A Study on the Stability of Circular Thin Plates by Nonlinear Analysis

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비선형 해석에 의한 원형박판의 안정성 연구

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Abstract

본 연구에서는 원형 박판 구조물의 안정성에 대하여 해석 하였다. 임계하중은 하중을 점차적으로 증가하여 구조물이 파괴가 발생하여 안정성을 상실 하는 상태에서 가장 작은 하중을 의미한다. 판구조의 안정성을 임계하중의 크기로 기초를 두고 해석 하였다. 원형 박판구조의 차분해석은 일반 판구조와 같으므로 최근에 많은 연구의 대상이 되어왔다. 차분법은 복잡한 구조물에서도 물론, 다양한 경계조건을 포함하는 문제에 이르기까지 효과적인 수치방법이다. 본 연구에서는 기본 박판구조의 지배방정식을 유도하고 차분화 하여 직접적으로 접근하였다. 원 둘레 의 지점이 힌지 받침으로, 등분포 하중을 받고 있는 박판을 기하학적 비선형 해석으로 수행하여 원형 박판의 처짐 및 응력을 해석 하였다.

Keywords : Circular plate, Polar coordinate, Stability, Finite Difference Method, Iteration method

1. Introduction

Plates are rigid planar structures, typically made of monolithic material, whose depths are small with respect to their other dimensions. A multidirectional dispersal of applied loads characterizes the way loads are carried to supports in plate structures. The advent of modern reinforced concrete has made the plate among the most common of all building element. Included among the more familiar examples of plates are roofs and windows of buildings, table tops, manhole covers and side panels. For instance, glass plates are widely used in modern buildings. With larger and larger sizes of glass plates being used in high-rise buildings, it is becoming important to be able to predict accurately the response of glass plates under lateral loads representing wind pressure.

Plates may be classified into two groups: thin plates with large deflections and thick plates. We shall consider only large deflections of thin plates, a simplification consistent with the magnitude of deformation commonly found in plate structures. A analysis of thin plates subjected to lateral loads are most commonly accomplished by using a linear theory in which one assumes that the lateral displacements or deflections due to the loads are small. However, this linear analysis will not be valid if deflection of the plate is large. As the deflections of the plate become large, the deformations in the middle surface of the plate increase in such a manner that errors in solutions using linear theory grow simultaneously. These errors become so large that

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linear solutions containing displacements and stresses totally disagree with experimental data. Thus, the nonlinear plate for theory developed by von Karman is used for the analysis of thin plates[5,7]. The nonlinear differential equations of the plate were first derived by von Karman in 1910 following initial work on large deflections by Kirchhoff. He coupled the effects of in-plane force with out-of-plane deflections. Closed form solutions for this theory, even for simple rectangular plates, are known. In 1936, Kaiser solved a uniformly laterally loaded, simply supported, square plate problem[1]. He solved the problem by using the finite difference method and supported solutions with experimental results using a thin square plate. A large number of research works has been reported in the area of finite difference of plate flexure problems such as large displacement analysis, plate vibrations, stress concentration problems etc[2,3,4,6]. The primary objective of this study is to develop a mathematical model which can analyze thin, circular plates which is subjected to lateral pressure. Complete finite difference representation of the circular plate is involved. Based on the von Karman theory of plates and using the finite difference method, we developed a computer program which determines the deflections and stresses in simply supported circular thin plates.

2. Nonlinear Differential Equations of Circular Plate

The governing differential equations of a thin plate under lateral load q per unit area in Cartesian coordinates is as follows:

$$D\Delta^4 w(x,y) = q(x,y) + t\,L(w,\phi) \tag{1}$$

$$\Delta^4\phi = -\frac{E}{2}L(w,w) \tag{2}$$

where the differential operator Δ^4 and L, applied to w and ϕ, are defined as

$$\Delta^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2\partial y^2} + \frac{\partial^4}{\partial y^4}$$

$$L(w,\phi) = \frac{\partial^2 w}{\partial x^2}\frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - 2\frac{\partial^2 w}{\partial x\partial y}\frac{\partial^2\phi}{\partial x\partial y}$$

