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Teachers Solving Mathematics Problems: Lessons from their Learning Journeys[1]

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This paper reports on the learning journeys in mathematical problem solving of 21 teachers enrolled on a Masters of Education course entitled Discrete Mathematics and Problem Solving. It draws from the reports written by these teachers on their personal journeys: the commonalities and differences among them in terms of how they look at their own problem solving experiences, what language they employ in talking about problem solving, and what impact the course has on their views about problem solving. One particular aspect of problem solving instruction, a pedagogical innovation called the Practical Worksheet, is addressed in some detail. These graduate students are full-time mathematics teachers with at least two years of classroom experience. They include primary

[1] A draft version of the article was presented at the 46th Korean National Meeting on Mathematics Education held at Soongsil University, Dongjak-gu, Seoul 156–743, Korea; April 2, 2011 (*cf.* Tay, Quek, Dindyal, Leong, & Toh, 2011).

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and secondary teachers.

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1. INTRODUCTION

Our research effort to introduce mathematical problem solving to all students recognises the crucial role teachers play in teaching problem solving successfully. An assumption of our research framework is that teachers must have had experience of genuine mathematical problem solving in order to teach it to their students. By genuine problem solving (as contrasted to solving routine problems), we mean that the teachers would not have a direct means to the solution of the problem; they would have to search for a solution, to retreat from dead ends, to ponder the perplexities and to attack the problem again. In doing so, the teachers would have had to deal not only with the cognitive impasse but also the affective—the frustration, disappointment, or exhilaration that is part-and-parcel of the problem solving endeavour. In our collaboration with schools to implement a problem solving curriculum (see Leong, Toh, Quek, Dindyal & Tay (2009); Leong, *et al*. (submitted)), teachers involved in the problem underwent training in solving non-routine problems. Our observations suggested that there was more to simply putting the teachers through the problem solving process. While the teachers were "doing" the problem, what were they thinking and feeling, and how do all these—doing, thinking and feeling—impact on the teaching of problem solving? In our discussions with these teachers, we became aware that the teachers needed a language to talk about problem solving, in the effort to describe their thoughts, feelings and behaviours when solving problems. To this end, we engaged a "convenient" group of practising teachers in genuine problem solving, had them report on that experience in the form of a learning journey, and tried to learn from their reported experience. The use of writing to learn mathematics is beginning to be recognized as a pedagogical technique (see Sterrett, 1990; Meier & Rishel, 1998; Crannell, LaRose & Ratliff, 2004). Before we begin our interpretations of the teachers' reports to learn from them, we will set out the background to the present inquiry and thereby reveal to readers our interpretative assumptions.

2. BACKGROUND

There are many issues that make the implementation of problem solving in the school curriculum difficult. Schoenfeld (2007) reported that, although in the 1980s, problem solving did become a fashionable term; its implementation in most classrooms was a "travesty". He provided three reasons for the narrow view of problem solving in American classrooms:

1. Problem-solving research was still in its infancy when the NCTM's Agenda for Action was published in 1980 and so there was little to guide curriculum development then.
2. Teachers are generally conservative and resist change, even when it is desirable. Tremendous effort and time is needed to make a significant change in the right direction
3. The mechanics of the publishing industry militate against change. Just as in the automobile industry, the current product is largely derived from the previous product and significant model changes are prohibitively expensive and thus come very rarely.

Schoenfeld reported that commercial textbook publishers merely made trivial modifications to their earlier drill-oriented texts and repackaged them as "problem solving" editions. In practice, "problem solving" in the primary schools was restricted to one- or two-step word problems such as

John had $5.00. He bought a pen for $.39 and a notebook for $2.19. How much money does John have left?

"In sum, "problem solving" in American classrooms in the 1980s came to mean "solving (simple) word problems.""

The three reasons hold true also in Singapore. However, one cannot fault the Singaporean schools for the level of difficulty of the word problems attempted in class. The problems in the given worksheets and school examinations are very hard and this is because the Primary School Leaving Examination (PSLE), the national high stakes examination at Primary 6 (for students aged 12 years), has very difficult problems. Below we give an example from the 2009 PSLE which raised uproar amongst anxious parents (Teo, 2009):

> Jim bought some chocolates and gave half of it to Ken. Ken bought some sweets and gave half of it to Jim. Jim ate 12 sweets and Ken ate 18 chocolates. The ratio of Jim's sweets to chocolates became 1:7 and the ratio of Ken's sweets to chocolates became 1:4. How many sweets did Ken buy?

