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Interval-Valued Fuzzy Almost α-연속 함수의 연구

Interval-Valued Fuzzy Almost α-Continuous Mappings

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요 약

IVF almost α-연속성의 개념을 소개하고 그 함수의 특성을 조사한다. 그리고 다른 IVF 연속함수와의 관계를 밝힌다.

Abstract

We introduce the concept of IVF almost α-continuity and investigate characterizations for such mappings on the interval-valued fuzzy topological spaces. We study the relationships between IVF almost α-continuous mappings and another types of IVF continuous mappings.

Key Words : IVF α-continuous, IVF weakly α-continuous, IVF almost α-continuous, IVF almost open mapping, IVF almost regular

1. Intorduction and Preliminaries

Zadeh [9] introduced the concept of fuzzy set and investigated basic properties. Gorzalczany [2] introduced the concept of interval-valued fuzzy set which is a generalization of fuzzy sets. In [8], Mondal and Samanta introduced the concepts of interval-valued fuzzy topology, continuity and compactness and studied some topological properties. The concept of interval-valued fuzzy topology is a generalization of fuzzy topology in sense of Chang's fuzzy topology [1]. In [3], Jun et al. introduced the concepts of IVF α-open sets and IVF α-open mappings and studied some results about them. The concept of IVF strong semi-continuous (or IVF α-continuous mapping) was introduced in [4]. The author introduced the concept of IVF weakly α-continuous mapping and investigate some properties for them in [6]. In this paper, we introduce the concept of IVF almost α-continuous mapping and investigate characterizations for such a mapping. We study the relationships among IVF α-continuous mapping, IVF weakly α-continuous mapping and IVF almost α-continuous mapping.

2. Preliminaries

Let I be the unit interval $[0,1]$ of the real line. A member A of I^X is called a fuzzy set of X. For any $A \in I^X$, A^c denotes the complement $1_X - A$. By 0_X and 1_X we denote constant maps on X with value 0 and 1, respectively.

A Chang's fuzzy topology τ [1] is a family $\tau \subseteq I^X$ satisfying the following conditions:
(O1) $0_X,\ 1_X \in \tau$;
(O2) for $A,\ B \in I^X$, if $A,\ B \in \tau$, then $(A \cap B) \in \tau$;
(O3) for every subfamily $\{A_i : i \in J\} \subseteq I^X$, if $A_i \in \tau$, then $\cup_{i \in J} A_i \in \tau$.

Let $D[0,1]$ be the set of all closed subintervals of the interval $[0,1]$. The elements of $D[0,1]$ are generally denoted by capital letters $M, N, \cdots$ and note that $M=[M^L, M^U]$, where M^L and M^U are the lower and the upper end points respectively. Especially, we denote 0 and 1 as follows: $0=[0,0]$, $1=[1,1]$. We also note that
(1) $(\forall M,\ N \in D[0,1])$ $(M=N \Leftrightarrow M^L=N^L,\ M^U=N^U)$.
(2) $(\forall M,\ N \in D[0,1])$ $(M \leq N \Leftrightarrow M^L \leq N^L,\ M^U \leq N^U)$.

For each $M \in D[0,1]$, the complement of M, denoted by M^c, is defined by $M^c=[1-M^U,\ 1-M^L]$.

Let X be a nonempty set. A mapping $A : X \rightarrow D[0,1]$ is called an *interval-valued fuzzy set* (simply, IVF set) in X. For each $x \in X$, $A(x)$ is a closed interval whose lower and upper end points are denoted by $[A(x)]^L$ and $[A(x)]^U$, respectively. For any $[a,b] \in D[0,1]$, the IVF set whose value is the interval $[a,b]$ for all $x \in X$ is denoted by $\widetilde{[a,b]}$. In particular, for any $a \in [0,1]$, the IVF set whose value is $a(x)=[a,a]$ for all $x \in X$ is denoted by simply $\tilde{a}$. For a point $p \in X$ and for $[a,b] \in$

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$D[0,1]$ with $b>0$, the IVF set which takes the value $[a,b]$ at p and 0 elsewhere in X is called an *interval-valued fuzzy point* (simply, IVF point) and is denoted by $[a,b]_p$. In particular, if $b=a$, then it is also denoted by a_p. We denote the set of all IVF sets in X by IVF(X). An IVF point M_x, where $M\in D[0,1]$ is said to belong to an IVF set A in X, denoted by $M_x \in A$, if $[A(x)]^L \geq M^L$ and $[A(x)]^U \geq M^U$. In [8], it has been shown that $A=\cup\{M_x : M_x \in A\}$.

