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N^p-SPACES

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Abstract. We introduce a new norm, called the N^p-norm ($1 \leq p < \infty$) on the space $N^p(V, W)$ where V and W are abstract operator spaces. By proving some fundamental properties of the space $N^p(V, W)$, we also discover that if W is complete, then the space $N^p(V, W)$ is also a Banach space with respect to this norm for $1 \leq p < \infty$.

Introduction

For abstract operator spaces V and W, a (bounded) linear map $\phi : V \to W$ provides another linear map $\phi_n : M_n(V) \to M_n(W)$ defined by

$$\phi_n((a_{i,j})) = (\phi(a_{i,j})),$$

where $n = 1, 2, \ldots$ and $M_n(V)$ denotes the normed linear space of $n \times n$ matrices with entries from a linear space V.

In this paper, $B(H)$ denotes the space of all bounded operators on a Hilbert space H with the operator norm.

Since ϕ is a bounded map, each ϕ_n is also bounded, and when $\|\phi\|_{cb} = \sup_n \|\phi_n\|$ is finite, we call ϕ a *completely bounded* map. That is, if a sequence $\{\|\phi_n\|\}_{n=1}^{\infty}$ belongs to l^∞, then ϕ is said to be a *completely bounded* map.

W. Arveson [1] and W. Stinespring [7] introduced operator space theory related to complete boundedness for a map $\phi : S \to B(K)$ where $S \subset B(H)$ and H and K are Hilbert spaces. It has also developed in the 1980s through the works of E. Effros ([2]), V. Paulsen ([3]), G. Pisier ([5]), Z. Ruan ([2]), and G. Wittstock ([8, 9]).

Then, naturally we have the following question:

Question. When does the sequence $\{\|\phi_n\|\}_{n=1}^{\infty}$ belong to l^p ($1 \leq p < \infty$)?

To answer this question, in this paper, we consider l^p-norm ($1 \leq p < \infty$) for the sequence $\{\|\phi_n\|\}_{n=1}^{\infty}$.

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Since

$$\|\phi_1\| \le \|\phi_2\| \le \|\phi_3\| \le \cdots,$$

there is no nonzero map ϕ such that $\{\|\phi_n\|\}_{n=1}^{\infty}$ belongs to l^p. To put it another way, we define a new norm

$$\|\phi\|_p = \sum_{n=1}^{\infty} \frac{\|\phi_n\|}{n^p}$$

and study the space $N^p(V,W)$ which is a vector space consisting of all linear maps $\phi : V \to W$ for which $\|\phi\|_p < \infty$. That is, we provide a new norm $\|\cdot\|_p$, called the N^p-norm, on the space $N^p(V,W)$ $(1 \le p < \infty)$.

In Section 2, we prove fundamental properties of the space $N^p(V,W)(1 \le p < \infty)$. In Proposition 2.2, we show that

(i) $N^p(V,W) \subset N^q(V,W)$ if $1 \le p \le q < \infty$.
(ii) If $\phi : V \to W$ is completely bounded, then $\phi \in N^p(V,W)$ for all $p > 1$,

and, in Proposition 2.6, we characterize a (bounded) linear map $\phi : V \to W$ by using the space $N^p(V,W)$, that is, the following statements are equivalent:
(a) $\phi : V \to W$ is a (bounded) linear map.
(b) $\phi \in N^p(V,W)$ for any $p > 2$.

The main results of this paper are given when W is complete as follows:

(i) (Theorem 2.8) If W is complete, then $N^p(V,W)$ is a Banach space for $1 \le p < \infty$.
(ii) (Corollary 2.9) If W is complete, then the space $B(V,W)$ with N^p-norm is a Banach space for $2 < p < \infty$.

1. Preliminaries and notation

Let $\mathbb{M}_{n,m}(V)$ denote the linear space of $n \times m$ matrices with entries from a linear space V and $B(H_1, H_2)$ be the space of all bounded operators $T : H_1 \to H_2$ where $H_i (i = 1, 2)$ is a Hilbert space. Any operator considered in this paper is bounded.