Many schools respond to the difficult problems in PSLE by making their students work on many similar problems, thereby hoping to 'routinise' the problems. Skewed teaching towards the exams often results. Some pare down a complete problem solving model to just a few specific heuristics (Tay *et al*., 2007). On the other hand, some ignore heuristics altogether. A recent study that involved mathematics lessons from three Primary schools and three Secondary schools, where a total of 106 lessons from the Primary classes and 53 lessons from the Secondary classes were observed, video-recorded and analysed by Teong *et al*. (2009), reported the following (p. 84):

> The teachers' approaches did not reflect an emphasis in the process of heuristics. The mostly closed routine word problems were not fully explicated in terms of the model proposed by Pólya (1957) and the range of heuristics used was limited. They generally read the problems, executed the solution and checked the answers. There was very little dwelling on the exploration or the planning aspect of the solutions. … The emphasis appeared to be more to address the skills and procedures [emphasis added] needed to solve problems than to tackle fresh problems anew where students have more chance of grappling with understanding and thinking about how to solve the problems.

Both the 'pared-down to heuristics' method and the 'skills and procedures' method, which often in Singapore comes down to the 'model method' (see for example, Ferrucci, Kaur, Carter & Yeap, 2008), however tend finally to a practise-and-practise approach to solving non-routine problems.

And yet the Singapore difficulty level is instructive in that it shows that schools and textbooks will move in tandem with, or more likely, be driven by the summative assessment of high stakes examinations. Based on the proposal of Holton, Anderson and Thomas (1997) for teaching problem solving in their schools, the New Zealand Ministry of Education developed a national numeracy project which emphasised a problem solving approach and that has now been introduced to the majority of primary schools in the country. However, success is so far limited to the primary level (Ministry of Education New Zealand, 2006) as high stakes examinations have blunted the problem solving approach in mathematics classes at the secondary level.

> First of all I think that you have to separate primary from secondary schools. There is a sense in which most primary schools are now using a problem solving approach and are being successful … Moving further into the secondary school, there are certainly some good teachers who use problem solving especially to introduce new topics but many teachers at that level feel intimidated by the exam system (we have national exams in each of the last three years of school) and so teach at that level in a more 'didactic' manner. (Holton, personal communication, 7 December 2006)

In view of the overhanging 'shadow' of national examinations in most countries, Toh, Quek, Leong, Dindyal and Tay (2009) add the following to Schoenfeld's three reasons for the generally unsuccessful implementation of problem solving in classrooms: the lack of suitable assessment for the processes, and not only the products, of problem solving. They juxtapose this lack of suitable assessment of problem solving processes against the unyielding demands of 'teaching to the test' imposed by high-stakes examinations. They acknowledge the unyielding nature of the 'shadow' and propose instead a solution that takes high stakes examinations into its consideration. Together with their companion article (Dindyal, Toh, Quek, Leong & Tay, 2009) they describe a project by their team to implement a problem solving module into the mathematics curriculum of a secondary school in Singapore.

Taking advantage of the fact that the school was ‘independent’ and adopted a modular system in its curriculum, the team ensured that assessment of the processes of problem solving was valued and was ‘seen to be valued’. To this end, a “Practical Worksheet” was designed on which the students would write their solutions. The worksheet contains sections explicitly guiding the students to use Polya’s stages and problem solving heuristics to solve a mathematics problem. A condensed format of the Practical Worksheet, with all the guiding instructions, is shown below. In the actual worksheet, each section takes up a page, and students may use more of each of sections I, II and III.

I	**Understand the problem (UP)** (You may have to return to this section a few times. Number each attempt to understand the problem accordingly as Attempt 1, Attempt 2, etc.) (a) Write down your feelings about the problem. Does it bore you? scare you? challenge you? (b) Write down the parts you do not understand now or that you misunderstood in your previous attempt. (c) Write down your attempt to understand the problem; and state the heuristics you used.
II	**Devise a plan (DP)** (You may have to return to this section a few times. Number each new plan accordingly as Plan 1, Plan 2, etc.) (a) Write down the key concepts that might be involved in solving the question. (b) Do you think you have the required resources to implement the plan? (c) Write out each plan concisely and clearly.
III	**Carry out the plan (CP)** (You may have to return to this section a few times. Number each implementation accordingly as Plan 1, Plan 2, etc., or even Plan 1.1, Plan 1.2, etc. if there are two or more attempts using Plan 1.) (i) Write down in the *Control* column, the key points where you make a decision or observation, for *e.g.*, go back to check, try something else, look for resources, or totally abandon the plan. (ii) Write out each implementation in detail under the *Detailed Mathematical Steps* column.
IV	**Check and Expand (C/E)** (a) Write down how you checked your solution. (b) Write down your level of satisfaction with your solution. Write down a sketch of any alternative solution(s) that you can think of (c) Give one or two adaptations, extensions or generalisations of the problem. Explain succinctly whether your solution structure will work on them.