For every A, $B\in$IVF(X), we define

$$A=B \Leftrightarrow (\forall x\in X)([A(x)]^L=[B(x)]^L,\ [A(x)]^U=[B(x)]^U).$$

$$A\subseteq B \Leftrightarrow (\forall x\in X)([A(x)]^L\subseteq[B(x)]^L,\ [A(x)]^U\subseteq[B(x)]^U).$$

The complement A^c of A is defined by $[A^c(x)]^L=1-[A(x)]^U$ and $[A^c(x)]^U=1-[A(x)]^L$ for all $x\in X$.

For a family of IVF sets $\{A_i : i\in J\}$ where J is an index set, the union $G=\cup_{i\in J}A_i$ and $F=\cap_{i\in J}A_i$ are defined by

$$[G(x)]^L=sup_{i\in J}[A_i(x)]^L,\ [G(x)]^U=sup_{i\in J}[A_i(x)]^U$$

and

$$[F(x)]^L=inf_{i\in J}[A_i(x)]^L,\ [F(x)]^U=inf_{i\in J}[A_i(x)]^U,$$

respectively, for all $x\in X$.

Let $f: X\rightarrow Y$ be a mapping and let A be an IVF set in X. Then the image of A under f, denoted by $f(A)$ [8], defined as follows

$$[f(A)(y)]^L=\begin{cases} sup_{z\in f^{-1}(y)}[A(z)]^L, & \text{if} f^{-1}(y)\neq\varnothing, \\ 0, & \text{otherwise}, \end{cases}$$

$$[f(A)(y)]^U=\begin{cases} sup_{z\in f^{-1}(y)}[A(z)]^U, & \text{if} f^{-1}(y)\neq\varnothing, \\ 0, & \text{otherwise}, \end{cases}$$

for all $y\in Y$.

Let B be an IVF set in Y. Then the inverse image of B under f, denoted by $f^{-1}(B)$ [8], defined as follows

$$[f^{-1}(B(x))]^L=[B(f(x))]^L,\ [f^{-1}(B(x))]^U=[B(f(x))]^U$$

for all $x\in X$. Then it follows that $f(M_x)=M_{f(x)}$ and $f^{-1}(M_y)=M_{f^{-1}(y)}$.

Definition 2.1 ([8]). A family τ of IVF sets in X is called an *interval-valued fuzzy topology* (simply IVFT) on X if it satisfies the following properties:
(1) $0,\ 1\in\tau$.
(2) $A,\ B\in\tau \Rightarrow A\cap B\in\tau$.
(3) For $i\in J$, $A_i\in\tau \Rightarrow \cup_{i\in J} A_i\in\tau$.

Every member of τ is called an *IVF open set*. An IVF set A is called an *IVF closed set* if the complement of A is an IVF open set. And the pair (X,τ) is alled an *interval-valued fuzzy topological space (simply, IVFTS)*.

In an IVF topological space (X,τ), for $A\in$IVF(X), the IVF closure and the IVF interior of A [8], denoted by cl(A) and int(A), respectively, are defined as

$$\text{cl}(A)=\cap\{B\in\text{IVF}(X):\ B^c\in\tau \text{ and } A\subseteq B\},$$
$$\text{int}(A)=\cup\{B\in\text{IVF}(X):\ B\in\tau \text{ and } B\subseteq A\}.$$

Theorem 2.2 ([8]). Let (X,τ) be an IVF topological space and $A,B\in$IVF(X). Then
(1) A is an IVF closed set iff $A=cl(A)$.
(2) cl(A)$\cup$cl(B)=cl($A\cup B$).
(3) cl(cl(A))=cl(A).
(6) int(A)=1-cl(1-A) and cl(A)=1-int(1-A).

Let A be an IVF set in an IVFTS (X,τ). Then A is said to be IVF α*-open* [3] (resp., IVF *semiopen* [3], IVF *preopen* [3], IVF *regular open* [5] and IVF β*-open* [5]) if $A\subseteq$int(cl(int(A))) (resp., $A\subseteq$cl(int(A)), $A\subseteq$int(cl(A)), $A=$ int(cl(A)) and $A\subseteq$cl(int(cl(A)))).

Let (X,τ_1) and (Y,τ_2) be two IVFTS's. Then $f: X\rightarrow Y$ is said to be IVF *continuous* [8] (resp., IVF α*-continuous* or IVF *strongly semi-continuous* [4]) if for every IVF open set B in Y, $f^{-1}(B)$ is IVF open (resp., IVF α-open) in X. And f is said to be IVF *weakly* α*-continuous* [6] if for every IVF point M_x and each IVF open set V containing $f(M_x)$, there exists an IVF α-open set U containing M_x such that $f(U)\subseteq$cl(V).