We write $\mathbb{M}_n(V) = \mathbb{M}_{n,n}(V)$ and if $V = \mathbb{C}$, we let $\mathbb{M}_{n,m} = \mathbb{M}_{n,m}(\mathbb{C})$. We will denote a typical element of $\mathbb{M}_n(V)$ by $(v_{i,j})$.

Definition 1.1. A (*concrete*) *operator space* V on a Hilbert space is a closed subspace of $B(H)$.

If V is a concrete operator space, then the inclusion

$$\mathbb{M}_n(V) \subset \mathbb{M}_n(B(H)) = B(H^n)$$

provides a norm $\|\cdot\|_{\mathbb{M}_n(V)}$ on $\mathbb{M}_n(V)$, and $M_n(V)$ denotes the corresponding normed space.

We define a *matrix norm* $\|\cdot\|$ on a linear space W to be an assignment of a norm $\|\cdot\|_{\mathbb{M}_n(W)}$ on the matrix space $\mathbb{M}_n(W)$ for each $n \in \mathbb{N}$.

Definition 1.2. An *abstract operator space* is a linear space W together with a matrix norm $\|\cdot\|$ for which

(i) $$\|v \oplus w\|_{\mathbb{M}_{m+n}(W)} = \max\{\|v\|_{\mathbb{M}_m(W)}, \|w\|_{\mathbb{M}_n(W)}\}$$

and

(ii) $$\|\alpha v \beta\|_{\mathbb{M}_n(W)} \leq \|\alpha\| \, \|v\|_{\mathbb{M}_m(W)} \, \|\beta\|$$

for all $v \in \mathbb{M}_m(W)$, $w \in \mathbb{M}_n(W)$ and $\alpha \in M_{n,m}$, $\beta \in M_{m,n}$.

By a *linear map* on an abstract operator space V, we mean a bounded linear map defined on V. The set of linear maps from V to W is denoted by $B(V, W)$ with $B(V, V)$ abbreviated by $B(V)$.

Given two abstract operator spaces V and W and a linear map $\phi : V \to W$, we also obtain a linear map $\phi_n : M_n(V) \to M_n(W)$ defined by

(1.1) $$\phi_n((v_{i,j})) = (\phi(v_{i,j})).$$

Since ϕ is a bounded map, each ϕ_n is also bounded.

Definition 1.3 ([3]). If $\sup_n \|\phi_n\|$ is finite, then ϕ is said to be a *completely bounded* map.

If ϕ is completely bounded, then we set

$$\|\phi\|_{cb} = \sup_n \|\phi_n\|,$$

and $CB(V, W)$ denotes the space of completely bounded maps from V to W.

Recall that l^∞ denotes the collection of all bounded complex functions on the positive integers. If f is a function in l^∞ and

$$\|f\|_\infty = \sup\{|f(n)| : n = 1, 2, \ldots\},$$

l^∞ is a Banach space with respect to this norm.

Therefore, in Definition 1.3, we can also define a *completely bounded* map as following:

If a sequence $\{\|\phi_n\|\}_{n=1}^\infty$ belongs to l^∞, then ϕ is said to be a *completely bounded* map.

Recall that, for $1 \leq p < \infty$, l^p is the set of all complex functions g on the positive integers such that

$$\sum_{i=1}^\infty |g(i)|^p < \infty;$$

and define

$$\|g\|_p^p = \sum_{n=1}^\infty |g(n)|^p.$$

Then, l^p is a Banach space with respect to this norm.

Thus, naturally, we have the following question:

Question. When does the sequence $\{\|\phi_n\|\}_{n=1}^\infty$ belong to l^p $(1 \leq p < \infty)$?

Since

$$\|\phi_1\| \le \|\phi_2\| \le \|\phi_3\| \le \cdots,$$

there is no nonzero map ϕ such that $\{\|\phi_n\|\}_{n=1}^{\infty}$ belongs to l^p $(1 \le p < \infty)$.

However, in the next section, we will introduce a new norm, called the N^p-norm, and a new space, called the N^p-space, to solve this problem.

2. The N^p-spaces

Let V and W be abstract operator spaces. For a linear map $\phi : V \to W$ and $1 \le p < \infty$, we introduce a new norm $\|\phi\|_p$ and the space $N^p(V, W)$ in the following definition.