Figure 1. Instructions on a Practical Worksheet (from Toh, et al., 2009)

A rubric (shown to the students), which clearly indicated that marks were allocated to applying Polya's 4-phase approach, making use of heuristics, and exhibiting 'control' during problem solving, was used to grade classwork, homework and a final exam. The initial pilot run of one elective class was conducted by the researchers, and the results were so promising that the school adopted the module and methods into their curriculum.

In addition, the team also describe a key component of their implementation, which is teacher professional development (Leong, Toh, Quek, Dindyal & Tay, 2009). This acknowledges the second reason of Schoenfeld (2007) as a major stone to be turned. They describe the many adaptations to the module that they had to make to get the teachers, who would finally be the ones teaching the module, on board. The team reports that teacher development in this area is still a work-in-progress (Leong *et al*., submitted).

An aspect of particular concern in teacher development for the teaching of problem solving is the ability of the teacher himself in solving problems. Leikin and Kawass (2005) described what happened to teachers' plans for teaching an unfamiliar mathematics problem after they themselves had attempted to solve the problem and when the teachers were shown a videotape of students successfully solving the problem. They recorded changes for a significant number of the teachers with regard to aspects such as goals and position in the curriculum, management of learning, mathematical challenge activities and prompting questions, and expectations of pupils' success, strategies, difficulties and ability to provide explanations. Lavy and Shriki (2010) state that for their institution, "one of the main guidelines used in the design of the [pedagogical method] course relates to the importance of self-experience of each method prior to discussing its possible implementation in the classroom." Silver and Marshal (1990) stated that although it is desirable that a teacher be an expert in problem solving, this is not a sufficient condition. On the other hand, this would still imply that a teacher who struggles in problem solving would also struggle in teaching problem solving. Altogether, these research and opinions suggest strongly that teacher professional development in teaching problem solving should include experiences of personal problem solving to enhance the teacher's own abilities and her understanding of the students who would be attempting similar problems.

Focusing then on this aspect of teacher development towards the teaching of problem solving, we ask: How can we enable teachers to experience problem solving, as genuinely as possible? What are their reactions to this experience and what do they learn from it? And, in what ways do they think their classroom practice or teaching of problem solving have been influenced by this experience? To obtain a feel for the answers to these questions, we looked into the self-reports or learning logs of a class of graduate students enrolled on a masters course—Discrete Mathematics and Problem Solving—taught by one of the authors of this paper. Complementing another course taught earlier in the same programme which discussed issues and directions in mathematical problem solving research, this Discrete Mathematics and Problem Solving course immersed the students in actual problem solving and required them to 'fit the research into the reality' themselves. The 13-week, 39-hour course included a final term paper entitled 'My Personal Journey in Mathematical Problem Solving', where the teachers were required to critique any model of problem solving in the light of their own experiences in solving mathematics problems, such as in the use of heuristics and value of extending a problem, in at least 5000 words.

We shall describe the course and its participants briefly and discuss the varied teachers' responses to learning problem solving, including the Practical Worksheet. We will also discuss the effects of personal perspectives and goals on the teachers' preferred mathematical pedagogy, especially with regards to problem solving. Finally, we will attempt to draw implications for mathematics teacher education and the teaching of problem solving in schools.

3. THE PROBLEM SOLVING CLASS

3.1. The programme and course descriptions

The Masters in Education programme has a specialisation in Mathematics Education. The part-time programme consists of 10 courses of 39 hours each, with an option to substitute two of the courses for a dissertation.

The objectives of the course Discrete Mathematics and Problem Solving stated in the course description were to extend combinatorial techniques beyond the high school level, to learn how to use a problem solving model, to develop skills in problem posing for mathematics competitions, and to develop an appreciation of and interest in discrete mathematics, in particular, graph theory, by showing its importance in solving practical problems. The content covered included Polya's model of problem solving, Schoenfeld's problem solving framework, topics in counting, and topics in graph theory. Assessment was by means of two in-class quizzes and a final term report entitled 'My Personal Journey in Mathematical Problem Solving'. The instruction for the report is as follows:

> Write a report titled: My personal journey in mathematical problem solving. You should discuss at least one model of mathematical problem solving and several heuristics. For the purposes of this module, ALL examples must be from Discrete Mathematics.

