# Standard Completeness for the Weak Uninorm Mingle Logic WUML＊ 

## Eunsuk Yang

【Abstract】 Fixed－point conjunctive left－continuous idempotent uninorms have been introduced（see e．g．［2，3］）．This paper studies a system for such uninorms．More exactly，one system obtainable from IUML（Involutive uninorm mingle logic）by dropping involution（INV），called here WUML （Weak Uninorm Mingle Logic），is first introduced．This is the system of fixed－point conjunctive left－continuous idempotent uninorms and their residua with weak negation．Algebraic structures corresponding to the system，i．e．， WUML－algebras，are then defined，and algebraic completeness is provided for the system．Standard completeness is further established for WUML and IUML in an analogy to that of WNM（Weak nilpotent minimum logic）and NM（Nilpotent minimum logic）in［4］．

【Key Words】（substructural）fuzzy logic，fuzzy logic，idempotent uninorm （based）logic

[^0]
## I. Introduction

In general a function $\mathrm{n}:[0,1] \rightarrow[0,1]$ is called negation function (briefly a negation) if and only if (iff) $n$ is non-increasing and satisfies $n(0)=1$ and $n(1)=0$. A negation $n$ satisfying $\mathrm{n}(\mathrm{n}(\mathrm{x})) \geq \mathrm{x}$ for all $\mathrm{x} \in[0,1]$ is said to be a weak negation; and a weak negation n is called a strong negation (or involutive negation) if n further satisfies $\mathrm{n}(\mathrm{n}(\mathrm{x}))=\mathrm{x}$ for all $\mathrm{x} \in$ $[0,1]$. In strong negations on $[0,1], n(x)$ can be defined as 1 x , i.e., $\mathrm{n}(\mathrm{x})=1-\mathrm{x}$, called the standard negation.

For the past 11 or 12 years idempotent uninorms, in particular, fixed-point conjunctive left-continuous idempotent uninorms, have been introduced (see e.g. [2, 3, 7]). Furthermore, the logic of the conjunctive left-continuous idempotent uninorm with strong negation and identity $\mathrm{e}=\mathrm{n}(\mathrm{e})=1 / 2$, i.e., IUML (Involutive uninorm mingle logic), has been recently introduced in [8]. But as far as the author knows, any other systems, which are logics of conjunctive left-continuous idempotent uinorms with weak negations (in place of strong negation) and identity $\mathrm{e} \in(0,1]$ introduced in $[2,3]$ have not yet been studied. We shall here introduce a system for such uninorms. More exactly, we first introduce WUML (Weak uninorm mingle logic) as a generalization of IUML having weak negation instead of strong negation. We then define the corresponding algebraic structures, WUML-algebras, and prove algebraic completeness for it. We further establish standard completeness for WUML and IUML in an analogy to that of WNM (Weak nilpotent minimum logic) and

NM (Nilpotent minimum logic) in [4].
For convenience, we shall adopt the notation and terminology similar to those in $[1,4,6,8]$, and assume being familiar with them (together with results found in them).

## II. Syntax

We base the weak uninorm mingle logic WUML on a countable propositional language with formulas FOR built inductively as usual from a set of propositional variables $V A R$, binary connectives $\rightarrow, \&, \wedge, \vee$, and constants $\mathbf{T}, \mathbf{F}, \mathbf{f}, \mathbf{t}$, with defined connectives:
df1. $\sim \Phi:=\Phi \rightarrow \mathbf{f}$, and
df2. $\phi \leftrightarrow \psi:=(\phi \rightarrow \psi) \wedge(\psi \rightarrow \phi)$.

We may define $\mathbf{t}$ as $\mathbf{f} \rightarrow \mathbf{f}$. We moreover define $\phi_{\mathbf{t}}$ as $\Phi \wedge \mathbf{t}$. For the remainder we shall follow the customary notation and terminology. We use the axiom systems to provide a consequence relation.

We start with the following axiomatization of WUML as a (substructural) fuzzy logic.

Definition 2.1 WUML consists of the following axiom schemes and rules:

A1. $\phi \rightarrow \phi$ (self-implication, SI)
A2. $(\Phi \wedge \psi) \rightarrow \phi,(\Phi \wedge \psi) \rightarrow \psi(\wedge$-elimination, $\wedge-E)$

A3. $((\Phi \rightarrow \Psi) \wedge(\Phi \rightarrow \chi)) \rightarrow(\Phi \rightarrow(\Psi \wedge \chi))(\wedge$-introduction, $\wedge-I)$
A4. $\phi \rightarrow(\phi \vee \Psi), \psi \rightarrow(\Phi \vee \Psi)(\vee$-introduction, $\vee$-I)
A5. $((\phi \rightarrow \chi) \wedge(\Psi \rightarrow \chi)) \rightarrow((\phi \vee \Psi) \rightarrow \chi)(\vee$-elimination, $\vee$ - $E)$
A6. $\Phi \rightarrow \mathbf{T}$ (verum ex quolibet, VE)
A7. $\mathbf{F} \rightarrow \Phi$ (ex falso quadlibet, EF )
A8. $(\Phi \& \Psi) \rightarrow(\Psi \& \Phi)(\&-c o m m u t a t i v i t y, \&-C)$
A9. $(\Phi \& \mathfrak{t}) \leftrightarrow \Phi$ (push and pop, PP)

$$
\begin{aligned}
& \text { A10. }(\phi \rightarrow \Psi) \rightarrow((\Psi \rightarrow X) \rightarrow(\Phi \rightarrow X) \text { ) (suffixing, SF) } \\
& \text { A11. }(\phi \rightarrow(\Psi \rightarrow \chi)) \leftrightarrow((\Phi \& \Psi) \rightarrow X) \text { (residuation, RE) }
\end{aligned}
$$

