[\mu]}}{[\mu]\lambda_k^{[\mu]}} \right) \right\} \tag{1.6}$$

and

(1.7)

$$E(\lambda, a) := \prod_{k=1}^{\infty} \left\{ \left(1 - \frac{\lambda}{\lambda_k + a}\right) \exp\left(\frac{\lambda}{\lambda_k + a} + \cdots + \frac{\lambda^{[\mu]}}{[\mu]\,(\lambda_k + a)^{[\mu]}} \right) \right\},$$

where $[\mu]$ denotes the greatest integer part in the order μ of the sequence $\{\lambda_k\}$.

There exists the following relationship between $E(\lambda)$ and $E(\lambda, a)$ (see Voros [20]):

$$E(\lambda, a) = \exp\left(\sum_{m=1}^{[\mu]} \mathcal{R}_{m-1}(-a) \frac{\lambda^m}{m!} \right) \frac{E(\lambda - a)}{E(-a)}, \tag{1.8}$$

where, for convenience,

$$\mathcal{R}_{[\mu]}(\lambda - a) := \frac{d^{[\mu]+1}}{d\lambda^{[\mu]+1}} \left\{ -\log E(\lambda, a) \right\}. \tag{1.9}$$

The shifted series $Z(s, a)$ of $Z(s)$ in (1.2) by a is given by

$$Z(s, a) := \sum_{k=1}^{\infty} \frac{1}{(\lambda_k + a)^s}. \tag{1.10}$$

Formally, indeed, we have

$$Z'(0, -\lambda) = -\sum_{k=1}^{\infty} \log(\lambda_k - \lambda),$$

which, if we define

$$D(\lambda) := \exp\left[-Z'(0, -\lambda) \right], \tag{1.11}$$

immediately implies that

$$D(\lambda) = \prod_{k=1}^{\infty} (\lambda_k - \lambda).$$

In fact, Voros [20] gave the relationship between $D(\lambda)$ and $E(\lambda)$ as follows:

$$\begin{aligned} D(\lambda) = {} & \exp[-Z'(0)]\, \exp\left[-\sum_{m=1}^{[\mu]} \mathrm{FP}Z(m) \frac{\lambda^m}{m} \right] \\ & \cdot \exp\left[-\sum_{m=2}^{[\mu]} C_{-m} \left(\sum_{k=1}^{m-1} \frac{1}{k} \right) \frac{\lambda^m}{m!} \right] E(\lambda), \end{aligned} \tag{1.12}$$

where an empty sum is understood to be nil and the *finite part* prescription is applied (as usual) as follows (*cf.* Voros [20, p. 446]):

(1.13)
$$\mathrm{FP} f(s) := \begin{cases} f(s), & \text{if } s \text{ is not a pole,} \\ \lim\limits_{\epsilon \to 0} \left(f(s+\epsilon) - \dfrac{\text{Residue}}{\epsilon} \right), & \text{if } s \text{ is a simple pole,} \end{cases}$$

and

$$Z(-m) = (-1)^m \, m! \, C_{-m}. \tag{1.14}$$

Now consider the sequence of eigenvalues on the standard Laplacian Δ_n on $\mathbf{S}^n$. It is known from the work of Vardi [19] (see also Terras [18]) that the standard Laplacian Δ_n $(n \in \mathbb{N})$ has eigenvalues

$$\mu_k := k(k+n-1) \tag{1.15}$$

with multiplicity

(1.16)
$$\begin{aligned} q_n(k) := \binom{k+n}{n} - \binom{k+n-2}{n} &= \frac{(2k+n-1)\,(k+n-2)!}{k!\,(n-1)!} \\ &= \frac{2k+n-1}{(n-1)!} \prod_{j=1}^{n-2} (k+j) \quad (k \in \mathbb{N}_0), \end{aligned}$$

where $\mathbb{N}$ denotes the set of positive integers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. From now on we consider the shifted sequence $\{\lambda_k\}$ of $\{\mu_k\}$ in (1.15) by $\left(\frac{n-1}{2}\right)^2$ as a fundamental sequence. Then the sequence $\{\lambda_k\}$ is written in the following simple and tractable form:

$$\lambda_k = \mu_k + \left(\frac{n-1}{2}\right)^2 = \left(k + \frac{n-1}{2}\right)^2 \tag{1.17}$$

with the same multiplicity as in (1.16).