3. IVF Almost α-continuous Mappings

Definition 3.1. Let $f: X\rightarrow Y$ be a mapping IVFTS's (X,τ_1) and (Y,τ_2). Then f is said to be *IVF almost* α*-continuous* if for each IVF point M_x and each IVF open set V containing $f(M_x)$, there exists an IVF α-open set U containing M_x such that $f(U)\subseteq$int(cl(V)).

Obviously the following implications are obtained but the converses are not true in general:

IVF continuous ⇒ IVF α-continuous ⇒ IVF almost α-continuous ⇒ IVF weakly α-continuous

Example 3.2. Let $X=I$ and let A, B, C, D and E be IVF sets defined as follows

$$A(x)=(\widetilde{\frac{2}{9}}),\ B(x)=(\widetilde{\frac{1}{3}}),\ C(x)=(\widetilde{\frac{2}{3}}),\ D(x)=(\widetilde{\frac{7}{9}}),$$

$$E(x)=(\widetilde{\frac{8}{9}}).$$

(1) Consider IVF topologies τ_1 and τ_2 on X as follows

$$\tau_1=\{0, A, B, 1\} \text{ and } \tau_2=\{0, B, E, 1\}.$$

Then the identity mapping $f:(X,\tau_1)\to(X,\tau_2)$ is an IVF almost α-continuous mapping but it is not IVF α-continuous.

(2) Consider IVF topologies τ_3 and τ_4 on X as follows

$$\tau_3=\{0, A, C, 1\} \text{ and } \tau_4=\{0, B, E, 1\}.$$

Then the identity mapping $f:(X,\tau_3)\to(X,\tau_4)$ is an IVF weakly α-continuous mapping but it is not IVF almost α-continuous.

Lemma 3.3 (Theorem 3.6 in [6]). Let (X,τ) be an IVFTS and A an IVF set in X. Then
(1) $A\cap$ int(cl(int(A))) is IVF α-open.
(2) $A\cup$ cl(int(cl(A))) is IVF α-closed.

Theorem 3.4. Let $f:X\to Y$ be a mapping between IVFTS's (X,τ_1) and (Y,τ_2). Then the following statements are equivalent:
(1) f is IVF almost α-continuous.
(2) $f^{-1}(B)\subseteq$ int(cl(int(f^{-1}(int(cl(B)))))) for each IVF open set B of Y.
(3) cl(int(cl(f^{-1}(cl(int(F))))))$\subseteq f^{-1}(F)$ for each IVF closed set F in Y.
(4) cl(int(cl(f^{-1}(cl(int(cl(B)))))))$\subseteq f^{-1}$(cl(B)) for each $B\in$IVF(Y).
(5) f^{-1}(int(B))$\subseteq$int(cl(int(f^{-1}(int(cl(int(B))))))) for each $B\in$IVF(Y).
(6) cl(int(cl(f^{-1}(cl(V)))))$\subseteq f^{-1}$(cl(V)) for an IVF regular open set V in Y.
(7) $f^{-1}(F)$ is IVF α-closed for an IVF regular closed set F in Y.
(1) $f^{-1}(V)$ is IVF α-open for an IVF regular open set V in Y.

Proof. (1) $\Rightarrow$ (2) Let B be an IVF open set in Y. Then for each $M_x\in f^{-1}(B)$, there exists an IVF α-open set U of M_x such that $f(U)\subseteq$int(cl(B)). Since U is IVF α-open, $M_x\in U\subseteq$int(cl(int(U)))$\subseteq$int(cl(int(f^{-1}(int(cl(B)))))). Hence the statement (2) is obtained.

(2) $\Rightarrow$ (1) For an IVF point M_x in X and V an IVF open set containing $f(M_x)$, by (2), $M_x\in f^{-1}$(V)$\subseteq$ int(cl(int(f^{-1}(int(cl(V)))))). Put $U=f^{-1}$(int(cl(V)))$\cap$ int(cl(int(f^{-1}(int(cl(V)))))); then by Lema 3.3, U is an IVF α-open set U containing M_x such that $M_x\in U\subseteq$ f^{-1}(int(cl(V))). This implies $f(U)\subseteq$int(cl(V)), and so hence f is IVF almost continuous.