A12. $(\Phi \rightarrow \Psi)_{\mathrm{t}} \vee(\Psi \rightarrow \Phi)_{\mathrm{t}}$ (prelinearity, PL)
A13. $(\Phi \& \phi) \leftrightarrow \Phi$
A14. $\mathbf{t} \leftrightarrow \mathbf{f}$
A15. $(\Phi \rightarrow \Psi) \vee((\Phi \rightarrow \psi) \rightarrow(\sim \Phi \wedge \Psi))$
A16. $(\Phi \rightarrow \Psi) \rightarrow(\sim \phi \vee \Psi)$
A17. $(\Phi \rightarrow \sim \Psi) \rightarrow((\Phi \& \Psi) \rightarrow(\Phi \wedge \Psi))$
A18. $(\Phi \& \Psi) \rightarrow((\phi \vee \Psi) \rightarrow(\Phi \& \Psi))$
A19. $\sim \mathbf{T} \rightarrow \mathbf{F}$
$\phi \rightarrow \Psi, \Phi \vdash \Psi$ (modus ponens, mp)
$\phi, \Psi \vdash \Phi \wedge \psi($ adjunction, adj).

The system having A1 to A12, mp, and adj is UL (Uninorm logic); the $\mathbf{U L}$ having A13 is $\mathbf{U M L}$ (Uninorm mingle logic); and the UML with A14 and (INV) $\sim \sim \Phi \rightarrow \Phi$ is IUML. These were introduced in [8]. We from now on call a system satisfying A1 to A14 a fixed-point uninorm mingle logic, briefly, a FUML.

Proposition 2.2 WUML proves:
(1) $(\Phi \&(\Psi \& X)) \leftrightarrow((\Phi \& \Psi) \& x)(\&$-associativity, AS)
(2) $\Phi \vee \sim \phi$ (excluded middle, EM)
(3) $\sim(\Phi \wedge \sim \phi)$ (non-contradiction, NC)
(4) $(\Phi \rightarrow \Psi) \rightarrow((\Phi \rightarrow \Psi) \leftrightarrow(\sim \Phi \vee \Psi))$
(5) $(\sim \Phi \wedge \Psi) \rightarrow(\Phi \rightarrow \Psi)$
(6) $((\phi \rightarrow \psi) \rightarrow(\sim \phi \wedge \psi)) \rightarrow((\sim \phi \wedge \psi) \rightarrow(\phi \rightarrow \psi))$
(7) $((\phi \rightarrow \psi) \rightarrow(\sim \phi \wedge \psi)) \rightarrow((\phi \rightarrow \psi) \leftrightarrow(\sim \phi \wedge \psi))$
(8) $(\Phi \rightarrow \sim \Psi) \vee(\Phi \& \Psi)$
(9) $(\Phi \rightarrow \sim \psi) \rightarrow((\Phi \& \psi) \leftrightarrow(\Phi \wedge \psi))$
$(10)(\Phi \& \Psi) \rightarrow((\Phi \& \Psi) \leftrightarrow(\Phi \vee \Psi))$
(11) $\sim \mathbf{T} \leftrightarrow \mathbf{F}$.

In WUML, f can be defined as $\sim \mathrm{t}$ and vice versa. A theory over WUML is a set $T$ of formulas. A proof in a sequence of formulas whose each member is either an axiom of WUML or a member of T or follows from some preceding members of the sequence using the rules mp and adj. $\mathrm{T} \vdash \phi$, more exactly $\mathrm{T} \vdash$ wuml $\phi$, means that $\phi$ is provable in T w.r.t. WUML, i.e., there is a WUML-proof of $\phi$ in T. The t -deduction theorem $\left(\mathrm{DT}_{\mathrm{t}}\right)$ for WUML is as follows:

Proposition 2.3 Let T be a theory, and $\phi, \Psi$ formulas. $\mathrm{T} \cup$ $\{\phi\} \vdash_{\text {wUML }} \psi$ iff $T \vdash_{\text {wUML }} \phi_{t} \rightarrow \psi$.

Proof: See [9].

A theory T is inconsistent if $\mathrm{T} \vdash \mathrm{F}$; otherwise it is consistent.

For convenience, " $\sim ", " \wedge ", " \vee "$, and $" \rightarrow "$ are used ambiguously as propositional connectives and as algebraic operators, but context should make their meaning clear.

## III. Semantics

Suitable algebraic structures for WUML are obtained as a subvariety of the variety of commutative residuated lattices in the sense of e.g. [5].

Definition 3.1 A pointed bounded commutative residuated lattice is a structure $\mathbf{A}=\left(\mathrm{A}, \top, \perp, \top_{\mathrm{t}}, \perp_{\mathrm{f}}, \wedge, \vee, *, \rightarrow\right)$ such that:
(I) $(\mathrm{A}, \top, \perp, \wedge, \vee)$ is a bounded lattice with top element $\top$ and bottom element $\perp$.
(II) $\left(\mathrm{A},{ }^{*}, \top_{\mathrm{t}}\right)$ satisfies for some $\top_{\mathrm{t}}$ and for all $\mathrm{x}, \mathrm{y}, \mathrm{z} \in \mathrm{A}$,
(a) $x * y=y * x$ (commutativity)
(b) $\top_{\mathrm{t}} * \mathrm{x}=\mathrm{x}$ (identity)
(c) $\mathrm{x} *(\mathrm{y} * \mathrm{z})=(\mathrm{x} * \mathrm{y}) * \mathrm{z}$ (associativity).
(III) $\mathrm{y} \leq \mathrm{x} \rightarrow \mathrm{z}$ iff $\mathrm{x}{ }^{*} \mathrm{y} \leq \mathrm{z}$, for all $\mathrm{x}, \mathrm{y}, \mathrm{z} \in \mathrm{A}$ (residuation).
$\left(\mathrm{A}, *, \top_{\mathrm{t}}\right)$ satisfying (II-b, c) is a monoid. Thus $\left(\mathrm{A}, *, \top_{\mathrm{t}}\right)$ satisfying (II-a, b, c) is a commutative monoid. To define the above lattice we may take in place of (III) a family of equations as in [6]. Using $\rightarrow$ and $\perp_{f}$ we can define $\top_{t}$ as $\perp_{f} \rightarrow \perp_{f}$, and $\sim$ as in (df1). Then, WUML-algebra whose class characterizes

