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φ-FRAMES AND φ-RIESZ BASES ON LOCALLY COMPACT ABELIAN GROUPS

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Abstract. We introduce φ-frames in $L^2(G)$, as a generalization of a-frames defined in [8], where G is a locally compact Abelian group and φ is a topological automorphism on G. We give a characterization of φ-frames with regard to usual frames in $L^2(G)$ and show that φ-frames share several useful properties with frames. We define the associated φ-analysis and φ-preframe operators, with which we obtain criteria for a sequence to be a φ-frame or a φ-Bessel sequence. We also define φ-Riesz bases in $L^2(G)$ and establish equivalent conditions for a sequence in $L^2(G)$ to be a φ-Riesz basis.

1. Introduction and preliminaries

The theory of frames was introduced by Duffin and Schaeffer [10] in the early 1950s to deal with problems in nonharmonic Fourier series. There has been renewed interest in the subject related to its role in wavelet theory and a lot of new applications. Several kinds of frames have been introduced up to now; e.g. frames in Hilbert C*-modules (modular frames) [14], frames of subspaces [7], G-frames [26], p-frames [1], frames for Banach spaces [6], a-frames [8], and many others for different purposes. In this paper we define and investigate φ-frames in $L^2(G)$, using the φ-bracket product, as a vector valued inner product on $L^2(G)$ introduced in [19], where G is a locally compact Abelian (which will be abbreviated to "LCA") group and φ is a topological automorphism on G. One of the nice things about φ-frames is the fact that they are useful in studying Gabor systems in the way that there is a close relationship between these frames and Gabor frames in $L^2(G)$. Indeed, our results relate Gabor frames in $L^2(G)$, which have become a paradigm for the spectral analysis associated with time frequency methods [6], to φ-frames. Our construction is related to an extension of Casazza and Lammers' definition of a-frames, $a > 0$, on $L^2(\mathbb{R})$ in [8], to the more general setting of $L^2(G)$, in a new and different approach. We characterize φ-frames in terms of the usual frames

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in $L^2(G)$ (Theorem 2.1 below), which reveals the above mentioned relation, and we show that φ-frames have several useful properties in common with frames. We also define φ-Riesz bases in $L^2(G)$ and establish equivalent conditions for a sequence to be a φ-Riesz basis, through which we establish a relation between φ-Riesz bases and usual Riesz bases in $L^2(G)$.

Let G be a LCA group and $\hat{G}$ denote the dual group of G. We refer the reader to the usual text books about locally compact groups [12, 16]. Let the Fourier transform $\hat{}: L^1(G) \longrightarrow C_0(\hat{G}),\ f \longrightarrow \hat{f}$, be defined by $\hat{f}(\xi) = \int_G f(x)\overline{\xi}(x)dx$. The Fourier transform can be extended to a unitary isomorphism from $L^2(G)$ to $L^2(\hat{G})$ known as the Plancherel transform [12, The Plancherel Theorem]. Let φ be a topological automorphism on G. Let L be a uniform lattice in G, that is, a discrete subgroup of G with compact quotient group G/L. Then obviously $\varphi(L)$ is also a uniform lattice in G. Denote by $\varphi(L)^{\perp}$ the annihilator of $\varphi(L)$ in $\hat{G}$, i.e., $\varphi(L)^{\perp} = \{\gamma \in \hat{G};\ \gamma(\varphi(L)) = \{1\}\}$, which is a uniform lattice in $\hat{G}$ (see [18, 21]). For a uniform lattice L in G, a fundamental domain is a measurable set S_L in G such that every $x \in G$ can be uniquely written in the form $x = ks$, where $k \in L$ and $s \in S_L$. The existence of a fundamental domain for a uniform lattice in an LCA group is guaranteed by [22, Lemma 2].

Choosing the counting measure on L, a relation between the Haar measures dx on G and $d\dot{x}$ on $G/\varphi(L)$ is given by the following special case of Weil's formula [12]:

For $f \in L^1(G)$, we have $\sum_{k\in L} f(x\varphi(k^{-1})) \in L^1(G/\varphi(L))$ and

$$\int_G f(x)dx = \int_{G/\varphi(L)} \sum_{\varphi(k^{-1})\in\varphi(L)} f(x\varphi(k^{-1}))d\dot{x}, \tag{1.1}$$

where $\dot{x} = x\varphi(L)$.

Let $f, g \in L^2(G)$. The *φ-bracket product* of f, g is defined by

$$[f,g]_{\varphi}(\dot{x}) = \sum_{k\in L} f\overline{g}(x\varphi(k^{-1})) \tag{1.2}$$

for all $x \in G$. We define the φ-norm of f as $\|f\|_{\varphi}(\dot{x}) = ([f,f]_{\varphi}(\dot{x}))^{1/2}$. The φ-bracket product is in fact a vector valued inner product on $L^2(G)$ (see [19, Proposition 2.4]). In particular, Cauchy Schwartz Inequality holds for it, i.e.,

$$|[f,g]_{\varphi}| \le \|f\|_{\varphi}\|g\|_{\varphi} \tag{1.3}$$

for $f,\ g \in L^2(G)$.

A sequence $(g_n)_{n\in\mathbb{N}} \subseteq L^2(G)$ is called *φ-orthonormal* if $[g_n, g_m]_{\varphi} = 0$ for all $n \neq m \in \mathbb{N}$ and $\|g_n\|_{\varphi} = 1$ for all $n \in \mathbb{N}$. A φ-orthonormal sequence $(g_n)_{n\in\mathbb{N}}$ is called a *φ-orthonormal basis* if $[f, g_n]_{\varphi} = 0$ a.e. for all $n \in \mathbb{N}$, implies $f = 0$ a.e..