We will exclude the zero mode, that is, start the sequence at $k = 1$ for later use. Furthermore, with a view to emphasizing n on $\mathbf{S}^n$, we choose the notations $Z_n(s)$, $Z_n(s,a)$, $E_n(\lambda)$, $E_n(\lambda,a)$, and $D_n(\lambda)$ instead of $Z(s)$, $Z(s,a)$, $E(\lambda)$, $E(\lambda,a)$, and $D(\lambda)$, respectively.

We readily observe from (1.11) that

$$D_n\left(\left(\frac{n-1}{2}\right)^2\right) = \det{}'\Delta_n, \tag{1.18}$$

where $\det' \Delta_n$ denote the *determinants of the Laplacians* on $\mathbf{S}^n$ $(n \in \mathbb{N})$.

Several authors (see Choi [5], Kumagai [11], Vardi [19], and Voros [20]) used the theory of multiple Gamma functions (see Barnes [1, 2, 3, 4]) to compute the determinants of the Laplacians on the n-dimensional unit sphere $\mathbf{S}^n$ ($n \in \mathbb{N}\}$). Quine and Choi [13] made use of zeta regularized products to compute $\det' \Delta_n$ and the determinant of the conformal Laplacian, $\det(\Delta_{\mathbf{S}^n} + n(n-2)/4)$. Choi and Srivastava [7, 8] and Choi *et al.* [6] made use of some known closed-form evaluations of the series involving Zeta function (see [16, Chapter 3]) for the computation of the determinants of the Laplacians on $\mathbf{S}^n$ ($n = 2, 3, 4, 5, 6, 7$). In the sequel, here, we aim at computing the determinants of the Laplacians on $\mathbf{S}^n$ ($n = 8, 9$) by mainly using ceratin known closed-form evaluations of series involving Zeta function.

2. Series associated with the Zeta functions

A rather classical (over two centuries old) theorem of Christian Goldbach (1690–1764), which was stated in a letter dated 1729 from Goldbach to Daniel Bernoulli (1700–1782), was revived in 1986 by Shallit and Zikan [15] as the following problem:

$$\sum_{\omega \in \mathcal{S}} (\omega - 1)^{-1} = 1, \tag{2.1}$$

where $\mathcal{S}$ denotes the set of all nontrivial integer kth powers, that is,

$$\mathcal{S} := \left\{ n^k \mid n, k \in \mathbb{N} \setminus \{1\} \right\}. \tag{2.2}$$

In terms of the Riemann Zeta function $\zeta(s)$ defined by

$$\zeta(s) := \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1-2^{-s}} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} & (\Re(s) > 1) \\ \displaystyle\frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} & (\Re(s) > 0;\ s \neq 1), \end{cases} \tag{2.3}$$

Goldbach's theorem (2.1) assumes the elegant form (*cf.* Shallit and Zikan [15, p. 403]):

$$\sum_{k=2}^{\infty} \{\zeta(k) - 1\} = 1 \tag{2.4}$$

or, equivalently,

$$\sum_{k=2}^{\infty} \mathcal{F}\left(\zeta(k)\right) = 1, \tag{2.5}$$

where $\mathcal{F}(x) := x - [x]$ denotes the *fractional* part of $x \in \mathbb{R}$. As a matter of fact, it is fairly straightforward to observe also that

$$\sum_{k=2}^{\infty} (-1)^k \, \mathcal{F}\left(\zeta(k)\right) = \frac{1}{2}, \tag{2.6}$$

$$\sum_{k=1}^{\infty} \mathcal{F}\left(\zeta(2k)\right) = \frac{3}{4}, \quad \text{and} \quad \sum_{k=1}^{\infty} \mathcal{F}\left(\zeta(2k+1)\right) = \frac{1}{4}. \tag{2.7}$$

The *Hurwitz* (or *generalized*) *Zeta function* $\zeta(s,a)$ is defined by

$$\zeta(s,a) := \sum_{k=0}^{\infty} (k+a)^{-s} \quad (\Re(s) > 1;\, a \notin \mathbb{Z}_0^-), \tag{2.8}$$

where $\mathbb{Z}_0^-$ denotes the set of nonpositive integers. It is noted that both the Riemann Zeta function $\zeta(s)$ and the Hurwitz Zeta function $\zeta(s,a)$ can be continued meromorphically to the whole complex s-plane except for a simple pole only at $s = 1$ with their residue 1. For easy reference, we recall some properties of $\zeta(s)$ and $\zeta(s,a)$ as in the following lemma.