The classes were conducted for 13 sessions of 3 hours each over a semester. The class met once a week from 4 pm to 7 pm.

3.2. The students

There were 21 students in the class. All of them were teachers in the schools, with seven at primary level and the rest at secondary level. All of them had at least 2 years teaching experience, with about half having 10 years or more. Most of the teachers had degrees in Mathematics while those who did not, had degrees in Science subjects. Most of the students had taken another course taught earlier in the same programme which discussed issues and directions in mathematical problem solving research.

4. DATA AND ANALYSIS

We relied on the teachers' self-reports to understand problem solving from their perspective. They transformed their "remembered experiences" (Campbell, 1988) into text, with its attendant risk of distorting them wittingly or unwittingly. We recognized that it will be difficult or even impossible to uncover the assumptions that the participants worked with in writing these reports. Neither would we be able to check on the severity of their deliberate distortions in reporting their 'lived" experiences. These reports, being a part of their course assessment, might be assumed to have been done with the intention to get as good a grade as the participants think they deserved for the work they put into. Our focus was on the "stories" the teachers told of their lived and remembered experiences, on the meanings and interpretations they attached (Kramp 2004, cited in Clandinin, 2007) to the problem solving experiences. Jerome Bruner argued in his *The Narrative Construction of Reality* (Bruner, 1991) and *What Is a Narrative Fact*? (Bruner, 1998) that narrative truth has a place alongside scientific or experimental truths.

With these cautions in mind, we interpreted these teachers' written reports with the unavoidable risk of distortion. We were also guarded against our biases from our own interpretative assumptions. We want the ideas or claims we have derived from the self-reports tested against empirical evidence that other researchers may gather.

We performed analytic induction (Goetz & LeCompte, 1984; Taylor & Bogdan, 1988) by first having two of the authors each read through all the reports. They then identified patterns and views that are shared or that are different. These were classified under four broad categories, viz., how much about Polya's problem solving model was learnt; the usefulness of the Practical Worksheet; the participants' descriptions of their personal feelings as they solved the problems; and the impact on their own teaching of problem solving.

Another researcher was then given the task of reading through all the reports again to quantify in some manner the responses of the teachers under each of the four categories.

Finally, the first two researchers read through all the reports a second time to glean out suitable quotes which give a richer insight into the teachers' responses under each of the four categories. In searching for categories, the researchers were vigilant to weed out idiosyncratic comments so as to identify the defining features that put the responses into the categories (Hammersley, 2008).

In the following, we present the analysis of the data under the four categories, beginning, where relevant, with a 'quantitative' description. Suitable quotes taken from the teachers' reports are then cited. (All the names are pseudonyms.) Our observations and comments are interspersed within the quotes.

4.1. Problem solving model

The teachers were rated on a scale of 1 = 'learnt nothing' to 5 = 'learnt a lot' by the researcher according to her impression of what they wrote about Polya's problem solving model. The table below shows the distribution of the scores.

Score	1	2	3	4	5
Number of teachers	1	0	4	5	11

Many of the teachers reported that they had learnt a lot from the course about how Polya's problem solving model works. Of interest is the low level of understanding of the Polya's model or other suggested systematic approaches to problem solving that they started off with.

Matt: The first time I heard of this model was in NIE [National Institute of Education]. Having been through [compulsory military service], I had been tuned to detect 'big name titles' and hence did not think much about this so and so. Along my teaching path, I heard it mentioned … but … still not very receptive. This module has opened my eyes with the explanation in detail.

Zak: In the course of this module, I was introduced to Polya's Problem Solving model - this is not a new model, of course, because I remembered vaguely being taught this model back in NIE … Unfortunately, the questions that were posed then were of the primary school difficulty and as such the rigour of the model did not really quite etch on my mind.

Pat: It is only recently that I learnt that I can use heuristics in 'understand the problem' phase … I no longer need to just tell my pupils to read, underline and reread only. So far, I have found 'Act it out' and 'Draw a diagram' heuristics useful in helping me grasp the problem situations.

The teachers recounted their familiarisation with and understanding of the model.

Zak: Polya problem solving model ... is built with such rigour that one requires patience to follow the steps. In short, current generations are too occupied with having immediate solution and to attempt the problem straight on, that thinking about your own thinking just seemed to be irrelevant.