[19, Proposition 14] asserts that $L^2(G)$ admits a φ-orthonormal basis.

One of the main tools in our studies is φ-factorable operators. For the sake of completeness, we recall some of our results on φ-factorable operators on

$L^2(G)$. For a detailed exposition of the φ-bracket product and φ-factorable operators confer [19, 20].

For $\gamma \in \hat{G}$, denote by M_γ the modulation operator on $L^2(G)$, i.e.,

$$M_\gamma f(x) = \gamma(x) f(x)$$

for all $f \in L^2(G)$. Let U be a bounded operator from $L^2(G)$ to $L^2(E)$, where E is a subgroup of G or $G/\varphi(L)$. U is called φ-factorable if

$$U(M_\gamma g) = M_\gamma U(g) \quad \text{for all } g \in L^2(G),\ \gamma \in \varphi(L)^\perp. \tag{1.4}$$

It is easily verified that if $U : L^2(G) \to L^2(G)$ is a bounded φ-factorable operator, then its adjoint U^* is also φ-factorable. Moreover,

$$[U(f), g]_\varphi = [f, U^*(g)]_\varphi, \quad \text{a.e. for all } f, g \in L^2(G). \tag{1.5}$$

We have the following Riesz Representation Theorem ([20, Theorem 2.4]), which characterizes all φ-factorable operators from $L^2(G)$ to $L^1(G/\varphi(L))$.

Theorem 1.1. *A bounded operator $U : L^2(G) \to L^1(G/\varphi(L))$ is φ-factorable if and only if there exists $g \in L^2(G)$ such that $U(f) = [f, g]_\varphi$ a.e. for all $f \in L^2(G)$. Moreover $\|U\| = \|g\|$.*

Let us now define a φ-frame and a φ-Bessel sequence.

Definition 1.2. A sequence $(f_n)_{n\in\mathbb{N}}$ in $L^2(G)$ is a said to be a φ-frame if there exist $0 < A, B < \infty$, such that for every $f \in L^2(G)$,

$$A\|f\|^2_\varphi(\dot{x}) \leq \sum_{n\in\mathbb{N}} |[f, f_n]_\varphi(\dot{x})|^2 \leq B\|f\|^2_\varphi(\dot{x}) \tag{1.6}$$

for a.e. $\dot{x} \in G/\varphi(L)$. A, B are called φ-frame bounds. Those sequences which satisfy only the upper inequality in (1.6), are called φ-Bessel sequences. In this case B is called φ-Bessel bound.

The rest of this paper is organized as follows: In Section 2 we investigate φ-frames and φ-Bessel sequences in $L^2(G)$, where G is a second countable LCA group and φ is a topological isomorphism on G. We characterize φ-frames in terms of frames in $L^2(G)$ (Theorem 2.1). We also define φ-pre-frame and φ-analysis operators. Then we study φ-frames and φ-Bessel sequences in terms of these operators. In Section 3 we introduce φ-Riesz bases and give equivalent conditions for a sequence in $L^2(G)$ to be a φ-Riesz basis (Theorem 3.4).

2. φ-Frames in $L^2(G)$

Throughout this paper we always assume that G is a second countable LCA group, L is a uniform lattice in G and φ is a topological isomorphism on G.

In this section we investigate φ-frames and characterize them with regard to standard frames in $L^2(G)$. We then define the associated φ-analysis and φ-preframe operators, with which we obtain criteria for a sequence to be a φ-frame or a φ-Bessel sequence.

Here is the characterization of φ-frames in terms of frames in $L^2(G)$.

Theorem 2.1. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $L^2(G)$. Then the following are equivalent.*

(1) *$(f_n)_{n\in\mathbb{N}}$ is a φ-frame.*
(2) *$(M_\gamma f_n)_{n\in\mathbb{N},\gamma\in\varphi(L)^\perp}$ is a frame.*

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a φ-frame with bounds A, B and $(g_n)_{n\in\mathbb{N}}$ be a φ-orthonormal basis for $L^2(G)$. Define $U : L^2(G) \to L^2(G)$ by $U(M_\gamma g_n) = M_\gamma f_n$ for $\gamma \in \varphi(L)^\perp$, $n \in \mathbb{N}$. Note that $M_\gamma g_n$'s form an orthonormal basis for $L^2(G)$, which guarantees that U is well defined. Then U is φ-factorable and so we have

$$[U^* f, g_n]_\varphi = [f, U(g_n)]_\varphi = [f, f_n]_\varphi, \tag{2.1}$$

a.e.. Since $(g_n)_{n\in\mathbb{N}}$ is a φ-orthonormal basis

$$\begin{aligned} \|U^* f\|_\varphi^2(\dot{x}) &= \sum_{n\in\mathbb{N}} |[U^* f, g_n]_\varphi(\dot{x})|^2 \\ &= \sum_{n\in\mathbb{N}} |[f, f_n]_\varphi(\dot{x})|^2 \\ &\le B\|f\|_\varphi^2(\dot{x}) \end{aligned} \tag{2.2}$$

