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Reference priors for nonregular Pareto distribution

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Abstract

In this paper, we develop the reference priors for the scale and shape parameters in the nonregular Pareto distribution. We derive the reference priors as noninformative priors and prove the propriety of joint posterior distribution under the general priors including reference priors in the order of inferential importance. Through the simulation study, we compare the reference priors with respect to coverage probabilities of parameter of interest in a frequentist sense.

Keywords: Nonregular case, Pareto distribution, reference prior, scale parameter, shape parameter.

1. Introduction

The Pareto distribution provides a statistical model which has an extensive variety of applications. It has been found in describing distributions of studies of income, property values, insurance risk, stock prices fluctuations, migration, size of cities and firms, word frequencies, occurrences of natural resources, business failures, service time in queuing systems, and error clustering in communications circuits and lifetime data, etc. (Arnold and Press, 1983; Fernández, 2008).

The Pareto distribution is reverse J-shaped and positively skewed with a decreasing hazard rate. Although this distribution was originally applied to analyzing certain socio-economic and natural phenomena with observations in long tails, this has been used potentially for modeling reliability and life time data as well (Arnold and Press, 1983). The Pareto distribution has been used by many authors in a Bayesian viewpoint (e.g., Arnold and Press, 1983, 1989; Geisser, 1984, 1985; Lwin, 1972; Nigm and Hamdy, 1987; Tiwari *et al.*, 1996; Ko and Kim, 1999; Fernández, 2008; Kim *et al.*, 2009; Kang, 2010).

Arnold and Press (1989) studied the Bayesian estimation problem using the independent conjugate prior and modified Lwin prior. Soliman (2001) studied the Bayesian estimation

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of Pareto distribution with the scale and the shape parameters using subjective priors such as conjugate priors and gamma-exponential priors. Many authors, in a Bayesian viewpoint, used the subjective priors such as a conjugate prior. However, in the absence of prior information or past data, Bayesian methods rely on the objective priors or the noninformative priors. Lee *et al.* (2003) developed the probability matching priors in Pareto distribution. But these authors have not considered the reference priors.

Reference priors initiated by Bernardo (1979) and studied intensively by Berger and Berarndo (1989, 1992) are priors which give as little influence as possible to posterior distribution in the sense of entropy. Many researchers have paid much attention to developing reference priors in many statistical models, but many of the researches were concentrated around regular family of distributions. However, nonregular families, such as the uniform or shifted exponential distributions, are also important in many practical problems. The method developed for regular families of distributions can not be applied to nonregular distributions. Ghosal and Samanta (1997) developed the reference priors for the case of one parameter families of discontinuous densities. And Ghosal (1997) derived the reference priors for the multiparameter nonregular cases where the family of densities has discontinuities at some points, which depend on one component of the parameter, while the family is regular with respect to the other parameters.

Nonregular Pareto distribution is very useful in analyzing data from economy or medical study. Arnold and Press (1989) analyzed annual income data using Pareto distribution. As mentioned above, they used various priors to describe annual income. Soliman (2001) used Pareto distribution for the purpose of analyzing Stanford heart translation data. Despite the usefulness of Pareto distribution, there is no result about reference priors for Bayesian analysis. Therefore, we feel a strong necessity for developing reference priors in this distribution.

Let X be a random variable distributed as the Pareto distribution with a shape parameter α and a scale parameter β. Denote it as $X \sim \mathcal{P}(\alpha, \beta)$. This paper focuses on developing the reference priors for the scale and the shape parameters.

The outline of the remaining sections is as follows. In Section 2, we develop reference priors for the scale and the shape parameters. In Section 3, we provide that the propriety of the posterior distribution for the general prior including the reference priors. In Section 4, simulated frequentist coverage probabilities under the derived priors are given for the purpose of comparing developed priors.

2. The reference priors

Reference priors introduced by Bernardo (1979), and extended further by Berger and Bernardo (1992) have become very popular over the years for the development of noninformative priors. Ghosal (1997) derived the reference prior in sense of Bernardo (1979) for multiparameter nonregular cases. In this section, we derive the reference priors for different groups of orderings of parameters by following Ghosal (1997).