w(x,y) is the transverse deflection of the plate, ϕ is the membrane stress function, q(x,y) is the intensity of the downward distributed load, and D is the stiffness of the plate related to the modulus of elasticity E, the thickness of the plate t, and the Poisson's ratio μ as $D = E\,t^3/12(1-\mu^2)$. Polar coordinates sometimes simplify problems with curved or circular boundaries. As the plate is circular, it is most suitable to use polar coordinates to express the governing equation and the finite differences. A point(x,y) in Cartesian coordinates is represented by the polar coordinates(ρ,α), where the transformation is given by

$$x = \rho\cos\alpha, \quad y = \rho\sin\alpha$$

or

$$\rho = \sqrt{x^2+y^2}, \qquad \alpha = \tan^{-1}\frac{y}{x}$$

Thus

$$\frac{\partial\rho}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{\rho} = \cos\alpha$$

$$\frac{\partial\rho}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} = \frac{y}{\rho} = \sin\alpha$$

$$\frac{\partial\alpha}{\partial x} = -\frac{y/x^2}{1+(y/x)^2} = -\frac{y}{\rho^2} = -\frac{\sin\alpha}{\rho}$$

$$\frac{\partial\alpha}{\partial y} = \frac{1/x}{1+(y/x)^2} = \frac{x}{\rho^2} = \frac{\cos\alpha}{\rho}$$

Consider a function $w(\rho,\alpha)$ in which ρ and α are functions of x and y. The partial derivatives of $w(\rho,\alpha)$ with respect to x and y are transformed to those in respect to ρ and α by

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial\rho}\frac{\partial\rho}{\partial x} + \frac{\partial w}{\partial\alpha}\frac{\partial\alpha}{\partial x} = \frac{\partial w}{\partial\rho}\cos\alpha - \frac{\partial w}{\partial\alpha}\frac{\sin\alpha}{\rho}$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial\rho}\frac{\partial\rho}{\partial y} + \frac{\partial w}{\partial\alpha}\frac{\partial\alpha}{\partial y} = \frac{\partial w}{\partial\rho}\sin\alpha + \frac{\partial w}{\partial\alpha}\frac{\cos\alpha}{\rho}$$

and

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial\rho^2}\cos^2\alpha + \frac{\partial w}{\partial\rho}\frac{\sin^2\alpha}{\rho^2}$$

$$-2\frac{\partial^2 w}{\partial\alpha\partial\rho}\frac{\sin\alpha\cos\alpha}{\rho}+2\frac{\partial w}{\partial\alpha}\frac{\sin\alpha\cos\alpha}{\rho^2}$$

$$\frac{\partial^2 w}{\partial y^2}=\frac{\partial^2 w}{\partial\rho^2}\sin^2\alpha+\frac{\partial w}{\partial\rho}\frac{\sin^2\alpha}{\rho^2}$$
$$+2\frac{\partial^2 w}{\partial\alpha\partial\rho}\frac{\sin\alpha\cos\alpha}{\rho}-2\frac{\partial w}{\partial\alpha}\frac{\sin\alpha\cos\alpha}{\rho^2}$$

$$\frac{\partial^2 w}{\partial x\partial y}=\frac{1}{2}sin2\alpha\frac{\partial^2 w}{\partial\rho^2}-\frac{1}{\rho^2}cos2\alpha\frac{\partial w}{\partial\alpha}-\frac{1}{2\rho^2}sin2\alpha$$
$$\frac{\partial^2 w}{\partial\alpha^2}-\frac{1}{2\rho}\sin2\alpha\frac{\partial w}{\partial\alpha}+\frac{1}{\rho}cos2\alpha\frac{\partial^2 w}{\partial\alpha\partial\rho}$$

Additional derivatives can be obtained by further differentiation of these basic expressions. The differential operator Δ^4 and L, applied to w and ϕ, are defined as

$$\Delta^4 w=\Delta^2(\Delta^2 w)$$
$$=\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\alpha^2}\right)$$
$$\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\alpha^2}\right)w \quad (3)$$

$$L(w,\phi)=\frac{\partial^2 w}{\partial\rho^2}\left(\frac{1}{\rho}\frac{\partial\phi}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\alpha^2}\right)+\left(\frac{1}{\rho}\frac{\partial w}{\partial\rho}+\frac{1}{\rho^2}\right.$$
$$\left.\frac{\partial^2 w}{\partial\alpha^2}\right)\frac{\partial^2\phi}{\partial\rho^2}-2\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial\phi}{\partial\alpha}\right)\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial w}{\partial\alpha}\right) \quad (4)$$