for $f \in L^2(G)$ and a.e. $\dot{x} \in G/\varphi(L)$. Integrating (2.2) over $G/\varphi(L)$ and using Weil's formula, we have $\|U^* f\|_2^2 \le B\|f\|_2^2$. That is, U^* is bounded. Also U^* is one-to-one. Indeed, if $U^* f = 0$ for some $f \in L^2(G)$, then $[U^* f, g_n]_\varphi = 0$. So by (2.1), $[f, f_n]_\varphi = 0$, which implies that $f = 0$, since $(f_n)_{n\in\mathbb{N}}$ is a φ-frame. Similarly ${U^*}^{-1}$ is bounded. Hence U^* is an isomorphism (note that U^* has dense range). Now by [3, Theorem 4.1], $\{M_\gamma f_n\}_{n\in\mathbb{N},\gamma\in\varphi(L)^\perp}$ is a frame. This completes the proof of (1) $\Rightarrow$ (2). Let $\{M_\gamma f_n\}_{n\in\mathbb{N},\gamma\in\varphi(L)^\perp}$ be a frame. By [3, Theorem 4.1], U^* is an isomorphism. Thus using (2.2) we have

$$A\|f\|_\varphi^2(\dot{x}) \le \sum_{n\in\mathbb{N}} |[f, f_n]_\varphi(\dot{x})|^2 \le B\|f\|_\varphi^2(\dot{x})$$

for a.e. $\dot{x} \in G/\varphi(L)$, in which $A = \|{U^*}^{-1}\|^{-2}, B = \|U^*\|^2$. That is, (2) implies (1). $\square$

We now intend to define φ-pre-frame and φ-analysis operators. First, we need to introduce a vector space which plays the role of $l^2(\mathbb{N})$ in the standard case. To this end, define $l_1^2(G/\varphi(L))$ as the space of the sequences in $L^2(G/\varphi(L))$ convergent in $L^1(G/\varphi(L))$, i.e.,

$$l_1^2(G/\varphi(L)) = \{\{g_i\}_{i\in\mathbb{N}} \subseteq L^2(G/\varphi(L));\ \int_{G/\varphi(L)} \sum_{i\in\mathbb{N}} |g_i(\dot{x})|^2 d\dot{x} < \infty\}. \tag{2.3}$$

$l_1^2(G/\varphi(L))$ is an inner-product space with the inner product defined as follows: $[\cdot,\cdot]_{l_1^2(G/\varphi(L))} :\ l_1^2(G/\varphi(L)) \times l_1^2(G/\varphi(L)) \to L^1(G/\varphi(L))$,

$$[\{g_i\}, \{h_i\}]_{l_1^2(G/\varphi(L))} = \sum_{i\in\mathbb{N}} g_i \overline{h_i}$$

for $\{g_i\}_{i\in\mathbb{N}}, \{h_i\}_{i\in\mathbb{N}} \in l_1^2(G/\varphi(L))$. Note that $\sum_{i\in\mathbb{N}} g_i\overline{h_i} \in L^1(G/\varphi(L))$. Indeed,

$$\begin{aligned}\|\sum_{i\in\mathbb{N}} g_i\overline{h_i}\|_{L^1(G/\varphi(L))} &\leq \int_{G/\varphi(L)} \sum_{i\in\mathbb{N}} |g_i(\dot{x})||h_i(\dot{x})|d\dot{x} \\ &\leq \left(\int_{G/\varphi(L)} \sum_{i\in\mathbb{N}} |g_i(\dot{x})|^2 d\dot{x}\right)^{1/2} \left(\int_{G/\varphi(L)} \sum_{i\in\mathbb{N}} |h_i(\dot{x})| d\dot{x}\right)^{1/2} \\ &< \infty.\end{aligned}$$

For $\{g_i\}_{i\in\mathbb{N}} \in l_1^2(G/\varphi(L))$, define the pointwise norm by

$$\|\{g_i\}_{i\in\mathbb{N}}\|_{l_1^2(G/\varphi(L))}(\dot{x}) = \left(\sum_{i\in\mathbb{N}} |g_i(\dot{x})|^2\right)^{1/2},$$

and the uniform norm by

$$\|\{g_i\}_{i\in\mathbb{N}}\|_{l_1^2(G/\varphi(L))} = \left(\int_{G/\varphi(L)} \sum_{i\in\mathbb{N}} |g_i(\dot{x})|^2 d\dot{x}\right)^{1/2}.$$

Let $\{f_n\}_{n\in\mathbb{N}}$ be a φ-bounded φ-Bessel sequence in $L^2(G)$. Define the *φ-analysis operator* as the mapping $T_\varphi : L^2(G) \to l_1^2(G/\varphi(L))$ given by

$$T_\varphi f = \{[f, f_n]_\varphi\}_{n\in\mathbb{N}}.$$

Define $\theta : L^2(G) \to L^1(G/\varphi(L))$ by $\theta(f) = [T_\varphi f, \{g_n\}_{n\in\mathbb{N}}]_{l_1^2(G/\varphi(L))}$ for some sequence $\{g_n\}_{n\in\mathbb{N}} \in l_1^2(G/\varphi(L))$. Note that if T_φ is bounded, then θ is a bounded φ-factorable operator. So by Riesz Representation Theorem for φ-factorable operators (Theorem 1.1), there exists $T_\varphi^*(\{g_n\}) \in L^2(G)$ with

$$\|T_\varphi^*(\{g_n\})\|_2 = \|\theta\|$$

such that $\theta(f) = [f, T_\varphi^*(\{g_n\})]_\varphi$. Note that $\|T_\varphi\| = \|T_\varphi^*\|$. Indeed,