Let X denote a random variable distributed as the Pareto distribution $\mathcal{P}(\alpha, \beta)$. Then the probability density function (p.d.f.) is given by

$$f(x|\alpha, \beta) = \alpha \beta^{\alpha} x^{-(\alpha+1)}, x \geq \beta > 0, \alpha > 0. \tag{2.1}$$

First, we derive the reference priors when the scale parameter β is the parameter of interest. The reference priors can be developed by considering a sequence of compact subsets of the parameter space and taking the limit of a sequence of priors as these compact subsets fill out of the parameter space. The compact subsets were taken to be Cartesian products of sets of the form $\alpha \in [a_1, b_1]$. The limit a_1 will tend to 0 and b_1 will tend to ∞. Here and below, a subscripted Q denotes a function that is constant and does not depend on any parameter, but any Q may depend on the ranges of the parameters.

The conditional reference prior for α given β is

$$\pi(\alpha|\beta) = [\det F(\beta,\alpha)]^{\frac{1}{2}} = \alpha^{-1}, \tag{2.2}$$

where the $F(\beta,\alpha)$ is given by

$$F(\beta,\alpha) = \alpha^{-2}.$$

Here $F(\beta,\alpha) = \{4J_{11}(\beta,\alpha)\}$, and J_{11} is defined by

$$J_{11}(\beta,\alpha) = \int g_\alpha(x;\beta,\alpha)^2 dx,$$

where $g_\alpha = \partial g/\partial\alpha$, $g = f^{\frac{1}{2}}$, and f is the p.d.f. (2.1). Thus the normalizing constant $K_l(\beta)$ of the reference prior $\pi(\alpha|\beta)$ is given by

$$K_l(\beta) = \left(\int_{a_1}^{b_1} [\det F(\beta,\alpha)]^{\frac{1}{2}} d\alpha\right)^{-1} = \left(\int_{a_1}^{b_1} \alpha^{-1} d\alpha\right)^{-1} = [\log(b_1/a_1)]^{-1}, \tag{2.3}$$

and so, the p.d.f. of the conditional reference prior of α given β is

$$p_l(\alpha|\beta) = K_l(\beta)\pi(\alpha|\beta) = [\log(b_1/a_1)]^{-1}\alpha^{-1}. \tag{2.4}$$

Thus the marginal reference prior for β is given by

$$\pi_l(\beta) = \exp\left\{\int_{a_1}^{b_1} p_l(\alpha|\beta)\log c(\beta,\alpha) d\alpha\right\} = \beta^{-1}Q(a_1,b_1), \tag{2.5}$$

where $c(\beta,\alpha) = E_{\beta,\alpha}[\partial \log f/\partial\beta] = \alpha/\beta$. Therefore the reference prior for (β,α) when β is parameter of interest is given by

$$\begin{aligned}\pi_1(\beta,\alpha) &= \lim_{l\to\infty}\left[\frac{K_l(\beta)\pi_l(\beta)}{K_l(\beta_0)\pi_l(\beta_0)}\right]\pi(\alpha|\beta)\\ &\propto \beta^{-1}\alpha^{-1},\end{aligned} \tag{2.6}$$

where β_0 is a fixed point. When both β and α are parameters of interest, the reference prior for (β,α) is given by

$$\begin{aligned}\pi_2(\beta,\alpha) &= c(\beta,\alpha)[\det F(\beta,\alpha)]^{\frac{1}{2}}\\ &\propto \beta^{-1}.\end{aligned} \tag{2.7}$$

Also the reference prior for (β, α) based on the appropriate penalty term of Ghosh and Mukerjee (1992) (and also see Ghosal, 1997) when β is parameter of interest given by

$$\pi_3(\beta, \alpha) = c(\beta, \alpha) = \beta^{-1}\alpha. \tag{2.8}$$

Remark 2.1 Lee *et al.* (2003) developed the probability matching priors for β. The developed reference prior π_1 is a probability matching prior, but the reference priors π_2 and π_3 do not satisfy a probability matching criterion.