WUML is defined as follows.

Definition 3.2 (WUML-algebra) A WUML-algebra is a pointed bounded commutative residuated lattice satisfying the conditions: for all $x, y$,

$$
(\mathrm{pl}) \top_{\mathrm{t}} \leq(\mathrm{x} \rightarrow \mathrm{y})_{T \mathrm{t}} \vee(\mathrm{y} \rightarrow \mathrm{x})_{T \mathrm{t}},
$$

(id) $\mathrm{x}^{*} \mathrm{x}=\mathrm{x}$,
(fp) $\top_{t}=\perp_{f}$,
$(\mathrm{w} 1) \top_{\mathrm{t}} \leq(\mathrm{x} \rightarrow \mathrm{y}) \vee((\mathrm{x} \rightarrow \mathrm{y}) \rightarrow(\sim \mathrm{x} \wedge \mathrm{y}))$,
(w2) $x \rightarrow y \leq \sim x \vee y$,
$(w 3) x \rightarrow \sim y \leq(x * y) \rightarrow(x \wedge y)$,
$(w 4) x^{*} y \leq(x \vee y) \rightarrow(x * y)$, and
(w5) $\sim \top=\perp$.

Note that UL-algebras are pointed bounded commutative residuated lattices satisfying ( pl ); UML-algebras are UL-algebras satisfying (id); and IUML-algebras are UML-algebras satisfying (fp) and $\sim \sim \mathrm{x} \leq \mathrm{x}$.

WUML-algebra is said to be linearly ordered if the ordering of its algebra is linear, i.e., $\mathrm{x} \leq \mathrm{y}$ or $\mathrm{y} \leq \mathrm{x}$ (equivalently, $\mathrm{x} \wedge \mathrm{y}$ $=\mathrm{x}$ or $\mathrm{x} \wedge \mathrm{y}=\mathrm{y}$ ) for each pair $\mathrm{x}, \mathrm{y}$.

Definition 3.3 (Evaluation) Let $A$ be an algebra. An A-evaluation is a function $\mathrm{v}:$ FOR $\rightarrow \notin$ satisfying:

$$
\begin{aligned}
& \mathrm{v}(\phi \rightarrow \Psi)=\mathrm{v}(\phi) \rightarrow \mathrm{v}(\Psi), \\
& \mathrm{v}(\Phi \wedge \Psi)=\mathrm{v}(\Phi) \wedge \mathrm{v}(\Psi),
\end{aligned}
$$

$$
\begin{aligned}
& \mathrm{v}(\Phi \vee \Psi)=\mathrm{v}(\Phi) \vee \mathrm{v}(\Psi) \\
& \mathrm{v}(\Phi \& \Psi)=\mathrm{v}(\Phi) * \mathrm{v}(\Psi) \\
& \mathrm{v}(\mathbf{F})=\perp \\
& \mathrm{v}(\mathbf{f})=\perp_{\mathbf{f}},
\end{aligned}
$$

(and hence $\mathrm{v}(\sim \phi)=\sim \mathrm{v}(\phi), \mathrm{v}(\mathbf{T})=\top$, and $\mathrm{v}(\mathrm{t})=\top_{\mathrm{t}}$ ).

Definition 3.4 Let \& be a WUML-algebra, $T$ a theory, $\Phi$ a formula, and K a class of WUML-algebras.
(i) (Tautology) $\Phi$ is a $T_{t}$-tautology in $A$, briefly an $A$-tautology (or $A$-valid), if $\mathrm{v}(\phi) \geq \top_{\mathbf{t}}$ for each $A$-evaluation v .
(ii) (Model) An $A$-evaluation $v$ is an $A$-model of $T$ if $v(\phi) \geq \top_{t}$ for each $\phi \in \mathrm{T}$. By $\operatorname{Mod}(T, A)$, we denote the class of A-models of T .
(iii) (Semantic consequence) $\Phi$ is a semantic consequence of $T$ w.r.t. K , denoting by $\mathrm{T} \vDash_{k} \phi$, if $\operatorname{Mod}(\mathrm{T}, \mathcal{A})=\operatorname{Mod}(\mathrm{T}$ $\{\Phi\}, A)$ for each $A \in \mathrm{~K}$.

Definition 3.5 (WUML-algebra) Let $A$, $T$, and $\Phi$ be as in Definition 3.4. A is a WUML-algebra iff whenever $\phi$ is WUML-provable in T (i.e. $\mathrm{T} \vdash_{\text {wuml }} \phi$ ), it is a semantic consequence of T w.r.t. the set $\{\notin\}$ (i.e. $T \vDash_{\{A\}} \phi$ ), \& a WUML-algebra. By $M O D^{(l)}(\boldsymbol{W} \boldsymbol{U} L)$, we denote the class of (linearly ordered) WUML-algebras. Finally, we write $T \vDash{ }^{(1)}$ wumL $\Phi$ in place of $\mathrm{T} \vDash{ }_{\text {MOD }}{ }^{(1)}{ }^{(\text {WUML })}$.