$$\begin{aligned}&\|[T_\varphi f, \{g_n\}_{n\in\mathbb{N}}]_{l_1^2(G/\varphi(L))}\|_{L^1(G/\varphi(L))} \\ &= \int_{G/\varphi(L)} |[T_\varphi f, \{g_n\}_{n\in\mathbb{N}}]_{l_1^2(G/\varphi(L))}(\dot{x})|d\dot{x} \\ &= \int_{G/\varphi(L)} |\sum_{n\in\mathbb{N}} [f, f_n]_\varphi(\dot{x})\overline{g_n}(\dot{x})|d\dot{x} \\ &\leq \left(\int_{G/\varphi(L)} \sum_{n\in\mathbb{N}} |[f, f_n]_\varphi(\dot{x})|^2 d\dot{x}\right)^{1/2} \left(\int_{G/\varphi(L)} \sum_{n\in\mathbb{N}} |g_n(\dot{x})|^2 d\dot{x}\right)^{1/2} \\ &= \|T_\varphi f\|_{l_1^2(G/\varphi(L))} \|\{g_n\}_{n\in\mathbb{N}}\|_{l_1^2(G/\varphi(L))}.\end{aligned}$$

Hence

$$\begin{aligned}\|T_\varphi^*(\{g_n\})\|_2 &= \|\theta\| \\ &= \sup_{\|f\|_2\leq 1} \|[T_\varphi f, \{g_n\}_{n\in\mathbb{N}}]_{l_1^2(G/\varphi(L))}\|_{L^1(G/\varphi(L))}\end{aligned}$$

$$\leq \|T_\varphi\| \|\{g_n\}\|_{l_1^2(G/\varphi(L))}.$$

That is, $\|T_\varphi^*\| \leq \|T_\varphi\|$. Also obviously, $T_\varphi = T_\varphi^{**}$. So $\|T_\varphi\| = \|T_\varphi^*\|$.
To obtain the φ-preframe operator T_φ^* explicitly, we calculate as follows.
Let $f \in L^2(G)$, $\{g_i\}_{i\in\mathbb{N}} \in l_1^2(G/\varphi(L))$. Then we have

$$\begin{aligned}
[f, T_\varphi^*(\{g_n\})]_\varphi(\dot{x}) &= [T_\varphi f, \{g_n\}_{n\in\mathbb{N}}]_{l_1^2(G/\varphi(L))}(\dot{x}) \\
&= \sum_{n\in\mathbb{N}} T_\varphi f(\dot{x})\overline{g_n}(\dot{x}) \\
&= \sum_{n\in\mathbb{N}} [f, f_n]_\varphi(\dot{x})\overline{g_n}(\dot{x}) \\
&= [f, \sum_{n\in\mathbb{N}} f_n g_n]_\varphi(\dot{x}).
\end{aligned}$$

Thus

$$\int_{G/\varphi(L)} [f, T_\varphi^*(\{g_n\})]_\varphi(\dot{x})d\dot{x} = \int_{G/\varphi(L)} [f, \sum_{n\in\mathbb{N}} f_n g_n]_\varphi(\dot{x})d\dot{x}.$$

That is,

$$\langle f, T_\varphi^*(\{g_n\})\rangle_{L^2(G)} = \langle f, \sum_{n\in\mathbb{N}} f_n g_n\rangle_{L^2(G)}.$$

Hence

$$T_\varphi^*(\{g_n\}) = \sum_{n\in\mathbb{N}} f_n g_n. \tag{2.4}$$

T_φ^* is called the *φ-preframe operator*.

In the following proposition we characterize φ-Bessel sequences in terms of the φ-preframe operator. To be more precise, we show that a φ-bounded sequence is φ-Bessel if and only if the φ-preframe operator is bounded.

Remark 2.2. (i) For $f \in L^2(G)$ we have

$$\|f\|_\varphi(\dot{x}) = \sup\{|[f, g]_\varphi(\dot{x})|;\ \|g\|_\varphi(\dot{x}) \leq 1\}$$

for a.e. $\dot{x} \in G/\varphi(L)$. Indeed, by Cauchy Schwartz Inequality (1.3) we have

$$\sup\{|[f, g]_\varphi(\dot{x})|;\ \|g\|_\varphi(\dot{x}) \leq 1\} \leq \|f\|_\varphi(\dot{x})$$

for a.e. $\dot{x} \in G/\varphi(L)$. Also

$$\sup\{|[f, g]_\varphi(\dot{x})|;\ \|g\|_\varphi(\dot{x}) \leq 1\} \geq |[f, \frac{f}{\|f\|_\varphi}]_\varphi(\dot{x})| = \|f\|_\varphi(\dot{x})$$

for a.e. $\dot{x} \in G/\varphi(L)$.

(ii) By a similar argument as in the standard L^2-space theory it is verified that $(L^2(G),\ \|\cdot\|_\varphi)$ is a Banach space.

We say $g \in L^2(G)$ is *φ-bounded* if there exists $M > 0$ so that $\|g\|_\varphi \leq M$ a.e.. Note that for $f, g \in L^2(G)$ the function $[f, g]_\varphi g$ need not generally be in $L^2(G)$. But if $f, g, h \in L^2(G)$ and g, h are φ-bounded, then $[f, g]_\varphi h \in L^2(G)$ (see [19]).

