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EXTREME PRESERVERS OF FUZZY MATRIX PAIRS DERIVED FROM ZERO-TERM RANK INEQUALITIES

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Abstract. In this paper, we construct the sets of fuzzy matrix pairs. These sets are naturally occurred at the extreme cases for the zero-term rank inequalities derived from the multiplication of fuzzy matrix pairs. We characterize the linear operators that preserve these extreme sets of fuzzy matrix pairs.

1. Introduction and Preliminaries

The linear preserver problems are one of the most active subjects in matrix theory during the past one hundred years, which concern the characterizations of linear operators on matrix spaces that leave certain functions, subsets, relations, etc., invariant. For survey of these types of problems, we refer to the article of Song([7]) and the papers in [6]. The specified frame of problems is of interest both for matrices with entries from a field and for matrices with entries from an arbitrary semiring such as Boolean algebra, nonnegative integers, and fuzzy semiring. It is necessary to note that there are several rank functions over a semiring that are analogues of the classical function of the matrix rank over a field. Detailed research and self-contained information about rank functions over semirings can be found in [1] and [7].

There are some results on the inequalities for the rank function of matrices([1] - [4]). Beasley and Guterman ([1]) investigated the rank inequalities of matrices over semirings. And they characterized the equality cases for some rank inequalities in [2]. The investigation of linear preserver problems of extreme cases of the rank inequalities of matrices over fields was obtained in [4]. The structure of matrix varieties which

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arise as extremal cases in the inequalities is far from being understood over fields, as well as semirings. A usual way to generate elements of such a variety is to find a matrix pairs which belongs to it and to act on this set by various linear operators that preserve this variety. Song and his colleagues ([4]) characterized the linear operators that preserve the extreme cases of column rank inequalities over semirings. There are some results on the linear operators that preserve zero-term rank([5]).

In this paper, we characterize linear operators that preserve the sets of matrix pairs which satisfy extreme cases for the zero term rank inequalities for the matrix multiplications over fuzzy semiring.

A *semiring* $\mathcal{S}$ consists of a set $\mathcal{S}$ and two binary operations, addition and multiplication, such that:

- $\mathcal{S}$ is an Abelian monoid under addition (identity denoted by 0);
- $\mathcal{S}$ is a semigroup under multiplication (identity, if any, denoted by 1);
- multiplication is distributive over addition on both sides;
- $s0 = 0s = 0$ for all $s \in \mathcal{S}$.

A semiring is called *antinegative* if the zero element is the only element with an additive inverse.

Let $\mathcal{R}$ be the field of reals, let $\mathcal{F}$=$\{\alpha \in \mathcal{R} \mid 0 \leq \alpha \leq 1\}$ denote a subset of reals. Define $a+b = \max\{a,b\}$ and $a \cdot b = \min\{a,b\}$ for all a,b in $\mathcal{F}$. Then $(\mathcal{F}, +, \cdot)$ is called a *fuzzy semiring.*

It is straightforward to see that the fuzzy semiring is commutative and antinegative.

Let $\mathcal{M}_{m,n}(\mathcal{F})$ denote the set of all $m \times n$ matrices with entries in a fuzzy semiring $\mathcal{F}$. If $m = n$, we use the notation $\mathcal{M}_n(\mathcal{F})$ instead of $\mathcal{M}_{n,n}(\mathcal{F})$. We call a matrix in $\mathcal{M}_{m,n}(\mathcal{F})$ as a *fuzzy matrix.*

Throughout we assume that $m \leq n$ for any $m \times n$ matrix. The matrix I_n is the $n \times n$ identity matrix, $J_{m,n}$ is the $m \times n$ matrix of all ones, $O_{m,n}$ is the $m \times n$ zero matrix. We omit the subscripts when the order is obvious from the context and we write I, J, and O, respectively. The matrix $E_{i,j}$, called a *cell*, denotes the matrix with exactly one nonzero entry, that being a one in the (i,j) entry. Let R_i denote the matrix whose i^{th} row is all ones and is zero elsewhere, and C_j denote the matrix whose j^{th} column is all ones and is zero elsewhere. A *line* of a matrix A is a row or a column of A.

A matrix $A \in \mathcal{M}_{m,n}(\mathcal{S})$ has *zero-term rank* k (denoted by $z(A) = k$) if the least number of lines needed to include all zero elements of A is equal to k.