Definition 2.1. Let V and W be abstract operator spaces. If $\phi : V \to W$ is a linear map and $1 \le p < \infty$, then define a norm

$$\|\phi\|_p = \sum_{n=1}^{\infty} \frac{\|\phi_n\|}{n^p} \tag{2.1}$$

and let the space $N^p(V, W)$ be a vector space consisting of all linear maps $\phi : V \to W$ for which $\|\phi\|_p < \infty$.

We can easily see that the equation (2.1) defines a norm on the $N^p(V, W)$-spaces, and we call $\|\phi\|_p$ the N^p-norm of ϕ.

Since we defined a new norm, called the N^p-norm, and a new space, called $N^p(V, W)$-space, naturally, we could ask the following question:

Whether the $N^p(V, W)$-space is a Banach space or not with respect to the N^p-norm?

We will discuss about this problem in Theorem 2.8, and before answering this question, we start comparing two spaces $N^p(V, W)$ and $N^q(V, W)$ for positive numbers p and q such that $q \ge p$. Furthermore, we compare two spaces $CB(V, W)$ and $N^p(V, W)$ for $p > 1$.

Note that, for any bounded linear operator $\varphi : V \to W$,

$$\|\varphi\| \left(\sum_{n=1}^{\infty} \frac{1}{n^p}\right) \le \|\varphi\|_p .$$

This implies that $N^1(V, W) = \{0\}$.

Proposition 2.2. *Let V and W be abstract operator spaces and $\phi : V \to W$ be a linear map. Then the following statements are true.*

(i) *If $\phi \in N^p(V, W)$ for some $1 \le p < \infty$, then $\phi \in N^q(V, W)$ for any $q \ge p$. Thus,*

$$N^p(V, W) \subset N^q(V, W) \tag{2.2}$$

if $1 \le p \le q < \infty$.

(ii) *If* $\phi : V \to W$ *is completely bounded, then* $\phi \in N^p(V,W)$ *for all* $p > 1$. *Thus,*

$$CB(V,W) \subset N^p(V,W)$$

for any $p > 1$.

(iii) *If*

$$\|\phi_n\| \leq n^{p-1-\epsilon} \tag{2.3}$$

for some $\epsilon > 0$ *and* $n = 1, 2, 3, \ldots,$ *then* $\phi \in N^p(V,W)$.

Proof. (i) Suppose that $\phi \in N^p(V,W)$ and $1 \leq p \leq q$. For any $n = 1, 2, \ldots,$

$$\frac{\|\phi_n\|}{n^q} \leq \frac{\|\phi_n\|}{n^p}.$$

It follows that

$$\|\phi\|_q \leq \|\phi\|_p. \tag{2.4}$$

Since $\phi \in N^p(V,W)$,

$$\|\phi\|_p < \infty.$$

Thus, from inequality (2.4),

$$\|\phi\|_q < \infty,$$

that is,

$$\phi \in N^q(V,W)$$

which proves the inclusion (2.2).

(ii) If $\phi : V \to W$ is completely bounded and

$$\|\phi\|_{cb} = m,$$

then

$$\|\phi\|_p = \sum_{n=1}^{\infty} \frac{\|\phi_n\|}{n^p} \leq m \sum_{n=1}^{\infty} \frac{1}{n^p}.$$

Since

$$\sum_{n=1}^{\infty} \frac{1}{n^p} < \infty$$

for any $p > 1$, we conclude that

$$\phi \in N^p(V,W)$$

for any $p > 1$.

(iii) By (2.3),

$$\|\phi\|_p = \sum_{n=1}^{\infty} \frac{\|\phi_n\|}{n^p} \leq \sum_{n=1}^{\infty} \frac{1}{n^{1+\epsilon}} < \infty.$$

Thus, $\phi \in N^p(V,W)$. $\square$

We can easily see that the following statements are equivalent:

(a) $\phi \in N^p(V, W)$.

(b) the sequence $\{\frac{\|\phi_n\|^{\frac{1}{p}}}{n}\}_{n=1}^{\infty}$ belongs to l^p for $1 \le p < \infty$.

Therefore, since, for $1 \le p \le q < \infty$,

$$l^p \subset l^q,$$

we can also provide another proof of Proposition 2.2(i). We will leave it as an exercise for the reader.