Next, we derive the reference priors when the shape parameter α is the parameter of interest. The compact subsets were taken to be Cartesian products of sets of the form $\beta \in [a_1, b_1]$. The reference prior for β given α is

$$\pi(\beta|\alpha) = c(\beta, \alpha) = \frac{\alpha}{\beta}. \tag{2.9}$$

The normalizing constant $K_l(\alpha)$ of the reference prior $\pi(\beta|\alpha)$ is given by

$$K_l(\alpha) = \left(\int_{a_1}^{b_1} c(\beta, \alpha) d\beta\right)^{-1} = \left(\int_{a_1}^{b_1} \alpha\beta^{-1} d\beta\right)^{-1} = \alpha^{-1}[\log(b_1/a_1)]^{-1}, \tag{2.10}$$

and so we obtain

$$p_l(\beta|\alpha) = K_l(\alpha)\pi(\beta|\alpha) = [\log(b_1/a_1)]^{-1}\beta^{-1}. \tag{2.11}$$

Thus the marginal reference prior for α is given by

$$\pi_l(\alpha) = \exp\left\{\int_{a_1}^{b_1} p_l(\beta|\alpha)[\det F(\beta, \alpha)]^{\frac{1}{2}} d\beta\right\} = \alpha^{-1}. \tag{2.12}$$

Therefore the reference prior for (α, β) when α is parameter of interest is given by

$$\begin{aligned}\pi_1(\alpha, \beta) &= \lim_{l\to\infty}\left[\frac{K_l(\alpha)\pi_l(\alpha)}{K_l(\alpha_0)\pi_l(\alpha_0)}\right]\pi(\beta|\alpha) \\ &\propto \alpha^{-1}\beta^{-1},\end{aligned} \tag{2.13}$$

where α_0 is a fixed point. Also when both α and β are parameters of interest, the reference prior for (α, β) is given by

$$\begin{aligned}\pi_2(\alpha, \beta) &= c(\beta, \alpha)[\det F(\beta, \alpha)]^{\frac{1}{2}} \\ &\propto \beta^{-1}.\end{aligned} \tag{2.14}$$

When α is the parameter of interest, the reference prior for (α, β) based on the appropriate penalty term of Ghosh and Mukerjee (1992) (and also see Ghosal, 1997) is given by

$$\pi_3(\alpha, \beta) = [\det F(\beta, \alpha)]^{\frac{1}{2}} = \alpha^{-1}. \tag{2.15}$$

Remark 2.2 Lee *et al.* (2003) developed the probability matching priors for α. The developed reference priors π_1 and π_3 satisfy a probability matching criterion, but the reference prior π_2 is not a probability matching prior.

3. Implementation of the Bayesian procedure

We investigate the propriety of posteriors for a general class of priors which includes the reference priors (2.6), (2.7), (2.8), (2.13), and (2.15). We consider the class of priors

$$\pi_g(\alpha, \beta) \propto \alpha^{-a} \beta^{-b}. \tag{3.1}$$

where $a \geq 0$ and $0 \leq b \leq 1$. The following general theorem can be proved.

Theorem 3.1 The posterior distribution of (α, β) under the general prior (3.1) is proper if $n - a > 0$.

Proof: Under the general prior (3.1), the joint posterior for α, β given $\mathbf{x}$ is

$$\pi(\alpha, \beta|\mathbf{x}) \propto \alpha^{n-a} \beta^{n\alpha - b} \prod_{i=1}^{n} x_i^{-\alpha}. \tag{3.2}$$

Then, integrating with respect to β in (3.2), we have the posterior

$$\pi(\alpha|\mathbf{x}) \propto \alpha^{n-a} (n\alpha - b + 1)^{-1} \prod_{i=1}^{n} \left(\frac{x_i}{z}\right)^{-\alpha}, \tag{3.3}$$

where $z = \min\{x_1, \cdots, x_n\}$. Therefore

$$\int_0^\infty \pi(\alpha|\mathbf{x}) \leq n^{-1} \alpha^{n-a-1} \prod_{i=1}^{n} \left(\frac{x_i}{z}\right)^{-\alpha} < \infty, \tag{3.4}$$

if $n - a > 0$. This completes the proof. □

Theorem 3.2 The marginal posterior distribution of α based on the general prior (3.1) is

$$\pi(\alpha|\mathbf{x}) \propto \alpha^{n-a} (n\alpha - b + 1)^{-1} \prod_{i=1}^{n} \left(\frac{x_i}{z}\right)^{-\alpha}, \tag{3.5}$$

where $z = \min\{x_1, \cdots, x_n\}$. Also, the marginal posterior distribution of β based on the general prior (3.1) is

$$\pi(\beta|\mathbf{x}) \propto \beta^{-b} \left[\sum_{i=1}^{n_1} \log \frac{x_i}{\beta}\right]^{-(n-a+1)}. \tag{3.6}$$

Note that normalizing constants for the marginal densities of α and β require a one dimensional integration. Therefore, we can have the marginal posterior densities of α and β, so we compute the marginal moments of α and β. In Section 4, we investigate the frequentist coverage probabilities for the shape and the scale parameters under the reference priors π_1, π_2 and π_3, respectively.