Example 1.1. Let

$$A = \begin{pmatrix} \frac{1}{2} & \frac{2}{3} & \frac{3}{4} \\ \frac{2}{3} & 0 & \frac{4}{5} \\ \frac{1}{2} & \frac{3}{4} & \frac{2}{3} \end{pmatrix}, \quad B = \begin{pmatrix} \frac{1}{2} & 0 & 1 \\ \frac{2}{3} & \frac{3}{4} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $z(A) = 1$ and $z(B) = 3$ for $A, B \in M_3(\mathcal{F})$. □

Let $\mathcal{F}$ be a fuzzy semiring. An operator $T : \mathcal{M}_{m,n}(\mathcal{F}) \to \mathcal{M}_{m,n}(\mathcal{F})$ is called *linear* if $T(X + Y) = T(X) + T(Y)$ and $T(\alpha X) = \alpha T(X)$ for all $X, Y \in \mathcal{M}_{m,n}(\mathcal{F})$, $\alpha \in \mathcal{F}$.

We say an operator, T, *preserves* a set $\mathcal{P}$ if $X \in \mathcal{P}$ implies that $T(X) \in \mathcal{P}$, or, if $(X, Y) \in \mathcal{P}$ implies that $(T(X), T(Y)) \in \mathcal{P}$ when $\mathcal{P}$ is a set of ordered pairs.

The matrix $X \circ Y$ denotes the *Hadamard* or *Schur product*, i.e., the (i, j) entry of $X \circ Y$ is $x_{i,j} y_{i,j}$.

An operator T is called a (P, Q)*-operator* if there exist permutation matrices P and Q such that $T(X) = PXQ$ for all $X \in \mathcal{M}_{m,n}(\mathcal{F})$, or, if $m = n$, $T(X) = PX^tQ$ for all $X \in \mathcal{M}_{m,n}(\mathcal{F})$.

It was shown in [2] and [4] that linear preservers for extremal cases of classical matrix inequalities over fields are types of (U, V)-operators where U and V are arbitrary invertible matrices. On the other side, linear preservers for various rank functions over semirings have been the object of much study during the last years, see for example [6], in particular zero-term rank was investigated in the last few years, see for example [5]. Also the fuzzy matrix has been the object of much research, see for examples [6] and [8].

We say that the matrix A *dominates* the matrix B if and only if $b_{i,j} \neq 0$ implies that $a_{i,j} \neq 0$, and we write $A \geq B$ or $B \leq A$.

If A and B are matrices and $A \geq B$ we let $A \backslash B$ denote the matrix C where

$$c_{i,j} = \begin{cases} 0 & \text{if } b_{i,j} \neq 0; \\ a_{i,j} & \text{otherwise.} \end{cases}$$

2. Zero-term rank inequality of fuzzy matrix multiplication

In this section, we obtain inequalities for the zero-term rank of matrix multiplication over fuzzy semiring. We also show that these inequalities are exact and best possible.

If $\mathcal{S}$ is a field then there is a usual rank function $\rho(A)$ for any matrix $A \in \mathcal{M}_{m,n}(\mathcal{S})$. The behavior of the usual rank function ρ with respect to matrix multiplication is given by the following inequalities: *Sylvester's law [2]*:

$$\rho(A) + \rho(B) - n \leq \rho(AB) \leq \min\{\rho(A), \rho(B)\},$$

where A and B are conformal matrices with coefficients from a field.

But for the zero-term rank, the Sylvester's law does not hold. Consider the following example:

Example 2.1. For $n > 2$, we take $A = C_1,\ B = R_1$ in $\mathcal{M}_n(\mathcal{F})$. Then we have $z(A) + z(B) - n = z(C_1) + z(R_1) - n = n - 2$, but $z(AB) = z(C_1R_1) = 0$. This pair(A,B) breaks the left inequality of the Sylvester's law for zero-term rank. Moreover, for

$$E = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix},$$

we have $z(EF) = z(\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}) = 2$ but $\min\{z(E), z(F)\} = 1$. This pair(E,F) breaks the right inequality of the Sylvester's law for zero-term rank. $\square$

Proposition 2.2. *Let $\mathcal{F}$ be a fuzzy semiring. For $A \in \mathcal{M}_{m,n}(\mathcal{F})$, $B \in \mathcal{M}_{n,k}(\mathcal{F})$ one has that*

$$0 \leq z(AB) \leq \min\{z(A) + z(B), k, m\}.$$

These bounds are exact and the best possible for $n > 2$.