Proposition 2.3 ([2]). *If V is an abstract operator space and $\varphi : V \to M_n$ is a linear map, then*

$$\|\varphi_n\| = \|\varphi\|_{cb}. \tag{2.5}$$

Therefore, every $\varphi : V \to M_n$ in $B(V, M_n)$ is completely bounded so that $CB(V, M_n) = B(V, M_n)$. Furthermore, in the next corollary, we will show that $CB(V, M_n) = N^p(V, M_n) = B(V, M_n)$ for $p > 1$.

Corollary 2.4. *If V is an abstract operator space and $\varphi : V \to M_n$ is a linear map, then for $p > 1$,*

$$\varphi \in N^p(V, M_n)$$

and

$$\|\varphi\|_p \le \|\varphi\|_{cb} \sum_{k=1}^{\infty} \frac{1}{k^p}. \tag{2.6}$$

Furthermore, for $p > 1$,

$$CB(V, M_n) = N^p(V, M_n) = B(V, M_n).$$

In particular, for $n = 1$,

$$\|\varphi\|_p = \|\varphi\| \sum_{k=1}^{\infty} \frac{1}{k^p}. \tag{2.7}$$

Proof. By Proposition 2.3,

$$\|\varphi_1\| \le \|\varphi_2\| \le \cdots \le \|\varphi_n\| = \|\varphi_{n+1}\| = \cdots = \|\varphi\|_{cb}.$$

It follows that

$$\|\varphi\|_p = \sum_{k=1}^{\infty} \frac{\|\varphi_k\|}{k^p} \le \sum_{k=1}^{\infty} \frac{\|\varphi\|_{cb}}{k^p} = \|\varphi\|_{cb} \sum_{k=1}^{\infty} \frac{1}{k^p}. \tag{2.8}$$

By Proposition 2.3 and (2.8), we conclude that

$$CB(V, M_n) = N^p(V, M_n) = B(V, M_n) \text{ for } p > 1.$$

If $n = 1$, then φ is a linear functional. Thus, by Proposition 2.3, clearly, the equation (2.7) is true. □

Therefore,

$$B(V,\mathbb{C}) \subset N^p(V,\mathbb{C})$$

for $p > 1$.

Proposition 2.5 ([5]). *Let V and W be abstract operator spaces and $\phi : V \to W$ be a linear map. Then,*

$$\|\phi_n\| \leq n\,\|\phi\|\,. \tag{2.9}$$

As an example, if we let τ denote the transpose map on $B(l^2)$, then τ is an isometry, but $\|\tau_n\| = n$. It follows that $\tau \in N^p(B(l^2))$ for $2 < p < \infty$, but τ is not contained in $N^p(B(l^2))$ for $1 < p \leq 2$.

Proposition 2.6. *Let V and W be abstract operator spaces. Then the following statements are equivalent*:

(i) *$\phi : V \to W$ is a linear map, that is, $\phi \in B(V,W)$.*

(ii) *$\phi \in N^p(V,W)$ for any $p > 2$.*

Proof. (i) $\Rightarrow$ (ii). By (2.9),

$$\frac{\|\phi_n\|}{n^p} \leq \frac{\|\phi\|}{n^{p-1}} \tag{2.10}$$

for any $p > 2$. From the inequality (2.10),

$$\|\phi\|_p = \sum_{n=1}^{\infty} \frac{\|\phi_n\|}{n^p} \leq \sum_{n=1}^{\infty} \frac{\|\phi\|}{n^{p-1}} \tag{2.11}$$

for any $p > 2$.

Since $\phi \in B(V,W)$, we have

$$\sum_{n=1}^{\infty} \frac{\|\phi\|}{n^{p-1}} < \infty$$

for any $p > 2$. From (2.11), we conclude that

$$\phi \in N^p(V,W)$$

for any $p > 2$.