When the plate is subjected to forces symmetrical to the origin, the deflection and membrane stress will be independent of α, and Equation(1) and(2) become

$$\Delta^4 w=\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)w$$
$$=\frac{\partial^4 w}{\partial\rho^4}+\frac{2}{\rho}\frac{\partial^3 w}{\partial\rho^3}-\frac{1}{\rho^2}\frac{\partial^2 w}{\partial\rho^2}+\frac{1}{\rho^3}\frac{\partial w}{\partial\rho}$$

$$L(w,\phi)=\frac{1}{\rho}\frac{\partial^2 w}{\partial\rho^2}\frac{\partial\phi}{\partial\rho}+\frac{1}{\rho}\frac{\partial w}{\partial\rho}\frac{\partial^2\phi}{\partial\rho^2}$$

From this expression $L(w,w)$ can be obtained by substituting w for ϕ. The moments M_r, M_α and $M_{r\alpha}$ then have the same values as the moments M_x, M_y, and M_{xy} at the same point, and by substituting $\alpha=0$. When the plate is subjected to forces symmetrical to the origin, the deflection will be independent of α and $M_{r\alpha}$ and Q_α are vanished. The radial moment M_r, the tangential moment M_α, the radial shear force Q_r per unit circumference become

$$M_r=-D\left(\frac{\partial^2 w}{\partial\rho^2}+\mu\frac{1}{\rho}\frac{\partial w}{\partial\rho}\right)$$

$$M_\alpha=-D\left(\frac{1}{\rho}\frac{\partial w}{\partial\rho}+\mu\frac{\partial^2 w}{\partial\rho^2}\right)$$

$$Q_r=-D\frac{\partial}{\partial\rho}\left(\frac{\partial^2 w}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial w}{\partial\rho}\right)$$

The notations and their positive directions are shown in Figure 1.

3. Finite Difference Discretization Equation

Many structural problems, such as bending of plates, involve partial differential equations. When these equations are associated with complicated loading or boundary conditions their exact solution presents a formidable problems. One powerful tool in solving these equations is the method of finite differences. The method of finite differences replaces the plate differential equation and the expressions defining the boundary conditions with equivalent difference equation of a set of algebraic equations, written for every nodal point within the plate.

Solutions of the governing differential equations as shown in Equation(3) and (4) by the finite difference method also need proper finite difference approximations for the boundary conditions. Since the central finite difference equations are used, some fictitious points outside the domain of the plate are required. If the pivotal point is located one point on the boundary, we must introduce four more fictitious point outside domain of the plate. Deflections of these fictitious points can be expressed in terms of deflections of the nearby points located on the plate using the boundary conditions. By using the generalized finite difference model, the finite difference expressions for the finite difference expressions for the governing equations, radial moment, tangential moment and transverse shear force are represented for a general point i within the domain can be presented as follows:

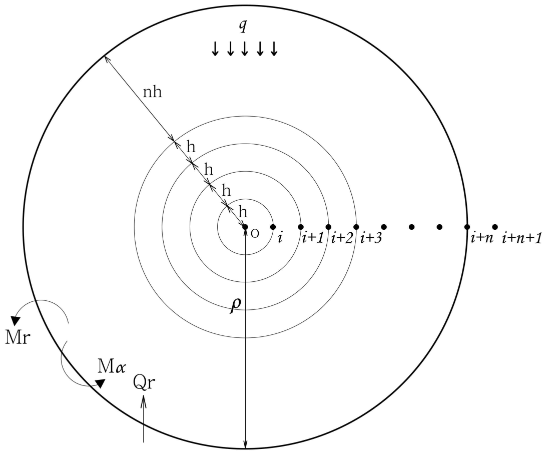
$$\Delta^4 w = \left(\frac{1}{h^4}+\frac{1}{\rho h^3}\right) w_{i+2} - \left(\frac{4}{h^4}+\frac{2}{\rho h^3}+\frac{1}{\rho^2 h^2}-\frac{1}{2\rho^3 h}\right) w_{i+1} + \left(\frac{6}{h^4}+\frac{2}{\rho^2 h^2}\right) w_i - \left(\frac{4}{h^4}-\frac{2}{\rho h^3}+\frac{1}{\rho^2 h^2}\right) w_{i-1} + \left(\frac{1}{h^4}-\frac{1}{\rho h^3}\right) w_{i-2}$$