Proof. The lower bound follows from the definition of the zero-term rank function. In order to show that this bound is exact and the best possible, let us consider the family of matrices: for each pair (r, s), $0 \leq r \leq \min\{m, n\}$, $0 \leq s \leq \min\{k, n\}$, we take $A_r = J\backslash(\Sigma_{i=1}^r E_{i,i})$, $B_s = J\backslash(\Sigma_{i=1}^s E_{i,i+1})$ if $s < \min\{k, n\}$ and $B_s = J\backslash(\Sigma_{i=1}^{s-1} E_{i,i+1} + E_{s,1})$ if

$s = \min\{k, n\}$. Then $z(A_r) = r$, $z(B_s) = s$ by definition and if $n > 2$ then A_rB_s does not have zero elements by antinegativity. Thus $z(A_rB_s) = 0$.

The upper bound follows directly from the definition of zero-term rank and from the antinegativity of $\mathcal{F}$. In order to show that this bound is exact and the best possible, let us consider the family of matrices: for each pair (r, s), $0 \leq r \leq \min\{m, n\}$, $0 \leq s \leq \min\{k, n\}$, we take $A_r = J \backslash (\Sigma_{i=1}^{r} R_i)$ and $B_s = J \backslash (\Sigma_{i=1}^{s} C_i)$. Then A_rB_s has zero elements in the first r rows and first s columns, which implies that $z(A_rB_s) \leq z(A_r) + z(B_s)$. $\square$

3. Extreme preservers of fuzzy matrix pairs derived from zero-term rank inequalities

In this section, we construct the sets of matrix pairs that arise as the extremal cases in the inequalities of zero-term rank of matrix multiplications shown in Proposition 2.2. And we obtain characterizations of the linear operators that preserve these extreme sets of matrix pairs.

Lemma 3.1. *Let $\mathcal{F}$ be a fuzzy semiring, $T : \mathcal{M}_{m,n}(\mathcal{F}) \to \mathcal{M}_{m,n}(\mathcal{F})$ be an operator which maps lines to lines and is defined by $T(E_{i,j}) = E_{\sigma(i,j)}$, where σ is a permutation on the set $\{(i, j) \mid i = 1, 2, \cdots, m; j = 1, 2, \cdots, n\}$. Then T is a (P, Q)-operator.*

Proof. Since no combination of u rows and v columns can dominate J where $u + v = m$ unless $v = 0$ (or if $m = n$, if $u = 0$) we have that either the image of each row is a row and the image of each column is a column, or $m = n$ and the image of each row is a column and the image of each column is a row. Thus, there are permutation matrices P and Q such that $T(R_i) \leq PR_iQ$ and $T(C_j) \leq PC_jQ$ or, if $m = n$, $T(R_i) \leq P(R_i)^tQ$ and $T(C_j) \leq P(C_j)^tQ$. Since each cell lies in the intersection of a row and a column and T maps nonzero cells to nonzero (weighted) cells, it follows that $T(E_{i,j}) = PE_{i,j}Q$, or, if $m = n$, $T(E_{i,j}) = PE_{j,i}Q = P(E_{i,j})^tQ$. Thus T is a (P, Q)-operator. $\square$

Lemma 3.2. *Let $T : \mathcal{M}_{m,n}(\mathcal{F}) \to \mathcal{M}_{m,n}(\mathcal{F})$ be a $(P, Q) -$ operator. Then T preserves all zero-term rank.*

Proof. Let π be a permutation corresponding P, μ be a permutation corresponding Q.

Let $z(A) = r$ with $A \in M_{m,n}(F)$. Then there are r lines such that those r lines cover all zero entries of A, say $r_1, r_2, \cdots, r_s, c_1, c_2, \cdots, c_t$ with $s + t = r$, covers all zero entries of A. Then all the zero entries of PAQ are covered by $r_{\pi(1)}, r_{\pi(2)}, \cdots, r_{\pi(s)}$ and $c_{\mu(1)}, c_{\mu(2)}, \cdots, c_{\mu(t)}$ with $s + t = r$.