Lemma 1. *Each of the following identities holds true.*

$$\zeta(s) = \zeta(s,1) = (2^s - 1)^{-1}\, \zeta\left(s, \frac{1}{2}\right) = 1 + \zeta(s,2). \tag{2.9}$$

$$\zeta(s,a) = \zeta(s, n+a) + \sum_{k=0}^{n-1} (k+a)^{-s} \quad (n \in \mathbb{N}). \tag{2.10}$$

$$\zeta(s) = \zeta(s, n+1) + \sum_{k=1}^{n} k^{-s} \quad (n \in \mathbb{N}_0), \tag{2.11}$$

where an empty sum is understood to be nil (as usual) throughout this paper.

$$\zeta(-n) = \begin{cases} -\dfrac{1}{2} & (n=0) \\[2ex] -\dfrac{B_{n+1}}{n+1} & (n \in \mathbb{N}), \end{cases} \tag{2.12}$$

where B_n are the Bernoulli numbers (see [16, Section 1.6]*).*

$$\zeta(-2n) = 0 \quad (n \in \mathbb{N}), \tag{2.13}$$

which are often referred to as the trivial zeros *of $\zeta(s)$.*

$$\zeta'(-2n) = \lim_{\epsilon \to 0} \frac{\zeta(-2n+\epsilon)}{\epsilon} = (-1)^n \frac{(2n)!}{2(2\pi)^{2n}} \zeta(2n+1) \quad (n \in \mathbb{N}). \tag{2.14}$$

$$\zeta'(0) = -\frac{1}{2} \log(2\,\pi). \tag{2.15}$$

$$\lim_{s \to 1} \left\{ \zeta(s,a) - \frac{1}{s-1} \right\} = -\psi(a), \tag{2.16}$$

where ψ denotes the Psi-(or Digamma) *function defined by $\psi(a) := \Gamma'(a)/\Gamma(a)$, Γ being the* Gamma *function.*

$$\psi(n) = -\gamma + \sum_{k=1}^{n-1} \frac{1}{k} \quad (n \in \mathbb{N}) \tag{2.17}$$

and

$$\psi\left(n + \frac{1}{2}\right) = -\gamma - 2\,\log\,2 + 2 \sum_{k=0}^{n-1} \frac{1}{2k+1} \quad (n \in \mathbb{N}_0), \tag{2.18}$$

where γ denotes the Euler-Mascheroni constant defined by

$$\gamma := \lim_{n \to \infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log\, n \right) \cong 0.57721\,56649\,01532\,86060\,6512 \cdots . \tag{2.19}$$

Employing the various methods and techniques used in the vast literature on the subject of the closed-form evaluations series associated with the Zeta functions, Srivastava and Choi (see [16, Chapter 3], [17, Chapter 3], and see also the related references therein) presented a rather extensive collection of closed-form sums of series involving the Zeta functions. For the use in the next section, we recall two general formulas as in the following lemma (see [17, p. 254]).

Lemma 2. *Each of the following identities holds true.*

(2.20)
$$\sum_{k=1}^{\infty} \frac{\zeta(2k,a)}{k+n}\, t^{2k+2n} = \sum_{k=0}^{2n} \binom{2n}{k} \left[\zeta'(-k,a-t) + (-1)^k\, \zeta'(-k,a+t)\right] t^{2\,n-k} - \sum_{\ell=0}^{n-1} \frac{\zeta(-2\ell,a)}{n-\ell}\, t^{2n-2\ell} - 2\,\zeta'(-2n,a) \quad (n \in \mathbb{N}_0;\ \ |t| < |a|)$$

and

(2.21)
$$\begin{aligned}\sum_{k=1}^{\infty} \frac{\zeta(2k+1,a)}{k+n+1}\, t^{2k+2n+2} &= \sum_{k=0}^{2n+1} \binom{2n+1}{k} \Big[\zeta'(-k,a-t) - (-1)^k \zeta'(-k,a+t)\Big]\, t^{2n+1-k} \\ &\quad - \sum_{\ell=1}^{n} \frac{\zeta(1-2\ell,a)}{n-\ell+1}\, t^{2n+2-2\ell} - \frac{t^{2n+2}}{n+1}\, [\psi(2n+2) - \psi(a) + \gamma] \\ &\quad - 2\,\zeta'(-2n-1,a) \qquad (n \in \mathbb{N}_0;\ \ |t| < |a|).\end{aligned}$$