Minah: Hence I was really glad that this module required me to solve mathematics problems using the mathematical problem solving model so that I will understand the procedure thoroughly.

Mary: This module has made me more aware of my own thinking and of consciously using the problem solving model.

Lee: I became more cognizant of the intricacies within each problem-solving stage and the need for systematic thinking, from understanding the problem, to checking, extension and generalisation. I also gradually became aware of the heuristics that can be applied at each stage and the important role of metacognition.

Kathy: Reflecting on the whole problem solving process, I realized that 'Looking Back' is a stage that must be consciously done even for an experienced mathematics classroom teacher like myself.

4.2. Practical Worksheet

The Practical Worksheet is an instructional tool to get the students to step through the recommended stages of problem solving. The worksheet contains sections explicitly guiding the students to use Polya's stages and problem solving heuristics to solve a mathematics problem. The teachers were rated on a scale of 1 = 'dislike a lot' to 5 = 'appreciate a lot' by the researcher according to her impression of what they wrote about the Practical Worksheet. 13 teachers did not mention the Practical Worksheet at all in their reports, probably because the instructions for the report did not explicitly mention the Practical Worksheet. The remaining 8 teachers who did write about the worksheet could then be safely assumed to have strong opinions about it. The table below shows the distribution of the scores of the remaining 8 teachers.

Score	1	2	3	4	5
Number of teachers	1	0	0	2	5

The Worksheet serves to steer the "novice" problem solvers through the desired processes which the novice would otherwise find challenging to adhere to.

Minah: The Practical Worksheet … initially forced me to think what I am thinking and to pen down my thoughts.…

The worksheet actually guides me to go through the various stages, be conscious of my thoughts and to be reflective. When we are stuck in one stage, we should be able to go back to the previous stage instead of being pressurised to go on to the next stage.

Jac: After being forced to verbalise my thoughts and having gone through the Practical Worksheets, I had understood the significance of [self-monitoring] and I was able to see myself more aware of the questions I asked myself when I couldn't move on in the problem.

Cris: As I worked through this second problem, I discovered that the Practical Worksheet was useful in helping me systematically think through my plan of actions and also to spot my gaps in thinking. So, to my frustration, I could not give up on the [problem] as each stage provided a clue to where my mistakes were and I had to rectify it and keep moving on.

Mel: Through the use of the Practical Worksheet, I felt that the process of trying to find the solution was more systematic. The worksheet restricted me into following certain protocols that were useful in helping me find the correct solution … The Practical Worksheet adopted Polya’s four step problem solving strategy and it has really helped me develop my thoughts and plan of attack.

This “use of force” on the novice problem solver to step through the process of problem solving by means of a template-like worksheet has been a matter of debate when we presented it at conferences. Nevertheless, the teachers here who have used the Practical Worksheet, despite their initial abhorrence of it, grew to appreciate it in developing their mental discipline towards problem solving.

Cris: The usefulness of the Practical Worksheet lies in its ability to push the problem solvers to the limit by its devious cycle of ‘understanding the problem - devise a plan - carry out the plan’.

Pat: Initially, I disliked using the Practical Worksheet but as I reflected, I could appreciate its usefulness in showing me what each phase of the Polya's model entails and when to incorporate Schoenfeld’s framework.

…

The worksheet helped me enter into the “problem-solving mode” relatively faster than my previous experience.

Chin: When I first started to solve problems using the Practical Worksheets provided, I find it a chore and difficult to make full use of it. I have to force myself to go through the different stages. However, after using it to solve a few problems, I realise that it can be very useful as it guides me through the four stages of problem solving.

At least one teacher however never took to the Practical Worksheet.

Jolene: There are times that I know how to solve the problem, but was bounded by the limitations of the worksheet, and it became more of a hindrance than an aid.

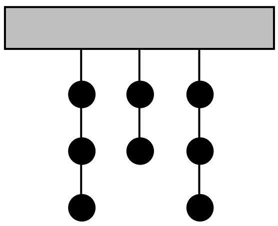
It is to be noted however that the Worksheet provides for the very real possibility of a person solving the problem without strict adherence to the format. This is done by allowing the problem solver to immediately write down the solution in Stage 3 (Carry out the Plan). However the instructions require that the student must fill up Stage 4 (Check and Expand).

4.3. Experiencing personal problem solving

The teachers wrote vividly about their emotions when solving the problems, *e.g.*, of frustration and struggle in grappling and re-searching for a solution; satisfaction, confidence-boosting and happiness on reaching a solution.