$$L(w,\phi) = \frac{1}{2\rho h^3}\left[(w_{i+1}-2w_i+w_{i-1})(\phi_{i+1}-\phi_{i-1}) + (w_{i+1}-w_{i-1})(\phi_{i+1}-2\phi_i+\phi_{i-1})\right]$$

$$\frac{M_r}{D} = -\left(\frac{1}{2\rho h}+\frac{\mu}{h^2}\right) w_{i+1} + \frac{2}{h^2} w_i - \left(\frac{1}{h^2}-\frac{\mu}{2\rho h}\right) w_{i-1}$$

$$\frac{M_\alpha}{D} = -\left(\frac{1}{2\rho h}+\frac{\mu}{h^2}\right) w_{i+1} + \frac{2\mu}{h^2} w_i + \left(\frac{1}{2\rho h}-\frac{\mu}{h^2}\right) w_{i-1}$$

$$\frac{Q_r}{D} = \frac{1}{2h^3}(w_{i+2}-2w_{i+1}+2w_{i-1}-w_{i-2}) + \frac{1}{2\rho^2 h}(w_{i+1}-w_{i-1}) - \frac{1}{\rho h^2}(w_{i+1}-2w_i+w_{i-1})$$



<Figure 1> Discretized circular plate and free body diagram

where subscripts i designates the location of pivotal points, as shown in Figure 1. Together with the appropriate boundary conditions, a set of simultaneous equations will be obtained to solve the deflections at the grid points.

4. Solution of the Nonlinear Equations

By performing a static condensation for a domain with i points with simple mathematical manipulations, we may get with the Equation(3) and (4) as a set of linear algebraic equations which can be represented in matrix form as

$$[K]\{w\}=\{q\}+\{f_1(w,\phi)\} \tag{5}$$

$$[M]\{\phi\}=\{f_2(w)\} \tag{6}$$

where [K] and [M] are biharmonic operators and matrices, and are symmetric and positive definite. w denotes the vector representing displacements. q is the vector representing lateral load. f is the Airy's stress function vector. f_1 and f_2 are the nonlinear functions representing part of right side of von Karman's equations. Since the above equations are solution. $f_1(w,\phi)$ and $f_2(w)$ are the vectors corresponding to the nonlinear terms in the equations.

The coefficient matrices [K] and [M] are coded as half banded matrices for computational efficiency. An initial value of is assumed in the equation and the deflections w are calculated. The first equation after $(n+1)$th iteration becomes

$$[A]\{w^{n+1}\}=\{Q\}\{f_1+(w^n,\phi^n)\} \tag{7}$$

Using this equation w^{n+1} can be determined. Knowing value of w^{n+1} and substituting in the right hand side of the second equation such as that Equation(6) we obtain

$$[B]\{\phi^{n+1}\}=\{f_2(w^{n+1})\} \tag{8}$$

And from this equation ϕ^{n+1} can be obtained. This iterative procedure is repeated until satisfactory convergence is achieved in the maximum values of deflection and Airy's functions in the plate. The iterative method developed by Vallabhan using a under-relaxation parameter ϵ to interpolate values between nth and $(n+1)$th iteration[8]. This method assures fast convergence. The values obtained in between nth and $(n+1)$th iterations are as follows.

$$w^{n+1} = (1.0 - \epsilon) w^n + \epsilon w^{n+1}$$

$$\phi^{n+1} = \frac{1}{2}(\phi^n + \phi^{n+1})$$

where

$$\epsilon = \frac{0.6}{(w-1.0)} + 0.05 \qquad \text{for} \quad w \geq 1.8$$
$$= 0.8 \qquad \text{for } w < 1.8$$