Note that since each condition for the WUML-algebra has a
form of equation or can be defined in equation (exercise), it can be ensured that the class of all WUML-algebras is a variety.

Let $\mathbf{A}$ be a WUML-algebra. We first show that classes of provably equivalent formulas form a WUML-algebra. Let T be a fixed theory over WUML. For each formula $\phi$, let $[\phi]_{\mathrm{T}}$ be the set of all formulas $\psi$ such that $T \vdash_{\text {wUML }} \phi \leftrightarrow \psi$ (formulas T-provably equivalent to $\phi) . \mathrm{A}_{\mathrm{T}}$ is the set of all the classes $[\phi]_{\mathrm{T}}$. We define that $[\phi]_{\mathrm{T}} \rightarrow[\psi]_{\mathrm{T}}=[\phi \rightarrow \psi]_{\mathrm{T}},[\phi]_{\mathrm{T}} *[\psi]_{\mathrm{T}}=[\phi \&$ $\psi]_{\mathrm{T}},[\phi]_{\mathrm{T}} \wedge[\psi]_{\mathrm{T}}=[\phi \wedge \psi]_{\mathrm{T}},[\phi]_{\mathrm{T}} \vee[\psi]_{\mathrm{T}}=[\phi \vee \psi]_{\mathrm{T}}, \perp=$ $[\mathbf{F}]_{\mathrm{T}}, \quad \mathrm{T}=[\mathbf{T}]_{\mathrm{T}}, \mathrm{T}_{\mathbf{t}}=[\mathbf{t}]_{\mathrm{T}}$, and $\perp_{\mathrm{f}}=[\mathbf{f}]_{\mathrm{T}}$. By $\boldsymbol{A}_{\mathrm{T}}$, we denote this algebra.

Proposition 3.6 For $T$ a theory over WUML, $\mathbf{A}_{T}$ is a WUML-algebra.

Proof: Note that A1 to A7 ensure that $\wedge$ and $\vee$ satisfy (I) in Definition 3.1; that AS, A8, A9 ensure that \& satisfies (II); that A11 ensures that (III) holds; and that A12 to A19 ensure that the conditions in Definition 3.2 hold. It is obvious that $[\phi]_{\mathrm{T}} \leq[\Psi]_{\mathrm{T}}$ iff $T \vdash_{\text {wUML }} \phi \leftrightarrow(\phi \wedge \Psi)$ iff $T \vdash_{\text {wuml }} \phi \rightarrow \psi$. Finally recall that $\mathbf{A}_{\mathrm{T}}$ is a WUML-algebra iff $\mathrm{T} \vdash_{\text {wumL }} \Psi$ implies $\mathrm{T} \vDash$ wuml $\psi$, and observe that for $\phi$ in $T$, since $T \vdash_{\text {wuml }} \mathbf{t} \rightarrow \phi$, it follows that $[\mathrm{t}]_{\mathrm{T}} \leq[\phi]_{\mathrm{T}}$. Thus it is a WUML-algebra.

We next note that the nomenclature of the prelinearity condition is explained by the subdirect representation theorem below.

Proposition 3.7 (Cf. [10]) Each WUML-algebra is a subdirect product of linearly ordered WUML-algebras.

Theorem 3.8 (Strong completeness) Let T be a theory, and $\Phi$ a formula. $\mathrm{T} \vdash_{\text {wuml }} \phi$ iff $\mathrm{T} \vDash_{\text {wuml }} \phi$ iff $\mathrm{T} \vDash^{1}{ }_{\text {wUML }} \phi$.

Proof: (i) $\mathrm{T} \vdash_{\text {wuml }} \Phi$ iff $\mathrm{T} \vDash_{\text {wuml }} \phi$. Left to right follows from definition. Right to left is as follows: from Proposition 3.6, we obtain $\mathbf{A}_{\mathrm{T}} \in \operatorname{MOD}(\mathrm{L})$, and for $\mathbf{A}_{\mathrm{T}}$-evaluation v defined as v $(\Psi)=[\Psi]_{\mathrm{T}}$, it holds that $\mathrm{v} \in \operatorname{Mod}\left(\mathrm{T}, \mathbf{A}_{\mathrm{T}}\right)$. Thus, since from T $\vDash_{\text {wUML }} \phi$ we obtain that $[\phi]_{\mathrm{T}}=\mathrm{v}(\phi) \geq \top_{\mathrm{t}}, \mathrm{T} \vdash_{\text {wUML }} \mathbf{t} \rightarrow \phi$. Then, since $\mathrm{T} \vdash_{\text {wuml }} \mathbf{t}$, by (mp) $\mathrm{T} \vdash_{\text {wuml }} \phi$, as required.
(ii) $\mathrm{T} \vDash{ }_{\text {wuml }} \Phi$ iff $\mathrm{T} \vDash{ }^{1}$ wuml $\phi$. It follows from Proposition 3.7. $\square$

## IV. Uninorms and their residua

In this section, using 1,0 , and some $1_{t}$, and $0_{f}$ in the real unit interval $[0,1]$, we shall express $\top, \perp$, $\top_{t}$, and $\perp_{f}$, respectively. We also define standard WUML-algebras and uninorms on $[0,1]$.

Definition 4.1 A WUML-algebra is standard iff its lattice reduct is $[0,1]$.

In standard WUML-algebras the monoid operator * is a uninorm.