By using the formulas in Lemma 1, we give some special cases of (2.20) and (2.21) for the direct use in the next section as in the following lemma.

Lemma 3. *Each of the following identities holds true.*

(2.22)
$$\sum_{k=4}^{\infty} \frac{\zeta(2k,5)}{k+1}\, 4^{2k+2} = \frac{13408900}{729} - 128\,\zeta(2) - \frac{4096}{3}\,\zeta(4) - 16384\,\zeta(6) - 16\,\log\pi - 32\,\log 2 - 13\,\log 3 + \log 5 + 9\,\log 7.$$

(2.23)
$$\sum_{k=3}^{\infty} \frac{\zeta(2k,5)}{k+2}\, 4^{2k+4} = \frac{1674592}{81} - \frac{4096}{3}\,\zeta(2) - \frac{48\,\zeta(3)}{\pi^2} - 16384\,\zeta(4) - 256\,\log\pi - 512\,\log 2 - 145\,\log 3 + \log 5 + 81\,\log 7.$$

(2.24)
$$\sum_{k=2}^{\infty}\frac{\zeta(2k,5)}{k+3}\,4^{2k+6}=\frac{350752}{9}-16384\,\zeta(2)-1920\,\frac{\zeta(3)}{\pi^2}+\frac{360\,\zeta(5)}{\pi^4}$$
$$-\,4096\,\log\pi+278528\,\log 2+63119\,\log 3+35841\,\log 5+50905\,\log 7.$$

(2.25)
$$\sum_{k=1}^{\infty}\frac{\zeta(2k,5)}{k+4}\,4^{2k+8}=238240-57344\,\frac{\zeta(3)}{\pi^2}+26880\,\frac{\zeta(5)}{\pi^4}-5040\,\frac{\zeta(7)}{\pi^6}$$
$$-\,65536\,\log\pi-131072\,\log 2-12865\,\log 3+\log 5+6561\,\log 7.$$

(2.26)
$$\sum_{k=1}^{\infty}\frac{\zeta(2k+1,\frac{9}{2})}{k+1}\,\left(\frac{7}{2}\right)^{2k+2}=\frac{1729}{60}-\frac{49}{4}\,\gamma+\frac{91}{12}\log 2+6\,\log 3+3\,\zeta'(-1).$$

(2.27)
$$\sum_{k=1}^{\infty}\frac{\zeta(2k+1,\frac{9}{2})}{k+2}\,\left(\frac{7}{2}\right)^{2k+4}=\frac{67669}{320}-\frac{2401}{32}\,\gamma+\frac{65519}{480}\log 2+\frac{70}{8}\,\log 3$$
$$-\,\frac{223}{8}\,\log 5-\frac{343}{8}\,\log 7+\frac{147}{2}\,\zeta'(-1)+2\,\zeta'(-3).$$

(2.28)
$$\sum_{k=1}^{\infty}\frac{\zeta(2k+1,\frac{9}{2})}{k+3}\,\left(\frac{7}{2}\right)^{2k+6}=-\frac{9057797}{11520}-\frac{117649}{192}\,\gamma-\frac{34291656}{4032}\log 2$$
$$+\,\frac{126985}{16}\,\log 3+\frac{106007}{32}\,\log 5+\frac{554631}{32}\,\log 7$$
$$+\,\frac{12005}{8}\,\zeta'(-1)+245\,\zeta'(-3)+\frac{31}{16}\,\zeta'(-5).$$