$$w = \frac{w^{n+1}}{t}$$

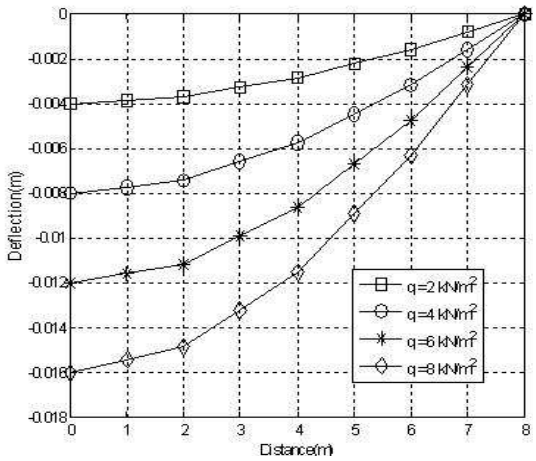
For the solution technique the procedures during a typical load increment can be summarized. First, assume initial values of w and ϕ. Let w^n and ϕ^n be the value for the n-th iteration. Use w^n and ϕ^n to determine the values of the vector $f_1(w^n, \phi^n)$ in Equation(5). Solve for $w^{n+1}$and use w^{n+1} to obtain the value of vector $f_2(w)$ in Equation(6). Solve for ϕ^{n+1}. Check the convergence and repeat for w^n and ϕ^n and continue through steps, if the results are not satisfied. The cycling is terminated when the nodal displacement reach sufficiently small values. If this is not achieved in a predetermined number of iterations, collapse conditions are deemed and the process is stopped.

5. Numerical Implementation

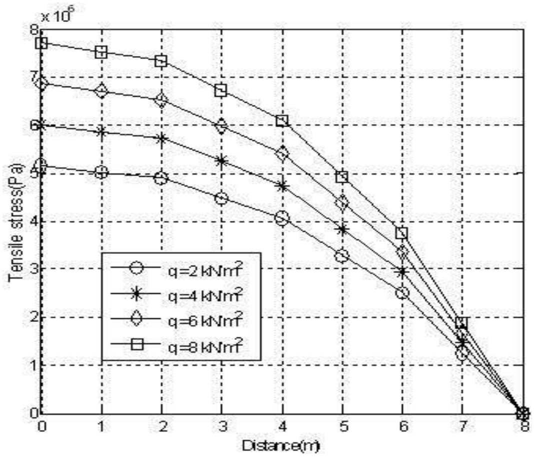
To investigate the validity of the proposed procedure, a distinct problem pertaining to circular plate is considered. As an example, a monolithic plate of size of radius $\rho = 8m$ is considered. The whole plate circumference edge is simply supported.

The plate is subjected in a state of uniformly distributed load with increasing of $0.5kN/m^2$. In this case Poisson's ratio μ and elastic modulus E are taken to be 0.15 and 20GPa. The plate has 8 divisions in the direction of radius.

The displacement distributions at various loads $(q = 2, 4, 6, 8kN/m^2)$are shown in Figure 2. This demonstrates the nonlinearity of the displacement patterns and the migration of the maximum displacement from the support towards the center of plate. Figure 3 shows the maximum principal tensile distributions at various loads. For increasing lateral pressures, the maximum principal tensile stress occurs at the center of the plate and migrates toward the corners as the load increases. It was deemed important to describe the locations of the maximum principal tensile stress as it occurs.



<Figure 2> Radial variation of deflection at various loads in the direction of radius



<Figure 3> Maximum principal tensile stress distribution in the direction of radius

6. Conclusions

A mathematical model for the nonlinear stress analysis of thin circular plates is developed. For the geometrically nonlinear, large deflection behavior of the circular plate, the classical von Karman equations are used. These equations are solved numerically by using the finite difference method. This method essentially consists of large sets of algebraic equations in terms of discrete values of the functions at discrete points. An Gauss elimination scheme has been employed to solve these linear algebraic equations.

The theoretical study of circular nonlinear plate provide interesting insights into the relative behaviors of these plates under lateral pressures. Since the elastic deflections are large compared to the plate thickness during loading, both bending and membrane stresses are developed and as such a nonlinear stress analysis is necessary, accounting for the effects of large deflection. The displacement behavior was described in terms of the membrane displacements.

Furthermore, The critical load in this case is that uniformly distributed live load will be in the range of $50kN/m^2$ to $60kN/m^2$ for the stability of this circular plate. This iterative scheme appears to be suitable for general nonlinear behavior because it relies on the fact that a unique deflection exists for an increment of load. Using the results from this analysis design curves can be developed for the design of laminated plate since laminated plate represents the layered.

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