Definition 4.2 A uninorm is a function $O:[0,1]^{2} \rightarrow[0,1]$ such that for some $1_{\mathrm{t}} \in[0,1]$ and for all $\mathrm{x}, \mathrm{y}, \mathrm{z} \in[0,1]$ :
(a) $\mathrm{x} \circ \mathrm{y}=\mathrm{y} \bigcirc \mathrm{x}$ (commutativity),
(b) $x \bigcirc(y \bigcirc z)=(x \bigcirc y) \bigcirc z$ (associativity),
(c) $\mathrm{x} \leq \mathrm{y}$ implies $\mathrm{x} \circ \mathrm{z} \leq \mathrm{y} \bigcirc \mathrm{z}$ (monotonicity), and
(d) $1_{t} O x=x$ (identity).

The function $O$ satisfying (1-identity) $1_{\mathrm{t}}=1$ is a $t$-norm. A uninorm $\bigcirc$ is said to be square increasing if it satisfies (square-increasingness) $\mathrm{x} \leq \mathrm{x} \bigcirc \mathrm{x}$, for all $\mathrm{x} \in[0,1]$; square decreasing if it satisfies (square-decreasingness) $\mathrm{x} O \mathrm{x} \leq \mathrm{x}$, for all $\mathrm{x} \in[0,1]$; and idempotent if it is both square increasing and square decreasing. A uninorm is called conjunctive if $0 \bigcirc 1=0$; disjunctive if $0 \bigcirc 1=1$; and fixed-point if $1_{\mathrm{t}}=0_{\mathrm{f}}$.

The left-continuity property of conjunctive uninorms is important in the sense that it gives a residuated implication and so plays an important role in standard completeness proof of WUML as in t-norm based logics such as MTL. O is said to be residuated iff there is $\rightarrow$ : 0,1$]^{2} \rightarrow[0,1]$ satisfying (residuation) on $[0,1]$. Given a uninorm $O$, residuated implication $\rightarrow$ determined by $O$ is defined as $\mathrm{x} \rightarrow \mathrm{y}:=\sup \{\mathrm{z}:$ $\mathrm{x} \circ \mathrm{z} \leq \mathrm{y}\}$ for all $\mathrm{x}, \mathrm{y} \in[0,1]$. Then, we can show that for any uninorm $O, O$ and its residuated implication $\rightarrow$ form a residuated pair iff $O$ is conjunctive and left-continuous in both arguments (see Proposition 5.4.2 [9]).

It is clear that the operator * of any standard UL-algebra is a
conjunctive uninorm with identity $T_{t}$ and residuum $\rightarrow$; conversely any residuated uninorm gives rise to a UL-algebra as follows: if $O$ is a uninorm with residuum $\rightarrow$ and identity $1_{\mathrm{t}}$, then for any $0_{\mathrm{f}} \in[0,1],\left([0,1], 1,0,1_{\mathrm{t}}, 0_{\mathrm{f}}, \min , \max , \mathrm{O}, \rightarrow\right)$ is a standard UL-algebra (see Proposition 17 in [8]).
We finally state some known interesting facts related to conjunctive left-continuous idempotent uninorms and their residua.

Fact 4.3 (i) ([2]) A binary operator $O$ is a conjunctive left-continuous idempotent uninorm with identity element $1_{t} \in(0$, 1] iff there is a left-continuous uniquely determined non-increasing function $\mathrm{n}:[0,1] \rightarrow[0,1]$ with $\mathrm{n}\left(1_{\mathrm{t}}\right)=1_{\mathrm{t}}$ and $\mathrm{n}(\mathrm{n}(\mathrm{x})) \geq \mathrm{x}$ for $x \in[0,1]$, such that for all $x, y \in[0,1]:$

$$
\begin{aligned}
\mathrm{x} \bigcirc \mathrm{y}= & \min (\mathrm{x}, \mathrm{y}) \text { if } \mathrm{y} \leq \mathrm{n}(\mathrm{x}), \\
& \max (\mathrm{x}, \mathrm{y}) \text { otherwise. }
\end{aligned}
$$

(ii) ([8]) Let $\mathbf{A}_{\mathrm{s}}=\left([0,1], 1,0,1 / 2,1 / 2, \min , \max , \mathrm{O}_{\mathrm{s}}, \rightarrow \rightarrow_{\mathrm{s}}\right)$, where:

$$
\begin{aligned}
\mathrm{x} \mathrm{O}_{s} \mathrm{y}= & \min (\mathrm{x}, \mathrm{y}) \text { if } \mathrm{x}+\mathrm{y} \leq 1, \\
& \max (\mathrm{x}, \mathrm{y}) \text { otherwise. }
\end{aligned}
$$

$\phi$ is valid in all standard IUML-algebras iff $\phi$ is valid in the IUML-algebra $\mathbf{A}_{s}$.

The operator $O_{s}$ in (ii) of Fact 4.3 is an example of (i) satisfying strong negation. We introduce an unknown further example of (i) with weak negation.

Example 4.4 Given a fixed-point weak negation n, i.e., a
negation n satisfying: for all $\mathrm{x} \in[0,1]$,
(a) $n\left(1_{t}\right)=1_{t}$,
(b) $n(n(x)) \geq x$, and
(c) $\mathrm{n}(0)=1$ and $\mathrm{n}(1)=0$,
we can construct a conjunctive left-continuous idempotent uninorm $\bigcirc$ given by, for all $x, y \in[0,1]$ :

$$
\begin{aligned}
\mathrm{x} \bigcirc \mathrm{y}= & \min (\mathrm{x}, \mathrm{y}) \text { if } \mathrm{y} \leq \mathrm{n}(\mathrm{x}) \\
& \max (\mathrm{x}, \mathrm{y}) \text { otherwise. }
\end{aligned}
$$

We call uninorms in Fact 4.3 (ii) and Example 4.4 IUML- and WUML-uninorms, respectively.