In the following, we highlight the emotional roller-coasters of two teachers. The problem that they worked on was the first in which the class used the Practical Worksheet. It is a problem from the American International Mathematics Examination and is shown below.

In a shooting match, eight clay targets are arranged in two hanging columns of three each and one column of two, as pictured.



A marksman is to break all eight targets according to the following rules:

(1) The marksman first chooses a column for which a target is to be broken.
(2) The marksman must break the lowest remaining unbroken target in the chosen column.

If these rules are followed, in how many different orders can the eight targets be broken?

Wynn: When I first encountered the problem, I was quite perplexed by it as I have not solved related problems before … My first plan was to attempt to list down all the orders … I had not even considered all the possible orders in Case 1 when I decided to abandon Plan 1 because there were just too many orders and the solution became tiresome and time-consuming. I also became confused as I continued listing as I was not sure whether all the orders had been considered.

…

I went back to Phase 2 and started to devise Plan 2. I decided to use a tree diagram … By the time I attempted to draw the tree diagram involving three columns of 2 targets each, I had to abandon the plan because it was just too tedious. At this juncture, I had already spent about three hours on the problem but heading to no way. Frustration started to build up and I began to lose my temper on my children. I knew I had to take a break and forget about the problem. But the problem kept on surfacing in my mind and I even dreamt about it that night. I could not sleep well. I knew that I had to persevere to find the problem, if not the problem would continue to "haunt" me. I woke up early the next morning to continue my "problem-solving journey". With my children all asleep, it was the best time to work on the problem with minimum disturbance.

…

I went on to devise Plan 3. I continued to apply the heuristics of starting with simpler problems but instead of using tree diagrams, I used the solutions obtained in Plan 2 for the simpler problems to find the solutions for the more complicated problems … FINALLY, the problem was solved. There was a great sense of achievement and satisfaction initially but as I analysed the solution further, I found that it could not be considered a perfect solution. What happened as the number of targets and the number of columns increased? … I attempted to identify a generalised pattern … but failed to do so. I was quite disappointed but I consoled myself that at least, I had persevered and arrived at the solution.

…

As discussed in the literature review, Schoenfeld (1985) has identified resources as the foundation upon which problem-solving performance is built … I truly agree with Schoenfeld because a few days after solving the above-mentioned problem, I was reading the topic on "permutations and combinations" and realised that this new mathematical knowledge can enable me to obtain a … more efficient solution … Therefore I started to devise Plan 4 … I considered the eight targets to be the eight entries in a row … Using the solution in Plan 4, I was able to make a generalisation of the problem.

…

Polya's model focuses mainly on the cognitive processes of problem solving but my own experiences told me that the affective domains were equally important. This is consistent with research findings. I feel that to be a successful problem-solver, belief and perseverance are very important … the problem-solving process could last for hours, or even days.

Pat: When I first read the problem, I felt quite detached from the problem. Probably, it was due to my personality. I am not the sort who likes to play shooting games … in order to feel more connected to the problem, I imagined that I was going to play a shooting game in a fun fair and I had to first figure out the rules … I interpreted the rules as first choosing any of the columns to shoot and I could proceed to another column only if I had finished shooting all the targets in the chosen column … I figured out that there were 6 possible cases. How could it be so simple? I had an uneasy feeling that I could have interpreted the question wrongly.

…

In retrospect, I felt that I fitted Schoenfeld's description of a poor problem solver who had focused on surface features … I lacked the metacognition to monitor if I was on the right track … I kept returning to my wrong way of interpretation and believed that it was right … Finally, I gave up and decided to ask my lecturer for help. It was then I realised the mistake I had made … I felt stupid but I was glad that I had done one right thing at least, that is, I had named the columns and the targets correctly.

…

Saturday is a day I do not need to work. It is a good day for problem solving. I started working at 10 a.m. … I decided to draw a tree diagram … It was indeed tedious to draw all the tree diagrams. I felt like giving up. However, as I observed how similar cases gave similar results, I believed I could use all these cases I had solved to solve the original problem … [By a recursion method] Behold, it was such a delight to know that by Addition Principle, by considering all possible cases, I finally found the required number of orders to shoot the eight targets was 560! … I felt that all my hard work had paid off. Looking back, it was my *belief* that had helped me to persevere especially during the "carry out the plan" phase and by exercising *control*, I verified my conjectures and stopped unproductive actions … I spent almost 5 hours going through the problem solving process. Though I was happy and relieved that I had accomplished something, I was not satisfied with my solution. There must be an easier way, I believe.