Proposition 2.3. *Let* $(f_n)_{n\in\mathbb{N}}$ *be a* φ*-bounded sequence in* $L^2(G)$*. Then* $(f_n)_{n\in\mathbb{N}}$ *is* φ*-Bessel with bound* B *if and only if* T_φ^* *is a well defined bounded operator from* $l^2(G/\varphi(L))$ *into* $L^2(G)$ *and* $\|T_\varphi\| \leq \sqrt{B}$*.*

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a φ-Bessel sequence with bound B in $L^2(G)$. Assume that $(g_n)_{n\in\mathbb{N}} \in l_1^2(G/\varphi(L)),\ n \in \mathbb{N}$. Then for $m, n \in \mathbb{N},\ n \geq m$, we have

$$\begin{aligned}
&\|\sum_{i=1}^{n} g_i f_i - \sum_{i=1}^{m} g_i f_i\|_\varphi(\dot{x}) \\
&= \|\sum_{i=m+1}^{n} g_i f_i\|_\varphi(\dot{x}) \\
&= \sup\nolimits_{\|g\|_\varphi \leq 1} |[\sum_{i=m+1}^{n} g_i f_i, g]_\varphi(\dot{x})| \\
&= \sup\nolimits_{\|g\|_\varphi \leq 1} |\sum_{i=m+1}^{n} g_i [f_i, g]_\varphi(\dot{x})| \\
&\leq \left(\sum_{i=m+1}^{n} |g_i(\dot{x})|^2\right)^{1/2} \sup\nolimits_{\|g\|_\varphi \leq 1} \left(\sum_{i=m+1}^{n} |[f_i, g]_\varphi(\dot{x})|^2\right)^{1/2} \\
&\leq \sqrt{B} \left(\sum_{i=m+1}^{n} |g_i(\dot{x})|^2\right)^{1/2}.
\end{aligned}$$

So $\sum_{i=1}^{n} g_i f_{i_{n\in\mathbb{N}}}$ is Cauchy in $(L^2(G),\ \|\cdot\|_\varphi)$ and therefore convergent. Thus T_φ^* is well defined. Also obviously $\|T_\varphi^*\| \leq B$. For the converse assume T_φ^* and so T_φ is bounded. Then $\|T_\varphi(hf)\|_{l^2(G/\varphi(L))} \leq \|T_\varphi\| \|hf\|_2$ for every $h \in L^\infty(G/\varphi(L))$. Therefore,

$$\int_{G/\varphi(L)} \sum_{n\in\mathbb{N}} |[hf, f_n]_\varphi(\dot{x})|^2 d\dot{x} \leq \int_{G/\varphi(L)} \|hf\|_\varphi^2(\dot{x}) \|T_\varphi\|^2 d\dot{x}.$$

That is,

$$\int_{G/\varphi(L)} |h(\dot{x})|^2 \sum_{n\in\mathbb{N}} |[f, f_n]_\varphi(\dot{x})|^2 d\dot{x} \leq \int_{G/\varphi(L)} |h(\dot{x})|^2 \|f\|_\varphi^2(\dot{x}) \|T_\varphi\|^2 d\dot{x}$$

for every $h \in L^\infty(G/\varphi(L))$. Hence

$$\sum_{n\in\mathbb{N}} |[f, f_n]_\varphi(\dot{x})|^2 \leq B\|f\|_\varphi^2(\dot{x})$$

for a.e. $\dot{x} \in G/\varphi(L)$, where $B = \|T_\varphi\|^2$. So $(f_n)_{n\in\mathbb{N}}$ is φ-Bessel. □

Let $(f_n)_{n\in\mathbb{N}}$ be a φ-frame. Assume that each f_n, $n \in \mathbb{N}$ is φ-bounded in $L^2(G)$. The φ-frame operator defined by $S_\varphi := T_\varphi^* T_\varphi$ is bounded. Indeed,

$$\begin{aligned}
[S_\varphi f, f]_\varphi &= [\sum_{n\in\mathbb{N}} [f, f_n]_\varphi f_n, f]_\varphi \\
&= \sum_{n\in\mathbb{N}} [f, f_n]_\varphi \overline{[f, f_n]}_\varphi \\
&= \sum_{n\in\mathbb{N}} |[f, f_n]_\varphi|^2.
\end{aligned}$$

So we have

$$A[f, f]_\varphi \leq [S_\varphi f, f]_\varphi \leq B[f, f]_\varphi,$$

which implies

$$A \int_{G/\varphi(L)} [f, f]_\varphi(\dot{x}) d\dot{x} \leq \int_{G/\varphi(L)} [S_\varphi f, f]_\varphi(\dot{x}) d\dot{x} \leq B \int_{G/\varphi(L)} [f, f]_\varphi(\dot{x}) d\dot{x}.$$

Therefore, $AI \leq S_\varphi \leq BI$. By a standard argument as in the frame theory S_φ is invertible and $B^{-1}I \leq S_\varphi^{-1} \leq A^{-1}I$.

We can now characterize φ-frames with the aid of the φ-preframe operator.

Proposition 2.4. *Let $(f_n)_{n\in\mathbb{N}}$ be a φ-bounded sequence in $L^2(G)$. Then $(f_n)_{n\in\mathbb{N}}$ is a φ-frame if and only if T_φ^* is well defined, bounded and onto.*

Proof. Let f_n be a φ-frame. Then by the above remarks S_φ is onto and so is T_φ^*. The rest follows from Proposition 2.3.