(2) $\Rightarrow$ (3) Let F be any IVF closed set of Y. Then since $1-F$ is IVF open, from (2) and Theorem 2.2, it follows

$$\begin{aligned}f^{-1}(1-F)&\subseteq \mathrm{int(cl(int}(f^{-1}(\mathrm{int(cl}(1-F)))))\\&=\mathrm{int(cl(int}(f^{-1}(1-\mathrm{cl(int}(F))))\\&=\mathrm{int(cl(int}(1-f^{-1}(\mathrm{cl(int}(F))))\\&=1-\mathrm{cl(int(cl}(f^{-1}(\mathrm{cl(int}(F))))).\end{aligned}$$

It implies cl(int(cl(f^{-1}(cl(int(F))))))$\subseteq f^{-1}(F)$.

(3) $\Rightarrow$ (4) Obvious.

(4) $\Rightarrow$ (5) For $B\in$IVF(Y), from hypothesis and Theorem 2.2, it follows

$$\begin{aligned}f^{-1}(\mathrm{int}(B))&=1-(f^{-1}(\mathrm{cl}(1-B)))\\&\subseteq 1-\mathrm{cl(int(cl}(f^{-1}(\mathrm{cl(int(cl}(1-B)))))))\\&=\mathrm{int(cl(int}(f^{-1}(\mathrm{int(cl(int}(B))))))).\end{aligned}$$

Hence f^{-1}(int(B))$\subseteq$int(cl(int(f^{-1}(int(cl(int(B)))))).

(5) $\Rightarrow$ (6) Let V be any IVF regular open set of Y. Then since $1-V$ is IVF regular closed, it follows

$$\begin{aligned}1-f^{-1}(\mathrm{cl}(V))&=f^{-1}(\mathrm{int}(1-V))\\&\subseteq \mathrm{int(cl(int}(f^{-1}(\mathrm{int(cl(int}(1-V)))))))\\&=\mathrm{int(cl(int}(f^{-1}(\mathrm{int}(1-V)))))\\&=\mathrm{int(cl(int}(1-(f^{-1}(\mathrm{cl}(V))))))\\&=1-\mathrm{cl(int(cl}(f^{-1}(\mathrm{cl}(V))))).\end{aligned}$$

Hence we have cl(int(cl(f^{-1}(cl(V)))))$\subseteq$ f^{-1}(cl(V)).

(6) $\Rightarrow$ (7) Let F be any IVF regular closed set in Y. Then int(F) is IVF regular open and by (6) and cl(int(F))=F, cl(int(cl($f^{-1}(F)$)))$\subseteq f^{-1}(F)$, and so F is IVF α-closed.

(7) $\Rightarrow$ (8) Obvious.

(8) $\Rightarrow$ (1) Let V be an IVF open set containing $f(M_x)$. Since int(cl(V)) is IVF regular open, by (8) and $V\subseteq$int(cl(V)), f^{-1}(int(cl(V))) is an IVF α-open set containing M_x. Set $U=f^{-1}$(int(cl(V))). Then U is an IVF α-open set satisfying $f(U)\subseteq$int(cl(V)). Thus f is an IVF almost α-continuous mapping.

Theorem 3.5. Let $f:X\to Y$ be a mapping between IVFTS's (X,τ_1) and (Y,τ_2). Then the following are equivalent:
(1) f is IVF almost α-continuous.
(2) cl(int(cl($f^{-1}(G)$)))$\subseteq f^{-1}$(cl(G)) for each IVF β-open set G in Y.
(3) cl(int(cl($f^{-1}(G)$)))$\subseteq f^{-1}$(cl(G)) for each IVF semi-open set G in Y.

Proof. (1) ⇒ (2) Let G be an IVF β-open set. Then $G \subseteq \text{cl}(\text{int}(\text{cl}(G)))$ and $\text{cl}(G)$ is an IVF regular closed set. Hence from the IVF almost α-continuity, it follows $\text{cl}(\text{int}(\text{cl}(f^{-1}(G)))) \subseteq \text{cl}(\text{int}(\text{cl}(f^{-1}(\text{cl}(G))))) \subseteq f^{-1}(\text{cl}(G))$.

(2) ⇒ (3) It is obvious since every IVF semiopen set is IVF β-open.

(3) ⇒ (1) Let F be an IVF regular closed set. Then F is IVF semiopen, and so from (3), we have

$$\text{cl}(\text{int}(\text{cl}(f^{-1}(F)))) \subseteq f^{-1}(\text{cl}(F)) = f^{-1}(F).$$

This implies $f^{-1}(F)$ is IVF α-closed. Hence, from Theorem 3.4 (7), f is an IVF almost α-continuous mapping.