…

[Alternative Solution 1] … By Multiplication Principle, the required number of ways was $56 \times 10 \times 1$ which equals 560. I took about two hours this time. By then it was already 5 plus in the evening. I decided to celebrate by having a good dinner.

…

[Alternative Solution 2] … Having two successful attempts, my confidence was boosted. I was unwilling to let go of the problem as I was curious why the answer stated behind the book was

$$\frac{8!}{3!2!3!} \ldots$$

I also revised the concepts of Combinations and Permutations as I felt the need to build up my *resources* to solve the problem … Though the solution looked short, I had spent another 3 hours to figure it out … Since I had spent 10 hours on the problem, I was reluctant to think of any extension. Nevertheless, I did a simple one eventually. I called it a day finally, satisfied that I had obtained 3 solutions to the problem and having experienced a range of emotions moving from hopelessness, feelings of frustration to stronger feelings of determination, to joy, relief and feelings of confidence.

Their stories also suggest that the teachers can describe their own mental states at different times in the process of solving a problem. We do not argue here about the existence of mental states and whether such introspective accounts are legitimate objects for (empirical) study. Rather, we assume the reader can identify with the mental state of being "perplexed", and of "realizing that one is facing an impasse and having to cycle back to understand the problem or to devise a plan (in Schoenfeld's words, exercising "control" during problem solving).

The teachers who found the problems quite beyond their ability expressed the new-found empathy that they had with their weaker students.

Mark: When relating this experience in problem solving with that of my pupils, I found an uncanny similarity … I felt incompetent [at not being able] to perform the tasks assigned to me and this feeling was amplified after the test. If you are able to perform a task well, you will be motivated to do it. The opposite is true as well.

Pat: However through this [demoralising] experience … helped [me] understand the frustrations [my] pupils faced and helped [me] be more empathetic … I think teachers not only need to teach problem solving, we need to equip our students with some coping mechanism to overcome the emotional stress.

Chin: Having gone through this module, I am able to understand how some of my students feel when they are solving problems: anxiety, lost and confusion.

The cyclic nature of problem solving (see Carlson & Bloom, 2005) is apparent. Some accounts contain descriptions of false starts, sudden enlightenment, or "Aha" type of realization that one has done something fruitful or rash (stupid) that led one to a surprise solution of problem or a painful sudden awareness of one misunderstanding the problem all the while. This cycling back could be a result encouraged by the use of the Practical Worksheet and the training that went with the course.

It is apparent from the students' written work that they have the language to talk about the nature and process of problem solving. We infer this from descriptions of how they went about solving the problems, references to the processes involved, and remarks on how the experience has affected them as problem solvers and as teachers. Having the language of problem solving is an essential first step in the professional development of mathematics teachers to teach problem solving to their own students. One particular aspect must certainly be the language to describe one's own problem solving experience. With the language in hand, teachers will be able to communicate with students to whom they are teaching problem solving.

4.4. Impact on teaching

All but two of the teachers reported that their experience of problem solving influenced their teaching in two ways—in shaping their beliefs of mathematics teaching and in the approach to teaching problem solving.

Some teachers related how their views of mathematics, particularly mathematical problem solving, have changed, and in turn, influenced the way they teach mathematics.

Matt: I find myself attempting the techniques and principles in class and when I explain to students why they … cannot solve a problem, I find myself tempted to explain the model - something I never thought I would do.

Jo: There needs to be ample … opportunities for 'imitation and practice' … the teachers have to role-model the process when solving problems in class.

Cindy: When they have problems with any of the questions, they will surface in the class and … in the past, I will almost immediately explain the solution. Now, I try to lead them to solving the problem themselves and I verbalise all my thoughts. I also ask them to read the question for information.

Jac: After being forced to verbalise my thoughts and having gone through the Practical Worksheets, I had understood the significance of [self-monitoring] and I was able to see myself more aware of the questions I asked myself when I couldn't move on in the problem.

…

… as a teacher, I have to build up necessary questioning techniques to scaffold and facilitate classroom discussions.

Minah: After witnessing the benefits of the looking back stage, I feel that teachers should actively encourage the students to come up with alternative methods or a generalisation … it also acts as a tool [for] a more accurate assessment of the student's understanding of that problem.

Mary: I realised that 'Looking Back' is a stage that must be consciously done even for an experienced mathematics teacher like me … I find this consolidation stage is often over-looked and not impressed upon the students sufficiently by the teachers.