Conversely, we have $f = S_\varphi S_\varphi^{-1} f = \sum_{n\in\mathbb{N}} [S_\varphi^{-1} f, f_n]_\varphi f_n$, so

$$\begin{aligned}
\|f\|_\varphi^2(\dot{x}) &= [f, f]_\varphi(\dot{x}) \\
&= [\sum_{n\in\mathbb{N}} [S_\varphi^{-1} f, f_n]_\varphi f_n, f]_\varphi(\dot{x}) \\
&= \sum_{n\in\mathbb{N}} [S_\varphi^{-1} f, f_n]_\varphi(\dot{x}) [f_n, f]_\varphi(\dot{x}) \\
&\leq \left(\sum_{n\in\mathbb{N}} |[S_\varphi^{-1} f, f_n]_\varphi(\dot{x})|^2 \right)^{1/2} \left(\sum_{n\in\mathbb{N}} |[f_n, f]_\varphi(\dot{x})|^2 \right)^{1/2} \\
&\leq \|T_\varphi(S_\varphi^{-1} f)\|_{l^2(G/\varphi(L))}(\dot{x}) \left(\sum_{n\in\mathbb{N}} |[f_n, f]_\varphi(\dot{x})|^2 \right)^{1/2} \\
&\leq \|T_\varphi\| \|S_\varphi^{-1}\| \|f\|_\varphi(\dot{x}) \left(\sum_{n\in\mathbb{N}} |[f_n, f]_\varphi(\dot{x})|^2 \right)^{1/2}
\end{aligned}$$

for a.e. $\dot{x} \in G/\varphi(L)$. That is,

$$A\|f\|_\varphi^2(\dot{x}) \leq \sum_{n\in\mathbb{N}} |[f_n, f]_\varphi(\dot{x})|^2,$$

where $A = \|T_\varphi\|^{-2}\|S_\varphi^{-1}\|^{-2}$. Now Proposition 2.3 completes the proof. $\square$

Next we consider the case when two φ-Bessel sequences may also be φ-frames.

Proposition 2.5. *Let $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ be two φ-bounded φ-Bessel sequences in $L^2(G)$. If $f = \sum_{n\in\mathbb{N}}[f, g_n]_\varphi f_n$, a.e. for all $f \in L^2(G)$, then both $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ are φ-frames.*

Proof. Let us denote by B the φ-Bessel bound of $(f_n)_{n\in\mathbb{N}}$. For all $f \in L^2(G)$, we have

$$\begin{aligned}
\|f\|_\varphi^4(\dot{x}) &= [f, f]_\varphi^2(\dot{x}) \\
&= [\sum_{n\in\mathbb{N}}[f, g_n]_\varphi f_n, f]_\varphi^2(\dot{x}) \\
&= (\sum_{n\in\mathbb{N}}[f, g_n]_\varphi(\dot{x})[f_n, f]_\varphi(\dot{x}))^2 \\
&\le \sum_{n\in\mathbb{N}}|[f, g_n]_\varphi(\dot{x})|^2 \sum_{n\in\mathbb{N}}|[f_n, f]_\varphi(\dot{x})|^2 \\
&\le B\|f\|_\varphi^2(\dot{x}) \sum_{n\in\mathbb{N}}|[f, g_n]_\varphi(\dot{x})|^2.
\end{aligned}$$

That is,

$$B^{-1}\|f\|_\varphi^2(\dot{x}) \le \sum_{n\in\mathbb{N}}|[f, g_n]_\varphi(\dot{x})|^2$$

for every $f \in L^2(G)$, for a.e. $\dot{x} \in G/\varphi(L)$. Hence $(g_n)_{n\in\mathbb{N}}$ is a φ-frame. A similar argument shows that $(f_n)_{n\in\mathbb{N}}$ is also a φ-frame. $\square$

It is clear that every φ-orthonormal basis is a Parseval φ-frame, but the converse is not true.

Example 2.6. Consider the LCA group $G = \mathbb{R}^+$. As a uniform lattice in G we choose $L = \{2^n;\ n \in \mathbb{Z}\}$. Then $L^\perp = \mathbb{Z}$. We can choose $S_L := [1, 2)$ as a fundamental domain for L in G. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be the topological automorphism defined by $\varphi(x) = x^2$. Let $(f_n)_{n\in\mathbb{N}}$ be a φ-orthonormal basis for $L^2(G)$ (e.g. consider the orthonormal basis $\{M_\gamma T_k \chi_{S_L};\ (k, \gamma) \in L \times L^\perp\}$, as in [23, Theorem 3.1.7] for $L^2(G)$, where M_γ is the modulation operator. By [19, Theorem 14], $\{T_k\chi_{S_L};\ k \in L\}$ is a φ-orthonormal basis for $L^2(G)$). Then $\{f_1, \frac{1}{\sqrt{2}}f_2, \frac{1}{\sqrt{2}}f_2, \frac{1}{\sqrt{3}}f_3, \frac{1}{\sqrt{3}}f_3, \frac{1}{\sqrt{3}}f_3, \ldots\}$ is a Parseval φ-frame but not a φ-orthonormal basis.

It is easy to see that if $(f_n)_{n\in\mathbb{N}}$ is a Parseval φ-frame and $\|f_n\|_\varphi = 1$ a.e. for every $n \in \mathbb{N}$, then $(f_n)_{n\in\mathbb{N}}$ is a φ-orthonormal basis.

3. φ-Riesz Bases in $L^2(G)$

Our goal in this section is to define and investigate φ-Riesz bases in $L^2(G)$, applying φ-factorable operators.

Riesz bases in $L^2(\mathbb{R})$ have several equivalent definitions (see [9, 15, 27]). The main result of this section (Theorem 3.4), sets out equivalent conditions for a sequence in $L^2(G)$ to be a φ-Riesz basis, where G is a second countable LCA group and φ is a topological automorphism on G. We start with a definition.