Left-continuity of the above uninorms ensures that the corresponding residuated implications can be obtained. The following are the known facts.

Fact 4.5 (i) ([3]) Consider a conjunctive left-continuous idempotent uninorm $\bigcirc$ with a negation $n$. Then its residuated implication $\rightarrow$ is given by

$$
\begin{aligned}
\mathrm{x} \rightarrow \mathrm{y}= & \max (\mathrm{n}(\mathrm{x}), \mathrm{y}) \text { if } \mathrm{x} \leq \mathrm{y} \\
& \min (\mathrm{n}(\mathrm{x}), \mathrm{y}) \text { otherwise }
\end{aligned}
$$

(ii) $([3,7])$ Consider an involutive negation $n_{s}$. Then the residiated implication of the corresponding conjunctive left-continuous idempotent uninorm $O_{s}$ is given by

$$
\begin{aligned}
\mathrm{x} \rightarrow_{\mathrm{s}} \mathrm{y}= & \max (1-\mathrm{x}, \mathrm{y}) \text { if } \mathrm{x} \leq \mathrm{y} \\
& \min (1-\mathrm{x}, \mathrm{y}) \text { otherwise }
\end{aligned}
$$

The residiated implication of the corresponding conjunctive
left-continuous idempotent uninorms with the negation in Example 4.4 is obtained as in Fact 4.5.

IUML is the system satisfying (ii) in Fact 4.3 and so (ii) in Fact 4.5. Any system satisfying (i) in Fact 4.3 corresponds to a FUML, i.e., a fixed-point uninorm mingle logic. But, as far as the author knows, such systems having weak negations in place of strong negation have not yet been introduced.

## V. Standard completeness

We here provide standard completeness results for WUML and IUML in an analogy to that of WNM and NM in [4]. First note that weak and strong negations are defined as in Section 1. Moreover,

Definition 5.1 ([4]) Given a bounded linearly ordered set (C, $\leq, \top, \perp)$, a non-increasing function $\mathrm{n}: \mathrm{C} \rightarrow \mathrm{C}$ is said to be symmetric w.r.t. the identity mapping if it satisfies:

1. if $x \in n(C)$, then $n(x)=y$ implies $x=n(y)$, and
2. if $x \notin n(C)$, then
(i) $n$ is constant in the interval $\left[x, n^{2}(x)\right]$ with value $n(x)$, and
(ii) for any $y>n(x)$, it is $n(y)<x$, i.e., $n(x)$ is a discontinuity point on the right with $n\left(n(x)^{-}\right)=n^{2}(x)$ and $n\left(n(x)^{+}\right)<x$.

We call linearly ordered WUML- and IUML-algebras WUMLand IUML-chains. We first note that given a bounded chain C , n $: \mathrm{C} \rightarrow \mathrm{C}$ is a weak negation iff it is non-increasing and
symmetric w.r.t. the identity mapping (see Proposition A. 3 in [4]). As a consequence, we then give WUML- and IUML-chains with weak and strong negations, respectively, as follows.

Proposition 5.2 (i) Given a weak negation $n$ on [ 0,1 ], we can define a residuated pair of operations $\mathrm{O}_{\mathrm{n}}, \rightarrow_{\mathrm{n}}$ such that ([0, 1], $\mathrm{O}_{\mathrm{n}}, \rightarrow_{\mathrm{n}}, 1,0,1_{\mathrm{t}}, 0_{\mathrm{f}}$ ) is a WUML-chain with negation n .
(ii) For any strong negation $\mathrm{n}_{\mathrm{s}}$ on [0, 1], we can define a residuated pair of operations $\mathrm{O}_{\mathrm{ns}}, \rightarrow_{\mathrm{ns}}$ such that ( $[0,1], \mathrm{O}_{\mathrm{ns}}, \rightarrow$ ns, $1,0,1_{\mathrm{t}}, 0_{\mathrm{f}}$ ) is an IUML-chain with negation $\mathrm{n}_{\mathrm{s}}$.

Proof: (i) Let n be the n in Example 4.4. We first note that given a weak negation $n$ on $[0,1]$, a uninorm $O_{n}$ for WUML, called the weak Gödel uninorm, is defined as in (i) of Example 4.4. Then, given a weak negation n on $[0,1]$, we can easily prove that:
(a) $O_{n}$ has residuum defined by
$\mathrm{x} \rightarrow_{\mathrm{n}} \mathrm{y}=\max (\mathrm{n}(\mathrm{x}), \mathrm{y})$ if $\mathrm{x} \leq \mathrm{y}$; $\min (n(x), y)$ otherwise (Weak Gödel u-implication), where u-implication is an abbreviation of uninormimplication,
(b) $\left([0,1], O_{\mathrm{n}}, \rightarrow_{\mathrm{n}}, 1,0,1_{\mathrm{t}}, 0_{\mathrm{f}}\right)$ is a WUML-algebra, and
(c) the corresponding negation is $n$.

Since the axioms of WUML are valid in a WUML-algebra, this algebra is a WUML-algebra as well.