(2.29)
$$\sum_{k=1}^{\infty}\frac{\zeta(2k+1,\frac{9}{2})}{k+4}\,\left(\frac{7}{2}\right)^{2k+8}=\frac{3968124853}{184320}-\frac{5764801}{1024}\,\gamma-\frac{272293951}{15360}\log 2$$
$$+\,\frac{80311}{128}\,\log 3+\frac{10039}{16}\,\log 5+\frac{823543}{64}\,\log 7$$
$$+\,\frac{823543}{32}\,\zeta'(-1)+\frac{84035}{8}\,\zeta'(-3)+\frac{1029}{2}\,\zeta'(-5)+\frac{255}{64}\,\zeta'(-7).$$

3. The determinants of the Laplacians on $\mathbf{S}^n$ $(n = 8,\ 9)$

Here, by using (1.18) and the results in the previous sections, we are ready to compute the determinants of the Laplacians on $\mathbf{S}^n$ $(n = 8,\ 9)$ as asserted by the following theorem.

Theorem. *The determinants of the Laplacians on* $\mathbf{S}^n$ $(n = 8,\ 9)$ *are given as follows*:

(3.1)
$$\det{}'\Delta_8 = 2^{-\frac{11394059}{483840}}\, 3^{\frac{8761093}{322560}}\, 3^{\frac{1226899}{107520}}\, 7^{\frac{2445697}{46080}} \exp\left[-\frac{178808399}{22118400} - \frac{159763}{26880}\,\zeta'(-1) - \frac{38257}{11520}\,\zeta'(-3) - \frac{49}{240}\,\zeta'(-5) - \frac{1}{1260}\,\zeta'(-7)\right].$$

(3.2)
$$\det{}'\Delta_9 = 2^{-\frac{506669}{2520}}\, 3^{-\frac{14285}{504}}\, 5^{-\frac{147}{10}}\, 7^{-\frac{3073}{90}}\, \pi \cdot \exp\left[\frac{24546672397}{1890} + \frac{16399\,\zeta(3)}{10080\,\pi^2} - \frac{2087\,\zeta(5)}{1920\,\pi^4} + \frac{31\,\zeta(7)}{128\,\pi^6} - \frac{\zeta(9)}{128\,\pi^8}\right].$$

Proof. $\det{}'\Delta_8$: In view of (1.15), the sequence $\{\mu_k\}$ of eigenvalues on the standard Laplacian Δ_8 on $\mathbf{S}^8$ is given as follows: $\mu_k := k(k+7)$ with multiplicity $q_8(k)$. Here we consider the shifted sequence $\{\lambda_k\}$ of $\{\mu_k\}$ by $\left(\frac{7}{2}\right)^2$ as a fundamental sequence. Then the sequence $\{\lambda_k\}$ is written in the following simple and tractable form:

$$\lambda_k = \mu_k + \left(\frac{7}{2}\right)^2 = \left(k + \frac{7}{2}\right)^2 \tag{3.3}$$

with the same multiplicity $q_8(k)$. It is noted that λ_k has the order $\mu = 8/2 = 4$. The involved Zeta function $Z_8(s)$ is given by

$$\begin{aligned} Z_8(s) &= \frac{1}{2520} \sum_{k=1}^{\infty} \frac{q_8(k)}{(k+7/2)^{2s}} \\ &= \frac{1}{2520} \sum_{k=1}^{\infty} \frac{(k+1)(k+2)(k+3)(k+4)(k+5)(k+6)}{(k+7/2)^{2s-1}}. \end{aligned}$$

It is seen that
(3.4)

$$Z_8(s) = \frac{1}{2520}\left\{\left(2^{2s-7}-1\right)\zeta(2s-7) - \frac{35}{4}\left(2^{2s-5}-1\right)\zeta(2s-5)\right.$$
$$+\frac{259}{8}\left(2^{2s-3}-1\right)\zeta(2s-3) - \frac{225}{64}\left(2^{2s-1}-1\right)\zeta(2s-1)$$
$$\left.-\frac{259}{128}2^{2s} - \frac{6993}{128}\left(\frac{2}{3}\right)^{2s} - \frac{32375}{128}\left(\frac{2}{5}\right)^{2s} - \frac{411397}{128}\left(\frac{2}{7}\right)^{2s}\right\}.$$