4. Numerical study

We investigate the frequentist coverage probability by investigating the credible intervals of the marginal posteriors densities of α and β under the noninformative priors π given in Section 3 for several configurations (α, β) and n. That is to say, the frequentist coverage of a $100(1-\eta)$%th posterior quantile should be close to $1-\eta$. This is done numerically. Table 4.1 and 4.2 give numerical values of the frequentist coverage probabilities of 0.05 (0.95) posterior quantiles for the proposed priors. The computation of these numerical values is based on the following algorithm for any fixed true (α, β) and any prespecified value η. Here η is 0.05 (0.95).

Let $\alpha^\pi(\eta|\mathbf{x})$ be the posterior η-quantile of α given $\mathbf{x}$. That is to say, $F(\alpha^\pi(\eta|\mathbf{x})|\ \mathbf{x}) = \eta$, where $F(\cdot|\mathbf{x})$ is the marginal posterior distribution of α. Then the frequentist coverage probability of this one sided credible interval of α is

$$P_{(\alpha,\beta)}(\eta, \alpha) = P_{(\alpha,\beta)}(0 < \alpha < \alpha^\pi(\eta|\mathbf{x})). \tag{4.1}$$

Table 4.1 Frequentist coverage probabilities of 0.05 (0.95) posterior quantiles for α

α	β	n	π_1	π_2	π_3
0.1	0.1	5	0.046 (0.952)	0.138 (0.983)	0.092 (0.979)
		10	0.051 (0.952)	0.098 (0.976)	0.069 (0.967)
		20	0.050 (0.947)	0.080 (0.968)	0.058 (0.956)
	1.0	5	0.049 (0.954)	0.135 (0.983)	0.093 (0.978)
		10	0.053 (0.951)	0.101 (0.973)	0.072 (0.966)
		20	0.051 (0.951)	0.079 (0.968)	0.059 (0.958)
	10.0	5	0.051 (0.951)	0.138 (0.982)	0.095 (0.977)
		10	0.048 (0.951)	0.099 (0.975)	0.068 (0.967)
		20	0.045 (0.951)	0.076 (0.970)	0.052 (0.959)
1.0	0.1	5	0.054 (0.951)	0.139 (0.980)	0.063 (0.962)
		10	0.054 (0.948)	0.097 (0.974)	0.056 (0.952)
		20	0.049 (0.949)	0.080 (0.968)	0.050 (0.951)
	1.0	5	0.046 (0.951)	0.134 (0.981)	0.056 (0.961)
		10	0.046 (0.950)	0.098 (0.976)	0.050 (0.953)
		20	0.048 (0.950)	0.077 (0.969)	0.049 (0.951)
	10.0	5	0.048 (0.949)	0.138 (0.980)	0.058 (0.959)
		10	0.050 (0.953)	0.096 (0.975)	0.053 (0.957)
		20	0.049 (0.948)	0.081 (0.966)	0.050 (0.949)
10.0	0.1	5	0.052 (0.948)	0.136 (0.980)	0.052 (0.949)
		10	0.044 (0.950)	0.094 (0.974)	0.044 (0.951)
		20	0.051 (0.951)	0.082 (0.969)	0.051 (0.951)
	1.0	5	0.051 (0.952)	0.142 (0.981)	0.051 (0.953)
		10	0.052 (0.952)	0.101 (0.975)	0.053 (0.953)
		20	0.052 (0.949)	0.084 (0.968)	0.052 (0.949)
	10.0	5	0.049 (0.952)	0.134 (0.983)	0.050 (0.954)
		10	0.052 (0.950)	0.105 (0.974)	0.053 (0.950)
		20	0.048 (0.952)	0.077 (0.969)	0.048 (0.952)

Also, we can give the frequentist coverage probability of this one sided credible interval of β by the above method. The estimated $P_{(\alpha,\beta)}(\eta, \alpha)$ and $P_{(\beta,\alpha)}(\eta, \beta)$ when $\eta = 0.05(0.95)$ are shown in Tables 4.1 and 4.2, respectively. In particular, for fixed (α, β), we take 10,000 independent random samples of $\mathbf{X} = (X_1, \cdots, X_n)$ from the model (2.1).