Thus $z(PAQ) = r$ and hence $z(T(A)) = r$. Therefore T preserves zero-term rank r, and hence T preserves all zero-term rank. □

Theorem 3.3. *Let $\mathcal{F}$ be a fuzzy semiring and $T : \mathcal{M}_{m,n}(\mathcal{F}) \to \mathcal{M}_{m,n}(\mathcal{F})$ be a linear operator. Then the following are equivalent:*

1. *T is bijective.*
2. *T is surjective.*
3. *There exists a permutation σ on $\{(i,j) \mid i = 1,2,\cdots,m; j = 1,2,\cdots,n\}$ such that $T(E_{i,j}) = E_{\sigma(i,j)}$.*

Proof. That 1) implies 2) and 3) implies 1) is straightforward. We now show that 2) implies 3).

We assume that T is surjective. Then, for any pair (i,j), there exists some X such that $T(X) = E_{i,j}$. Clearly $X \neq O$ by the linearity of T. Thus there is a pair of indexes (r,s) such that $X = x_{r,s}E_{r,s} + X'$ where (r,s) entry of X' is zero and the following two conditions are satisfied: $x_{r,s} \neq 0$ and $T(E_{r,s}) \neq O$. Indeed, if in the contrary for all pairs (r,s) either $x_{r,s} = 0$ or $T(E_{r,s}) = O$ then $T(X) = O$ which contradicts with the assumption $T(X) = E_{i,j} \neq O$. Hence

$$T(x_{r,s}E_{r,s}) \leq T(x_{r,s}E_{r,s}) + T(X \setminus (x_{r,s}E_{r,s})) = T(X) = E_{i,j}.$$

Thus $x_{r,s}T(E_{r,s}) = T(x_{r,s}E_{r,s}) \leq E_{i,j}$, and hence $T(x_{r,s}E_{r,s}) = \alpha E_{i,j}$ for a certain $\alpha \in \mathcal{F}$. That is, there is some permutaion σ on $\{(i,j) \mid i = 1,2,\cdots,m; j = 1,2,\cdots,n\}$ such that for some scalars $b_{i,j}$, $T(E_{i,j}) = b_{i,j}E_{\sigma(i,j)}$. We now only need show that the $b_{i,j}$ are all units. Since T is surjective and $T(E_{r,s}) \not\leq E_{\sigma(i,j)}$ for $(r,s) \neq (i,j)$, there is some α such that $T(\alpha E_{i,j}) = E_{\sigma(i,j)}$. But then, since T is linear, $T(\alpha E_{i,j}) = \alpha T(E_{i,j}) = \alpha b_{i,j}E_{\sigma(i,j)} = E_{\sigma(i,j)}$. That is, $\alpha b_{i,j} = 1$, or $b_{i,j}$ is a unit. But 1 is the only unit over fuzzy semiring. Thus $b_{i,j} = 1$ and $T(E_{i,j}) = E_{\sigma(i,j)}$. □

Now, we construct the sets of matrix pairs that arise as the extremal cases in the inequalities of zero-term rank of matrix multiplications listed at Proposition 2.2 in section 2:

$$\mathcal{Z}_1(\mathcal{F}) = \{(X,Y) \in M_n(F)^2 | z(XY) = 0\},$$
$$\mathcal{Z}_2(\mathcal{F}) = \{(X,Y) \in M_n(F)^2 | z(XY) = z(X) + z(Y)\}.$$

Now, we characterize the linear operators that preserve set $\mathcal{Z}_1(\mathcal{F})$ in the following theorem.

Theorem 3.4. *Let $\mathcal{F}$ be a fuzzy semiring, $T : M_n(\mathcal{F}) \to M_n(\mathcal{F})$ be a linear surjective map. Then T preserves the set $\mathcal{Z}_1(\mathcal{F})$ if and only if T is a nontransposing $(P, P^t) - operators$, where P is a permutation matrix.*

Proof. By Theorem 3.3 we have that $T(E_{i,j}) = E_{\sigma(i,j)}$ for all i, j, $1 \le i \le m$, $1 \le j \le n$, σ is a permutation on the set of pairs (i, j).

Let us show that T maps lines to lines. Suppose that the images of two cells are in the same line, but the cells are not in the same line, say, $E_{i,j}, E_{i,k}$ are the cells such that $T^{-1}(E_{i,j}), T^{-1}(E_{i,k})$ are not in the same line.