Theorem 3.6. Let $f: X \to Y$ be a mapping between IVFTS's (X,τ_1) and (Y,τ_2). Then f is IVF almost α-continuous if and only if $\text{cl}(\text{int}(\text{cl}(f^{-1}(\text{cl}(G))))) \subseteq f^{-1}(\text{cl}(G))$ for each IVF preopen set G in Y.

Proof. Suppose f is IVF almost α-continuous. Let G be an IVF preopen set in Y. Then we have $\text{cl}(G) = \text{cl}(\text{int}(\text{cl}(G)))$. Set $U = \text{int}(\text{cl}(G))$; then by Theorem 3.4 (4), $\text{cl}(\text{int}(\text{cl}(f^{-1}(\text{cl}(\text{int}(\text{cl}(U))))))) \subseteq f^{-1}(\text{cl}(U))$.

From $\text{cl}(U) = \text{cl}(G)$ and $\text{cl}(G) = \text{cl}(\text{int}(\text{cl}(G)))$, it follows $\text{cl}(\text{int}(\text{cl}(f^{-1}(\text{cl}(G))))) \subseteq f^{-1}(\text{cl}(G))$.

For the converse, let A be an IVF regular closed set in Y. Then $\text{int}(A)$ is IVF preopen. From hypothesis and $\text{cl}(\text{int}(A)) = A$, it follows $\text{cl}(\text{int}(\text{cl}(f^{-1}(A)))) \subseteq f^{-1}(A)$. This implies $f^{-1}(A)$ is IVF α-closed and hence f is IVF almost α-continuous.

Theorem 3.7. Let $f: X \to Y$ be a mapping between IVFTS's (X,τ_1) and (Y,τ_2). Then f is IVF almost α-continuous if and only if $f^{-1}(G) \subseteq \text{int}(\text{cl}(\text{int}(f^{-1}(\text{int}(\text{cl}(G))))))$ for each IVF preopen set G in Y.

Proof. Suppose f is IVF almost α-continuous and let G be an IVF preopen set in Y. Then $\text{int}(\text{cl}(G))$ is IVF regular open. From Theorem 3.4, we have

$$f^{-1}(G) \subseteq f^{-1}(\text{int}(\text{cl}(G))) \subseteq \text{int}(\text{cl}(\text{int}(f^{-1}(\text{int}(\text{cl}(G)))))).$$

For the converse, let U be IVF regular open. Then U is obviously IVF preopen. By hypothesis, $f^{-1}(U) \subseteq \text{int}(\text{cl}(\text{int}(f^{-1}(\text{int}(\text{cl}(U)))))) = \text{int}(\text{cl}(\text{int}(f^{-1}(U))))$. So $f^{-1}(U)$ is IVF α-open, and hence f is IVF almost α-continuous.

Definition 3.8 ([7]). An IVFTS (X,τ) is said to be IVF semi-regular if for each IVF open set U of X and each IVF point $M_x \in U$ there exists an IVF regular open set V of X such that $M_x \in V \subseteq U$.

Theorem 3.9. Let $f: X \to Y$ be a mapping on IVFTS's (X,τ_1) and (Y,τ_2). If f is IVF almost α-continuous and Y is IVF semi-regular, then f is IVF α-continuous.

Proof. Let M_x be an IVF point in X and U be an IVF open set in Y containing $f(M_x)$. By the IVF semi-regularity of Y, there exists an IVF regular open V of Y such that $f(x_\alpha) \in V \subseteq U$. Since f is IVF almost α-continuous, $f^{-1}(V)$ is an IVF α-open set containing M_x. Set $G = f^{-1}(V)$; then G is an IVF α-open set containing M_x such that $f(M_x) \in f(G) \subseteq U$. Hence f is IVF α-continuous.

Definition 3.10. Let $f: X \to Y$ be a mapping on IVFTS's (X,τ_1) and (Y,τ_2). Then f is said to be IVF almost α^*-open if $f(U) \subseteq \text{int}(\text{cl}(f(U)))$ for each IVF α-open set U in X.

Definition 3.11. Let $f: X \to Y$ be a mapping on IVFTS's (X,τ_1) and (Y,τ_2). If f is IVF weakly α-continuous and IVF almost α^*-open, then f is IVF almost α-continuous.

Proof. Let M_x be an IVF point in X and U be an IVF open set in Y containing $f(M_x)$. By the IVF weakly α-continuity, there exists an IVF α-open set V in Y such that $f(V) \subseteq \text{cl}(U)$. Since f is IVF almost α^*-open, $f(V) \subseteq \text{int}(\text{cl}(f(V))) \subseteq \text{int}(\text{cl}(U))$. This implies f is IVF almost α-continuous.

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