(ii) $\Rightarrow$ (i). Since $\phi \in N^p(V,W)$ for any $p > 2$,

$$\|\phi\|_p = \sum_{n=1}^{\infty} \frac{\|\phi_n\|}{n^p} = \|\phi_1\| + \sum_{n=2}^{\infty} \frac{\|\phi_n\|}{n^p} < \infty.$$

It follows that

$$\|\phi_1\| = \|\phi\| < \infty.$$

Thus, $\phi \in B(V,W)$. □

By Proposition 2.6, we have the following conclusion:

Corollary 2.7. *Let V and W be abstract operator spaces. Then,*

$$B(V,W) = N^p(V,W) \quad \textit{for} \quad 2 < p < \infty.$$

From Proposition 2.2(i) and Proposition 2.6, for every bounded map $\phi : V \to W$, we can find a real number $r_\phi \geq 1$ defined by

$$r_\phi = \inf\{p : \phi \in N^p(V, W) \text{ and } 1 \leq p < \infty\}.$$

The number r_ϕ is called the *index* of ϕ. Clearly,

$$1 \leq r_\phi \leq 2.$$

Finally, in the next theorem, we provide a sufficient condition for the space $N^p(V, W)$ to be complete with respect to the N^p-norm.

Theorem 2.8. *Let V and W be abstract operator spaces. If W is complete, then $N^p(V, W)$ is a Banach space for $1 \leq p < \infty$.*

Proof. Suppose that W is complete. Let $\{\varphi_{(l)}\}_{l=1}^\infty$ be a Cauchy sequence in $N^p(V, W)$ for a fixed $p \in [1, \infty)$ and $\epsilon > 0$ be given. Then there is a natural number $N(\epsilon)$ such that for all natural numbers $n, m \geq N(\epsilon)$, we have

$$\left\|\varphi_{(n)} - \varphi_{(m)}\right\|_p = \sum_{k=1}^\infty \frac{\left\|(\varphi_{(n)} - \varphi_{(m)})_k\right\|}{k^p} < \epsilon. \tag{2.12}$$

Since

$$\left\|\varphi_{(n)} - \varphi_{(m)}\right\| \leq \left\|\varphi_{(n)} - \varphi_{(m)}\right\|_p, \tag{2.13}$$

$\{\varphi_{(l)}\}_{l=1}^\infty$ is also a Cauchy sequence in $B(V, W)$.

Since W is complete, so is $B(V, W)$. It follows that there is a bounded operator $\varphi \in B(V, W)$ such that

$$\lim_{l\to\infty} \left\|\varphi_{(l)} - \varphi\right\| = 0. \tag{2.14}$$

Let $k \in \{1, 2, 3, \ldots\}$ be given. It follows from (2.12) that

$$\left\|(\varphi_{(n)} - \varphi_{(m)})_k\right\| \leq k^p\epsilon$$

for all natural numbers $n, m \geq N(\epsilon)$.

Thus, for any $v = [v_{ij}] \in M_k(V)$,

$$\left\|(\varphi_{(n)} - \varphi_{(m)})_k(v)\right\| \leq \left\|(\varphi_{(n)} - \varphi_{(m)})_k\right\| \|v\| \leq k^p\epsilon \|v\| \tag{2.15}$$

if $n, m \geq N(\epsilon)$.

Since $\varphi_{(n)}(v_{i,j})$ converges to $\varphi(v_{i,j})$ in W, (2.15) implies that

$$\left\|(\varphi - \varphi_{(m)})_k(v)\right\| \leq k^p\epsilon \|v\| \tag{2.16}$$

if $m \geq N(\epsilon)$. It follows from (2.16) that

$$\left\|(\varphi - \varphi_{(m)})_k\right\| \leq k^p\epsilon \tag{2.17}$$

if $m \geq N(\varepsilon)$.

Since ϵ is arbitrary, we have

$$\lim_{m\to\infty} \left\|(\varphi_{(m)} - \varphi)_k\right\| = \lim_{m\to\infty} \left\|(\varphi - \varphi_{(m)})_k\right\| = 0 \tag{2.18}$$

for any $k \in \{1, 2, 3, \ldots\}$.