Let us consider $A = T^{-1}(J \setminus R_i)$. Thus there are no zero rows of A since T is a permutation on the set of cells and not all elements of the preimage of the i'th row of J lie in one row by the choice of i. Hence AJ does not have zero elements by the additions and multiplications in $\mathcal{F}$ and $z(AJ) = 0$. Thus $(A, J) \in Z_1(\mathcal{F})$ as far as $(T(A), T(J)) = (T(T^{-1}(J \setminus R_i)), T(J)) = (J \setminus R_i, T(J)) \notin Z_1(F)$, since $z((J \setminus R_i)(T(J))) = z(J \setminus R_i) = 1$, a contradiction to the assumption that T preserves the set $\mathcal{Z}_1(\mathcal{F})$.

Moreover, since σ is bijective on the set of pairs (i, j) and each row intersects each column and does not intersect rows, T maps rows to rows and columns to columns, or , it is also possible that T maps all rows to columns and all columns to rows. Thus there are permutation matrices P and Q such that $T(E_{i,j}) = PE_{i,j}Q$, or, $T(E_{i,j}) = PE_{j,i}Q = P(E_{i,j})^tQ$, i.e, T is a (P, Q)-operator where P and Q are permutation matrices of order n. Let us show that $Q = P^t$. Assume on the contrary that $QP \neq I$. Thus there exist indexes i, j such that QP transforms i'th column into j'th column. In this case we take matrices $A = J \setminus (E_{1,1} + \cdots + E_{1,n}) + E_{1,i}$, $B = J \setminus E_{j,1}$. Thus AB has no zero elements, i.e, $z(AB) = 0$. However, the $(1, j)^{th}$ element of $T(A)T(B)$ is zero, i.e, $z(T(A)T(B)) \neq 0$, which contradicts the fact that T preserves $\mathcal{Z}_1(\mathcal{F})$. This contradiction implies that $QP = I$. Thus $Q = P^t$. Hence T is a $(P, P^t) - operator$.

Conversely (P, Q)-operators preserve zero term rank by Lemma 3.2. Thus $(P, P^t) - operators$ preserve the set $\mathcal{Z}_1(\mathcal{F})$. □

Example 3.5. Let $\mathcal{F}$ be a fuzzy semiring, $T : M_4(\mathcal{F}) \to M_4(\mathcal{F})$ be a surjective map such that $T(X) = PXQ$, where

$$P = I_4, Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{and } QP = Q \neq I_4.$$

Then T maps rows to themselves. But T maps 1_{st} column of X to itself, 2_{nd} column of X to 3_{rd} column, 3_{rd} column of X to 2_{nd} column and 4_{th} column of X to 4_{th} column.
Let $A = J_4 - C_3$ and $B = R_2$ in $M_4(\mathcal{F})$. Then $AB = J_4$. Thus $z(AB) = 0$ and hence $(A, B) \in Z_1(\mathcal{F})$. But $T(A) = J_4 - C_2$ and $T(B) = R_2$ in $M_4(\mathcal{F})$. Then $T(A)T(B) = O_4$ and hence $z(T(A)T(B)) = 4$. Thus $(T(A), T(B)) \notin Z_1(\mathcal{F})$.

This example shows that linear operator T does not preserve $Z_1(\mathcal{F})$, because T is not $(P, P^t) - operator$. □

Now, we characterize the linear operator that preserves the extremal set $\mathcal{Z}_2(\mathcal{F})$.

Theorem 3.6. *Let $\mathcal{F}$ be a fuzzy semiring and $T : M_n(\mathcal{F}) \to M_n(\mathcal{F})$ be a linear surjective map. Then T preserves the set $\mathcal{Z}_2(\mathcal{F})$ if and only if T is a nontransposing $(P, P^t) - operator$, where P is a permutation matrix.*

Proof. By Theorem 3.3 we have that $T(E_{i,j}) = E_{\sigma(i,j)}$ for all i, j, $1 \le i \le m$, $1 \le j \le n$, σ is a permutation on the set of pairs (i, j).

Let us show that T maps lines to lines. Suppose that the images of two cells are not in the same line, but the cells are in the same line, say, $E_{i,j}, E_{i,k}$ are the cells such that $T(E_{i,j}), T(E_{i,k})$ are not in the same line.