For the cases presented in Table 4.1, we see that the reference prior π_1 matches the target coverage probability much more accurately than do the reference priors π_2 and π_3 for values of (α, β) and values of (n_1, n_2). Also, the reference prior π_3 gives good coverage probabilities for the sample sizes 10 and 20. Note that the reference priors π_1 and π_3 are probability matching priors. The results of the table are not much sensitive to change of the values of (α, β) under the reference prior π_1. Thus, we recommend the use of the reference prior π_1 in the sense of asymptotic frequentist coverage property for the shape parameter α.

For the cases presented in Table 4.2, we see that the reference prior π_1 matches the target coverage probability much more accurately than do the reference priors π_2 and π_3 for values of (α, β) and values of (n_1, n_2). Note that the reference prior π_1 is a probability matching prior, and the results of the table are not much sensitive to change of the values of (α, β). Thus, we recommend the use of the reference prior π_1 in the sense of asymptotic frequentist coverage property for the scale parameter β.

Table 4.2 Frequentist coverage probabilities of 0.05 (0.95) posterior quantiles for β

β	α	n	π_1	π_2	π_3
0.1	0.1	5	0.048 (0.952)	0.088 (0.961)	0.133 (0.969)
		10	0.051 (0.951)	0.069 (0.955)	0.087 (0.959)
		20	0.054 (0.946)	0.063 (0.949)	0.070 (0.952)
	1.0	5	0.047 (0.944)	0.091 (0.955)	0.135 (0.962)
		10	0.051 (0.952)	0.068 (0.957)	0.088 (0.961)
		20	0.049 (0.948)	0.058 (0.950)	0.066 (0.952)
	10.0	5	0.050 (0.950)	0.093 (0.960)	0.137 (0.966)
		10	0.051 (0.948)	0.068 (0.953)	0.086 (0.957)
		20	0.048 (0.951)	0.056 (0.954)	0.065 (0.957)
1.0	0.1	5	0.051 (0.949)	0.095 (0.959)	0.140 (0.966)
		10	0.047 (0.948)	0.064 (0.952)	0.082 (0.956)
		20	0.049 (0.951)	0.057 (0.953)	0.065 (0.956)
	1.0	5	0.047 (0.951)	0.087 (0.961)	0.133 (0.968)
		10	0.050 (0.948)	0.069 (0.955)	0.086 (0.959)
		20	0.048 (0.952)	0.057 (0.955)	0.066 (0.958)
	10.0	5	0.051 (0.950)	0.093 (0.960)	0.134 (0.967)
		10	0.049 (0.950)	0.065 (0.954)	0.084 (0.959)
		20	0.048 (0.950)	0.058 (0.952)	0.067 (0.954)
10.0	0.1	5	0.051 (0.946)	0.093 (0.956)	0.140 (0.964)
		10	0.050 (0.950)	0.069 (0.955)	0.088 (0.958)
		20	0.047 (0.954)	0.056 (0.957)	0.064 (0.959)
	1.0	5	0.052 (0.948)	0.093 (0.959)	0.138 (0.966)
		10	0.049 (0.948)	0.068 (0.953)	0.088 (0.957)
		20	0.055 (0.947)	0.063 (0.950)	0.071 (0.953)
	10.0	5	0.052 (0.950)	0.095 (0.960)	0.141 (0.965)
		10	0.048 (0.949)	0.068 (0.954)	0.084 (0.958)
		20	0.049 (0.949)	0.057 (0.953)	0.065 (0.955)

5. Concluding remarks

In the nonregular Pareto distribution, we have found the reference priors for the shape and the scale parameters. For each parameter, we derived the reference priors when α or β are the parameter of interest, and both α and β are parameters of interest. We showed that the reference prior π_1 for each parameter performs better than do the reference priors π_2 and

π_3, in matching the target coverage probabilities. Also, we revealed that the reference prior π_1 satisfies a probability matching criterion. Thus, we recommend the use of the reference prior π_1 for the Bayesian inference of the shape and the scale parameters.

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