Note that the condition $\mathrm{x} \leq \mathrm{n}(\mathrm{y})$ is symmetric in the sense that it is equivalent to $\mathrm{y} \leq \mathrm{n}(\mathrm{x})$ because $\mathrm{x} \leq \mathrm{n}(\mathrm{y})$ implies $\mathrm{n}^{2}(\mathrm{y})$
$\leq \mathrm{n}(\mathrm{x})$, and $\mathrm{y} \leq \mathrm{n}^{2}(\mathrm{y})$.
(ii) Proof of the case of strong negation is analogous to that of the negation in (i), i.e., given a strong negation $\mathrm{n}_{\mathrm{s}}$ on $[0,1]$, we can easily prove that:
(a) $O_{\text {ns }}$ has residuum defined by
$\mathrm{x} \rightarrow_{\mathrm{ns}} \mathrm{y}=\max (1-\mathrm{x}, \mathrm{y})$ if $\mathrm{x} \leq \mathrm{y}$; $\min (1-x, y)$ otherwise (Strong Gödel u-implication), where u-implication is an abbreviation of uninormimplication,
(b) $\left([0,1], \bigcirc_{\mathrm{ns}}, \rightarrow_{\mathrm{ns}}, 1,0,1 / 2,1 / 2\right)$ is an IUML-algebra, and
(c) the corresponding negation $n$ is the strong one. $\qquad$

Fact 5.3 ([4]) (1) Weak negation functions on the real unit interval $[0,1]$ are not all isomorphic, and yet (2) strong negation functions on it are.

Then because of Fact 5.3 we can say that

Proposition 5.4 (i) Weak negation functions on [0, 1] are not all isomorphic so that WUML-chains defined by WUML-uninorms are not.
(ii) Strong negation functions on $[0,1]$ are all isomorphic so that IUML-chains defined by IUML-uninorms are.

Theorem 5.5 (Weak standard completeness) (i) $\vdash_{\text {wUML }} \phi$ iff $\phi$ is a tautology in all standard WUML-algebras.
(ii) $\vdash_{\text {IUML }} \Phi$ iff $\Phi$ is a tautology w.r.t. the standard

IUML-algebra $\left([0,1], \mathrm{O}_{\mathrm{ns}}, \rightarrow_{\mathrm{ns}}, 1,0,1 / 2,1 / 2\right)$.

Proof: Each proof of (i) and (ii) is analogous to that of Theorems 3 and 4, respectively, in [4].
(i) Soundness is obvious. For completeness, let $\vdash_{\text {wuml }} \phi$. We show that there is a WUML-chain C and an evaluation v on C such that $\mathrm{v}(\phi)<1_{\mathrm{t}}$. We prove the case $\mathrm{n}_{\mathrm{C}}$. Let $\mathrm{n}_{\mathrm{C}}$ be the negation in $C$ and take $X$ as the finite subset of $C$ consisting of all the values $\mathrm{v}(\Psi), \mathrm{n}_{\mathrm{C}}(\mathrm{v}(\Psi))$, and $\mathrm{n}_{\mathrm{C}}\left(\mathrm{n}_{\mathrm{C}}(\mathrm{v}(\Psi))\right.$ ) for all subformulas $\psi$ of $\phi$, plus $0_{\mathrm{c}}, 1_{\mathrm{C}}$, and $1_{\mathrm{tC}}\left(=0_{\mathrm{fC}}\right)$. Let

$$
X \cap \mathrm{n}_{\mathrm{C}}(\mathrm{C})=\left\{0_{\mathrm{C}}=\mathrm{a}_{0}<\mathrm{a}_{1}<\cdots<1_{\mathrm{tC}}=0_{\mathrm{fC}}<\cdots<\mathrm{a}_{\mathrm{m}}=\right.
$$ $\left.1_{C}\right\}$,

where $1_{\mathrm{tC}}=\mathrm{a}_{\mathrm{m} / 2}$ if m is even and otherwise $1_{\mathrm{tC}}=\mathrm{a}_{(\mathrm{m}+1) / 2}$. Let $\mathrm{f}: \mathrm{X} \rightarrow[0,1]$ be an ordered mapping $(\mathrm{x}<\mathrm{y}$ implies $\mathrm{f}(\mathrm{x})<$ $f(y))$ such that $f\left(a_{i}\right)=i / m$. Define on [0, 1] the weak negation function n as follows: taking $1_{\mathrm{t}}=\mathrm{f}\left(1_{\mathrm{tc}}\right)$, and letting $1_{\mathrm{t}}^{-}=1_{\mathrm{t}}$ $1 / \mathrm{m}$,

$$
\begin{aligned}
\mathrm{n}(\mathrm{x})=1-\mathrm{x} & \text { if } \mathrm{x} \in\{\mathrm{i} / \mathrm{m}: 0 \leq \mathrm{i} \leq \mathrm{m}\} \text { and } \mathrm{x} \neq 1_{\mathrm{t}} \\
1_{\mathrm{t}} & \text { if } 1_{\mathrm{t}}^{-}<\mathrm{x} \leq 1_{\mathrm{t}} \\
(\mathrm{~m}-\mathrm{i}-1) / \mathrm{m} & \text { otherwise, i.e., } \\
& \text { if } \mathrm{x} \in(\mathrm{i} / \mathrm{m},(\mathrm{i}+1) / \mathrm{m}) \text { where } \mathrm{i} / \mathrm{m} \neq 1_{\mathrm{t}}^{-}
\end{aligned}
$$