Note that $z((J \setminus R_i)J) = z(J \setminus R_i) = 1 = 1+0 = z(J \setminus R_i)+z(J)$. Thus $(J \setminus R_i, J) \in \mathcal{Z}_2(\mathcal{F})$. Now, $T(J \setminus R_i$ has no zero rows by above argument, and $T(J) = J$ over $M_n(\mathcal{F})$. Hence $T(J \setminus R_i)T(J) = T(J \setminus R_i)J = J$ on $M_n(\mathcal{F})$ by the sums and products over $\mathcal{F}$. Thus $z(T(J \setminus R_i)T(J)) = 0$.

On the other hand, $(T(J\setminus R_i), T(J)) \notin \mathcal{Z}_2(\mathcal{F})$. This contradiction shows that T maps lines to lines.

It follows from Lemma 3.1 that T is a (P,Q)-operator where P and Q are permutation matrices of order n.

To show that transposition operator does not preserve $\mathcal{Z}_2(\mathcal{F})$, it suffices to take the pair of matrices $A = J \setminus R_i$, $B = J \setminus C_i$. Consider $A = J\setminus R_1$, $B = J\setminus C_1$. Then $z(AB) = 2 = 1+1 = z(A)+z(B)$, hence $(A,B) \in \mathcal{Z}_2(\mathcal{F})$. But $z(A^tB^t) = z(J) = 0$ and $z(A^t) = z(B^t) = 1$. Hence $z(A^tB^t) \neq z(A^t) + z(B^t)$, that is, $(A^t, B^t) \notin \mathcal{Z}_2(\mathcal{F})$. Thus $(T(A), T(B)) \notin \mathcal{Z}_2(\mathcal{F})$. This show that transposing operator does not preserve $\mathcal{Z}_2(\mathcal{F})$. Therefore T is a nontransposing (P,Q)-operator.

Let us show that $Q = P^t$ now. Assume on the contrary that $QP \neq I$. Thus there exists indexes i, j such that QP transforms i'th column into j'th column. But then consider $A = J \setminus C_i$, $B = R_i$. We have $z(AB)$ $= z(0) = n$ $=1 + (n-1) = z(A) + z(B)$. Hence $(A,B) \in \mathcal{Z}_2(\mathcal{F})$. But $z(AQPB) = z((J\setminus C_j)R_i) = z(J) = 0$ and $z(AQP) + z(B) = 1+(n-1) = n$. Thus $(T(A), T(B)) \notin \mathcal{Z}_2(\mathcal{F})$, which contradicts the fact that T preserves $\mathcal{Z}_2(\mathcal{F})$. This contradiction implies that $QP = I$, and $Q = P^t$. We have T is a nontransposing (P, P^t)-operator.

Conversely, (P,Q)-operator preserves zero term rank by Lemma 3.2. Thus (P, P^t)-operator preserve the set $\mathcal{Z}_2(\mathcal{F})$. $\square$

Example 3.7. Let $\mathcal{F}$ be a fuzzy semiring, $T : M_4(\mathcal{F}) \to M_4(\mathcal{F})$ be a surjective map such that $T(X) = PXQ$, where

$$P = I_4, Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{and } QP = Q \neq I_4,$$

which is the same linear operator as in Example 3.5.
Let $E = J_4 - C_2$ and $F = R_2$ in $M_4(\mathcal{F})$. Then $EF = O_4$. Thus $z(EF) = 4 = 1+3 = z(E) + z(F)$ and hence $(E,F) \in Z_2(\mathcal{F})$. But $T(E) = J_4 - C_3$ and $T(F) = R_2$ in $M_4(\mathcal{F})$. Then $T(E)T(F) = J_4$ and hence $z(T(E)T(F)) = 0 \neq 4 = 1+3 = z(T(E)) + z(T(F))$. Thus $(T(E), T(F)) \notin Z_2(\mathcal{F})$.

This example shows that linear operator T does not preserve $Z_2(\mathcal{F})$, because T is not $(P, P^t) - operator$. $\square$

As a concluding remark, we have characterized the linear operator T that preserve the extreme sets of the zero-term rank inequalities of the matrix multiplications over fuzzy semiring as $(P, P^t) - operator$.

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