Definition 3.1. A sequence $(f_n)_{n\in\mathbb{N}}$ in $L^2(G)$ is said to be a *φ-Riesz basis* if there exists a φ-orthonormal basis $(g_n)_{n\in\mathbb{N}}$ and a φ-factorable operator $U : L^2(G) \to L^2(G)$, which is a topological automorphism such that $U(g_n) = f_n$ for every $n \in \mathbb{N}$.

We introduce a φ-complete (φ-total) sequence in $L^2(G)$ as follows:

Definition 3.2. Given a sequence $(f_n)_{n\in\mathbb{N}} \subseteq L^2(G)$, by $\overline{\text{span}}^{\|\cdot\|_\varphi}(f_n) = L^2(G)$ we mean that for every $f \in L^2(G)$ there exists a sequence $\{h_n\}_{n\in\mathbb{N}} \in l^2_1(G/\varphi(L))$, such that $f = \sum_{n=1}^{\infty} h_n f_n$, a.e. We say a sequence $(f_n)_{n\in\mathbb{N}} \subseteq L^2(G)$ is *φ-complete* (*φ-total*) in $L^2(G)$, if $\overline{\text{span}}^{\|\cdot\|_\varphi}(f_n) = L^2(G)$.

The following lemma will be needed in the proof of Theorem 3.4.

Lemma 3.3. *Suppose U is a bounded φ-factorable operator on $L^2(G)$. For every $f \in L^2(G)$, we have $\|Uf\|_\varphi \le \|U\|\|f\|_\varphi$ a.e.*

Proof. For every φ-periodic $h \in L^\infty(G)$, we have

$$\begin{aligned}
&\int_{G/\varphi(L)} |h(\dot{x})|^2 \|U(f)\|_\varphi^2(\dot{x})d\dot{x} \\
&= \int_{G/\varphi(L)} \sum_{k\in L} |U(f)(x\varphi(k^{-1}))|^2 |h(x\varphi(k^{-1}))|^2 d\dot{x} \\
&= \int_{G/\varphi(L)} \sum_{k\in L} |U(hf)(x\varphi(k^{-1}))|^2 d\dot{x} \\
&= \|U(hf)\|_2^2 \\
&\le \|U\|^2 \|hf\|_2^2 \\
&= \|U\|^2 \int_G |hf(x)|^2 dx \\
&= \|U\|^2 \int_{G/\varphi(L)} \sum_{k\in L} |hf(x\varphi(k^{-1}))|^2 d\dot{x} \\
&= \|U\|^2 \int_{G/\varphi(L)} |h(\dot{x})|^2 \|f\|_\varphi^2(\dot{x})d\dot{x},
\end{aligned}$$

which obviously completes the proof. $\square$

In the following theorem we establish equivalent conditions for a sequence in $L^2(G)$ to be a φ-Riesz basis. As a matter of fact Theorem 3.4 gives a characterization of φ-Riesz bases with regard to standard Riesz bases in $L^2(G)$, which implies that a φ-Riesz basis shares many useful properties with a Riesz basis.

Theorem 3.4. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $L^2(G)$. The following are equivalent.*

(1) *$(f_n)_{n\in\mathbb{N}}$ is φ-complete, and there exist positive constants A and B such that for any sequence $\{h_n\}_{n\in\mathbb{N}} \in l_1^2(G/\varphi(L))$ one has*

$$A\sum_{n=1}^{\infty}|h_n|^2 \leq \|\sum_{n=1}^{\infty}h_nf_n\|_{\varphi}^2 \leq B\sum_{n=1}^{\infty}|h_n|^2 \quad a.e. \tag{3.1}$$

(2) *$(f_n)_{n\in\mathbb{N}}$ is a φ-Riesz basis.*

(3) *$(M_\gamma f_n)_{\gamma\in\varphi(L)^\perp, n\in\mathbb{N}}$ is a Riesz basis in $L^2(G)$.*

Proof. (1)$\Rightarrow$ (2) Let $(e_n)_{n\in\mathbb{N}}$ be a φ-orthonormal basis in $L^2(G)$. Then by [19, Theorem 14], $\overline{\text{span}}^{\|\cdot\|_\varphi}(e_n) = L^2(G)$. Define $U : L^2(G)(=\overline{\text{span}}^{\|\cdot\|_\varphi}(e_n)) \to L^2(G)$ by $U(\sum_{n=1}^{\infty}h_ne_n) = \sum_{n=1}^{\infty}h_nf_n$, where $\{h_n\}_{n\in\mathbb{N}} \in l_1^2(G/\varphi(L))$. Then U is bounded. In fact, by (3.1)

$$\begin{aligned}\|U(\sum_{n=1}^{\infty}h_ne_n)\|_{\varphi}^2 &= \|\sum_{n=1}^{\infty}h_nf_n\|_{\varphi}^2\\ &\leq B\sum_{n=1}^{\infty}|h_n|^2\\ &= B\|\sum_{n=1}^{\infty}h_ne_n\|_{\varphi}^2, \quad \text{a.e.,}\end{aligned}$$

and so

$$\begin{aligned}\|U(\sum_{n=1}^{\infty}h_ne_n)\|_2^2 &= \int_{G/\varphi(L)}\|U(\sum_{n=1}^{\infty}h_ne_n)\|_{\varphi}^2(\dot{x})d\dot{x}\\ &\leq B\int_{G/\varphi(L)}\|\sum_{n=1}^{\infty}h_ne_n\|_{\varphi}^2(\dot{x})d\dot{x}\\ &= B\|\sum_{n=1}^{\infty}h_ne_n\|_2^2\end{aligned}$$