It is clear that n is a fixed-point weak negation on $[0,1]$ and $f$ is a morphism w.r.t. minimum, maximum, implication, and
negation. Note that we take $1_{\mathrm{t}}=\mathrm{f}\left(1_{\mathrm{tc}}\right)$. Thus, by defining the evaluation $\mathrm{v}^{\prime}(\mathrm{p})=\mathrm{v}(\mathrm{f}(\mathrm{p}))$ for any propositional variable p occurring in $\phi$, we get $\mathrm{v}^{\prime}(\phi)=\mathrm{v}(\mathrm{f}(\phi))<1_{\mathrm{t}}$, as desired.
(ii) Soundness is obvious. For completeness, let $\vdash_{\text {rumL }} \phi$. Then there is an IUML-chain C and an evaluation v on C such that v $(\phi)<1_{t}$. Let $n_{s C}$ be the strong negation in $C$ and take $X$ as the finite subset of C , whose number of elements being odd, consisting of all the values $\mathrm{v}(\Psi)$ and $\mathrm{n}_{\mathrm{sc}}(\mathrm{v}(\Psi))$ for all subformulas $\Psi$ of $\phi$ together with $0_{\mathrm{C}}, 1_{\mathrm{C}}$, and $1_{\mathrm{tC}}\left(=0_{\mathrm{fC}}\right)$. Suppose that X has $\mathrm{m}+1$ elements and let $\mathrm{X}=\left\{0_{\mathrm{C}}=\mathrm{a}_{0}<\mathrm{a}_{1}<\cdots<1_{\mathrm{tC}}=0_{\mathrm{fC}}<\right.$ $\left.\cdots<a_{m}=1_{C}\right\}$, where $1_{\mathrm{tC}}=a_{\mathrm{m} / 2}$ and m is even. Now let $\mathrm{f}\left(\mathrm{a}_{\mathrm{i}}\right)=$ $\mathrm{i} / \mathrm{m}, 0 \leq \mathrm{i} \leq \mathrm{m}$. It is clear that f , as a mapping from X to the set $\{\mathrm{i} / \mathrm{m}: 0 \leq \mathrm{i} \leq \mathrm{m}\}$, is a morphism w.r.t. minimum, maximum, implication, and negation, and we take $1_{t}=f\left(1_{\mathrm{c}}\right)$. Thus, defining $\mathrm{v}^{\prime}(\mathrm{p})=\mathrm{v}(\mathrm{f}(\mathrm{p})$ ) for any propositional variable p occurring in $\phi$, we get $\mathrm{v}^{\prime}(\phi)=\mathrm{v}(\mathrm{f}(\phi))<1_{\mathrm{t}}$, as required. Thus, $\vdash_{\text {rumL }} \phi$ iff $\phi$ is a tautology in all standard IUML-algebras.
Finally note that Proposition 5.4 (ii) ensures that $\phi$ is a tautology in all standard IUML-algebras iff $\phi$ is a tautology in the standard $\left([0,1], O_{\mathrm{ns}}, \rightarrow_{\mathrm{ns}}, 1,0,1 / 2,1 / 2\right)$.

Furthermore, we can show strong standard completeness for WUML and IUML.

Theorem 5.6 (Strong standard completeness) For $\mathbf{L} \in$ (WUML, IUML $\}$, let $T$ be a theory over $\mathbf{L}$, and $\phi$ a formula. $T \vdash_{\mathbf{L}} \phi$ iff $\phi$ is true in each standard L -model of T .

Proof: Its proof is analogous to that of Theorem 4.2.17 in [6].

## VI. Concluding remark

In this paper we introduced the system WUML, which is a generalization of IUML having weak negation in place of strong negation. After defining the corresponding algebraic structures, we provided algebraic completeness for it. Furthermore, we established standard completeness for WUML and IUML.

In fact we can also introduce systems for other fixed-point conjunctive left-continuous idempotent uninorms. We shall investigate this in some subsequent paper.

## References

[1] Cintula, P. (2006), "Weakly Implicative (Fuzzy) Logics I: Basic properties", Archive for Mathematical Logic, pp. 673-704.
[2] De Baets, B. (1999), "Idempotent uninorms", European Journal of Operational Research 118, pp. 631-642.
[3] De Baets, B. and Fodor, J. (1999), "Residual operators of uninorms", Soft Computing 3, pp. 89-100.
[4] Esteva, F., and Godo, L. (2001), "Monoidal t-norm based logic: towards a logic for left-continous t-norms", Fuzzy Sets and Systems 124, pp. 271-288.
[5] Galatos, N., Jipsen, P., Kowalski, T., and Ono, H. (2007), Residuated lattices: an algebraic glimpse at substructural logics, Amsterdam, Elsevier.
[6] Hájek, P. (1998), Metamathematics of Fuzzy Logic, Amsterdam, Kluwer.
[7] Klement, E. P., Mesiar, R., and Pap, E. (2000), Triangular Norms, Dordrecht, Kluwer.
[8] Metcalfe, G and Montagna, F. (2007), "Substructural Fuzzy Logics", Journal of Symbolic Logic 72, pp. 834-864.
[9] Meyer, R. K., Dunn, J. M., and Leblanc, H. (1976), "Completeness of relevant quantification theories", Notre Dame Journal of Formal Logic 15, pp. 97-121.
[10] Tsinakis, C., and Blount, K. (2003), "The structure of residuated lattices", International Journal of Algebra and Computation 13, pp. 437-461.

서울시립대
Division of Liberal Arts and Teacher Education
University of Seoul
Email: eunsyang@uos.ac.kr

## WUML의 표준적 완전성

## 양 은 석

이 논문은 uninorm의 한 체계인 WUML(Weak Uninorm Mingle Logic)을 다룬다. 먼저 WUML을 도입하고, 그 체계에 상응하는 대 수적 구조를 정의한다. 그런 다음 그 체계의 대수적 완전성을 증명 한다. 끝으로 WUML과 IUML의 표준적 완전성을 증명한다.

주요어: (준구조) 퍼지 논리, 퍼지 논리, 멱등 유니놈 논리


[^0]:    ＊접수일：2010．3．22．심사 및 수정완료일：2010．12．13．게재확정일：2011．1． 16.