It is observed that $Z_8(s)$ has simple poles at $s = 1$, 2, 3, and 4 with their respective residues $-\frac{5}{7168}$, $\frac{37}{5760}$, $-\frac{1}{576}$, and $\frac{1}{5040}$. So it is found that

$$\mathrm{FP}Z_8(1) = -\frac{5383489}{40642560} - \frac{5}{3584}\gamma - \frac{5}{1792}\log 2;$$
$$\mathrm{FP}Z_8(2) = -\frac{23520979}{829785600} + \frac{37}{2880}\gamma + \frac{37}{1440}\log 2 - \frac{5}{512}\zeta(3);$$
$$\mathrm{FP}Z_8(3) = -\frac{62186808029}{1143548280000} - \frac{1}{288}\gamma - \frac{1}{144}\log 2 + \frac{259}{2880}\zeta(3) - \frac{155}{3584}\zeta(5);$$
$$\mathrm{FP}Z_8(4) = -\frac{40683977191604}{196994059171875} + \frac{1}{2520}\gamma + \frac{1}{1260}\log 2$$
$$-\frac{7}{288}\zeta(3) + \frac{1147}{2880}\zeta(5) - \frac{635}{3584}\zeta(7).$$

It is seen that

$$\log E_8\left(\frac{49}{4}\right) = -\frac{823543}{92160}\sum_{k=1}^{\infty}\frac{\zeta(2k+1,\frac{9}{2})}{k+4}\left(\frac{7}{2}\right)^{2k} + \frac{117649}{18432}\sum_{k=1}^{\infty}\frac{\zeta(2k+1,\frac{9}{2})}{k+3}\left(\frac{7}{2}\right)^{2k}$$
$$-\frac{88837}{46080}\sum_{k=1}^{\infty}\frac{\zeta(2k+1,\frac{9}{2})}{k+2}\left(\frac{7}{2}\right)^{2k} + \frac{35}{2048}\sum_{k=1}^{\infty}\frac{\zeta(2k+1,\frac{9}{2})}{k+1}\left(\frac{7}{2}\right)^{2k}$$
$$-\frac{364776727}{4423680}\zeta(3) + \frac{6534074197}{2949120}\zeta(5) - \frac{522949805}{524288}\zeta(7) - \frac{2608692457019}{2187000000}.$$

It follows from (1.12) and (1.18) that
(3.5)

$$\det{}'\Delta_8 = D_8\left(\frac{49}{4}\right) = \exp\left[-Z_8'(0)\right]\exp\left[-\sum_{m=1}^{4}\mathrm{FP}Z_8(m)\frac{1}{m}\left(\frac{49}{4}\right)^m\right]$$
$$\cdot\exp\left[-\sum_{m=2}^{4}C_{-m}\left(\sum_{k=1}^{m-1}\frac{1}{k}\right)\frac{1}{m!}\left(\frac{49}{4}\right)^m\right]E_8\left(\frac{49}{4}\right).$$

Now it is easy to compute $\det'\Delta_8$ by putting the results in this and previous sections in (3.5).

$\det'\Delta_9$: Similarly as in obtaining $\det'\Delta_8$, we can compute $\det'\Delta_9$. Therefore we just present the following two essential parts:

$$\begin{aligned}
Z_9(s) &= \sum_{k=1}^{\infty} \frac{q_9(k)}{(k+4)^{2s}} \\
&= \frac{1}{20160}\,[\zeta(2s-8) - 14\,\zeta(2s-6) + 49\,\zeta(2s-4) - 36\,\zeta(2s-2)] \\
&= \frac{1}{20160}\,[\zeta(2s-8,5) - 14\,\zeta(2s-6,5) + 49\,\zeta(2s-4,5) - 36\,\zeta(2s-2,5)]\,;
\end{aligned}$$

$$\begin{aligned}
\log\, E_9(16) = -\frac{1}{20160} &\left\{ \sum_{k=1}^{\infty} \frac{\zeta(2k,5)}{k+4}\, 4^{2k+8} - 14 \sum_{k=2}^{\infty} \frac{\zeta(2k,5)}{k+3}\, 4^{2k+6} \right. \\
&\left. +49 \sum_{k=3}^{\infty} \frac{\zeta(2k,5)}{k+2}\, 4^{2k+4} - 36 \sum_{k=4}^{\infty} \frac{\zeta(2k,5)}{k+1}\, 4^{2k+2} \right\}.
\end{aligned}$$

□

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