By triangle inequality,

$$\left\|(\varphi_{(n)} - \varphi_{(m)})_k\right\| - \left\|(\varphi_{(n)} - \varphi)_k\right\| \le \left\|(\varphi_{(m)} - \varphi)_k\right\|$$

and

$$-\left\|(\varphi_{(m)} - \varphi)_k\right\| \le \left\|(\varphi_{(n)} - \varphi_{(m)})_k\right\| - \left\|(\varphi_{(n)} - \varphi)_k\right\|,$$

that is,

$$\left|\,\left\|(\varphi_{(n)} - \varphi_{(m)})_k\right\| - \left\|(\varphi_{(n)} - \varphi)_k\right\|\,\right| \le \left\|(\varphi_{(m)} - \varphi)_k\right\|.$$

By the equation (2.18), we conclude that

$$\lim_{m\to\infty} \left\|(\varphi_{(n)} - \varphi_{(m)})_k\right\| = \left\|(\varphi_{(n)} - \varphi)_k\right\| \tag{2.19}$$

for any n and k in $\{1, 2, 3, \ldots\}$.

Let $n \ge N(\epsilon)$ be given and $\{u_k\}_{k=1}^{\infty}$ be a sequence of functions defined on $\{1, 2, 3, \ldots\}$ by

$$u_k(m) = \frac{\left\|(\varphi_{(n)} - \varphi_{(m)})_k\right\|}{k^p}.$$

Since $\{\varphi_{(l)}\}_{l=1}^{\infty}$ is a Cauchy sequence in $N^p(V, W)$, the equations (2.12), (2.18), and (2.19) imply that if $n \ge N(\epsilon)$, then

$$\begin{aligned}\lim_{m\to\infty} \left\|\varphi_{(n)} - \varphi_{(m)}\right\|_p &= \lim_{m\to\infty} \sum_{k=1}^{\infty} \frac{\left\|(\varphi_{(n)} - \varphi_{(m)})_k\right\|}{k^p} = \lim_{m\to\infty} \sum_{k=1}^{\infty} u_k(m)\\ &= \sum_{k=1}^{\infty} \lim_{m\to\infty} u_k(m) = \sum_{k=1}^{\infty} \lim_{m\to\infty} \frac{\left\|(\varphi_{(n)} - \varphi_{(m)})_k\right\|}{k^p}\\ &= \sum_{k=1}^{\infty} \frac{\left\|(\varphi_{(n)} - \varphi)_k\right\|}{k^p},\end{aligned}$$

that is, if $n \ge N(\epsilon)$ and $p \in [1, \infty)$,

$$\lim_{m\to\infty} \left\|\varphi_{(n)} - \varphi_{(m)}\right\|_p = \left\|\varphi_{(n)} - \varphi\right\|_p. \tag{2.20}$$

From (2.12) and (2.20), we can conclude that

$$\lim_{n\to\infty} \left\|\varphi_{(n)} - \varphi\right\|_p = 0,$$

and so $\varphi_{(n)} \to \varphi$ in N^p-norm.

Thus, there is a natural number n_0 such that

$$\left\|\varphi_{(n_0)} - \varphi\right\|_p \le \epsilon, \tag{2.21}$$

and so by triangle inequality and the inequality (2.21), we have

$$\begin{aligned}\|\varphi\|_p = \sum_{k=1}^{\infty} \frac{\|\varphi_k\|}{k^p} &\le \sum_{k=1}^{\infty} \frac{\left\|(\varphi_{(n_0)})_k - \varphi_k\right\| + \left\|(\varphi_{(n_0)})_k\right\|}{k^p}\\ &= \left\|\varphi_{(n_0)} - \varphi\right\|_p + \left\|\varphi_{(n_0)}\right\|_p \le \epsilon + \left\|\varphi_{(n_0)}\right\|_p.\end{aligned}$$

Since $\varphi_{(n_0)} \in N^p(V, W)$, i.e., $\left\|\varphi_{(n_0)}\right\|_p < \infty$, we have

$$\|\varphi\|_p < \infty.$$

Thus,

$$\varphi \in N^p(V,W).$$

Therefore, $N^p(V,W)$ is complete for $1 \leq p < \infty$. □

Corollary 2.9. *Let V and W be abstract operator spaces and W be complete. Then the space $B(V,W)$ with N^p-norm is a Banach space for $2 < p < \infty$.*

Proof. By Corollary 2.7 and Theorem 2.8, it is clear. □

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