Lee: These learning points on the problem solving process can be translated into teaching strategies. Self-questioning can be infused into teaching … students should be taught how to generate new problems from existing problems, and investigate if their methods continue to be applicable.

Pat: The impact on me was evident as I found myself repeatedly telling my pupils in class, "Spend time understanding the problem before doing it, otherwise all your efforts might be wasted."

Other teachers mentioned how the experience of problem solving led them to explore ways of teaching problem solving in their own classroom.

Jo: I would prefer Polya's model and using the Practical Worksheet as compared to journal writing. By the time the students settle down to write the journal, the feelings or emotions evoked when solving the problem would have subsided … However in [the Practical Worksheet], the thinking process is part of the assessment and ... gives ... the most accurate and up-to-date picture of the thinking and emotional process of the child during problem solving.

Minah: … give [students] a simplified version of the Practical Worksheet … ensure that they make use of the worksheet often.

Mary: … all that I have gained in this problem solving journey will be put into practice in my own teaching in class.

Lee: Drill and practice coupled with problems that students are motivated to solve would be a potent combination to produce superior performance in problem solving.

Some teachers felt strongly that teachers' personal experience in problem solving is crucial for understanding problem solving and then teaching it.

Minah: Hence I was really glad that this module required me to solve mathematics problems using the mathematical problem solving model so that I will understand the procedure thoroughly.

…

… recent research … focused on identifying the attributes of the [successful] problem solver … As a teacher, I can easily read up on these attributes. But when I have gone through the whole mathematical problem solving process myself, I am more aware of these attributes ... and it will be easier for me to relate to my students regarding these attributes.

Pat: Learning about problem solving is indeed very different from experiencing it yourself. Never have I felt so intensely about the problem-solving experiences before.

Sue: I believe that in order for me to facilitate effectively the problem solving process with my pupils, I (as the teacher) need to go through it myself. Through this exercise, I found new insights that I was oblivious to previously and which I could adapt and adopt in my teaching.

A teacher went as far as to support the contention of Toh, Quek, Leong, Dindyal and Tay (2009) that the lack of success of any attempt to teach problem solving within the curriculum lay in the failure to assess problem solving processes at national high stakes examinations.

Cris: The reality as it is, examination drives instruction and students' learning attitudes. Anything that is not tested would be sidelined till there is time! A Practical Worksheet that demands our students to develop desirable problem-solving processes would meet its slow natural death against the ominous examinations, unless of course it becomes mandatory!

5. DISCUSSION AND CONCLUSION

This paper highlights some observations from the self-reports of a class of teachers as they attempted to solve mathematics problem. The authors are aware that the weaknesses of data gleaned from self-reports (*e.g.*, possible distortions by participants, knowingly or unknowingly). With this caution in mind, there are nevertheless, a number of interesting observations that can be starting points for a more thorough investigation.

Enabling teachers to experience mathematical problem solving as genuinely as possible appears to have a critical role in the professional development of teachers in general and in the teaching of problem solving in school in particular. By “genuine” we mean that they now face a problem unlike those that they encounter in school mathematics. These school mathematics "problems" would be "exercises" to the practising teachers. Many of the teachers enrolled on the Discrete Mathematics course might not have opportunities to engage in "genuine" problem solving in their study of mathematics. In particular, the group of school teachers in the course reported discomfort, lack of confidence, puzzlement, determination, disappointment, frustration, joy, satisfaction and accomplishment.

The Practical Worksheet is apparently useful in bringing about awareness of the problem solving process. It is clear from the accounts that the Worksheet has a critical role in cutting a groove (so to speak) at the initial stages of learning problem solving, in which the learner could roll along in later attempts to solve problems. For example, one student wrote, "I must now do a check and extend." The Worksheet brings to the forefront of consciousness the essential stages to pass through in mathematical problem solving. This instructional crutch may be dispensed with once the desired mental discipline towards problem solving is learned.

Literature on problem solving attempts to distinguish between problem and exercise to students, and in the well-cited definition of a problem, a problem is one where there is no easy access to a procedure to solve it (*e.g*., Schoenfeld, 1985, p. 11). We feel however that the affective aspect of something being a problem—to feel "frustrated" or "challenged"—has not been sufficiently emphasized. The teachers' reports show that their feelings had much impact on their problem solving efforts. This suggests that instruction in problem solving must take into consideration affect. One outcome would be that suitable/accessible problems should be used. Considerations of encouragement and time would also be sensible in this light.

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