for any $\{h_n\}_{n\in\mathbb{N}} \in l_1^2(G/\varphi(L))$. That is, $\|U\| \leq \sqrt{B}$. Now define $S : L^2(G)(=\overline{\text{span}}^{\|\cdot\|_\varphi}(f_n)) \to L^2(G)$ by $S(\sum_{n=1}^{\infty}h_nf_n) = \sum_{n=1}^{\infty}h_ne_n$, where $\{h_n\}_{n\in\mathbb{N}} \in l_1^2(G/\varphi(L))$. Hence by (3.1) we get

$$\|S(\sum_{n=1}^{\infty}h_nf_n)\|_{\varphi}^2 = \|\sum_{n=1}^{\infty}h_ne_n\|_{\varphi}^2$$

$$= \sum_{n=1}^{\infty} |h_n|^2$$
$$\leq 1/A \| \sum_{n=1}^{\infty} h_n f_n \|_{\varphi}^2, \quad \text{a.e.}$$

This implies that S is bounded on $L^2(G)$ and $\|S\| \leq \sqrt{1/A}$. Also obviously, $SU = I$ and $US = I$ on $L^2(G)$. Hence U is a topological isomorphism, which is clearly φ-factorable and $U(e_n) = f_n$ for every $n \in \mathbb{N}$.

(2)$\Rightarrow$ (3) Choose a φ-orthonormal basis $(e_n)_{n\in\mathbb{N}}$ for $L^2(G)$ and the corresponding topological automorphism U which is a φ-factorable operator and $U(e_n) = f_n$ for every $n \in \mathbb{N}$, as in Definition 3.1. By [19, Theorem 14], $(M_\gamma f_n)_{\gamma\in\varphi(L)^\perp, n\in\mathbb{N}}$ is an orthonormal basis for $L^2(G)$, and since U is φ-factorable

$$U(M_\gamma e_n) = M_\gamma U(e_n) = M_\gamma f_n$$

for every $n \in \mathbb{N}$, $\gamma \in \varphi(L)^\perp$. So $(M_\gamma f_n)_{\gamma\in\varphi(L)^\perp, n\in\mathbb{N}}$ is a Riesz basis.

(3)$\Rightarrow$ (2) Let $S_{\varphi(L)}$ be a fundamental domain for $\varphi(L)$. By [23, Theorem 3.1.7], the system $(M_\gamma T_{\varphi(k)} \chi_{S_{\varphi(L)}})_{k\in L, \gamma\in\varphi(L)^\perp}$ is an orthonormal basis for $L^2(G)$, where $T_{\varphi(k)}\chi_{S_{\varphi(L)}}$ is the translation of $\chi_{S_{\varphi(L)}}$ by $\varphi(k)$. Define $U : L^2(G) \to L^2(G)$ by $U(M_{\gamma_m} T_{\varphi(k_n)} \chi_{S_{\varphi(L)}}) = M_{\gamma_m} f_n$, $m, n \in \mathbb{N}$. Obviously, U is a φ-factorable operator. Moreover, by [19, Theorem 14], $(T_{\varphi(k)}\chi_{S_{\varphi(L)}})_{k\in L}$ is a φ-orthonormal basis for $L^2(G)$, and obviously $U(T_{\varphi(k_n)}\chi_{S_{\varphi(L)}}) = f_n$ for every $n \in \mathbb{N}$. Finally since $(M_\gamma f_n)_{\gamma\in\varphi(L)^\perp, n\in\mathbb{N}}$ is a Riesz basis, U is a topological automorphism.

(2)$\Rightarrow$ (1) Suppose $(e_n)_{n\in\mathbb{N}}$ is a φ-orthonormal basis and U is the corresponding topological automorphism which is a φ-factorable operator and $U(e_n) = f_n$ for every $n \in \mathbb{N}$, as in the Definition 3.1. Let $\{h_n\}_{n\in\mathbb{N}} \in l_1^2(G/\varphi(L))$. Then using Lemma 3.3

$$\| \sum_{n=1}^{\infty} h_n f_n \|_{\varphi}^2 = \| \sum_{n=1}^{\infty} h_n U(e_n) \|_{\varphi}^2$$
$$= \| U(\sum_{n=1}^{\infty} h_n e_n) \|_{\varphi}^2$$
$$\leq \|U\|^2 \| \sum_{n=1}^{\infty} h_n e_n \|_{\varphi}^2$$
$$= \|U\|^2 \sum_{n=1}^{\infty} |h_n|^2, \quad \text{a.e.}$$

On the other hand

$$\sum_{n=1}^{\infty} |h_n|^2 = \| \sum_{n=1}^{\infty} h_n e_n \|_{\varphi}^2$$

$$= \|U^{-1}U(\sum_{n=1}^{\infty} h_n e_n)\|_{\varphi}^2$$
$$\leq \|U^{-1}\|^2 \|U(\sum_{n=1}^{\infty} h_n e_n)\|_{\varphi}^2$$
$$= \|U^{-1}\|^2 \|\sum_{n=1}^{\infty} h_n f_n\|_{\varphi}^2, \quad \text{a.e.}$$

So (3.1) holds. Moreover $(f_n)_{n\in\mathbb{N}}$ is φ-complete. Indeed, given any $f \in L^2(G)$, there exists a unique $g \in L^2(G)$ with $U(g) = f$ (since U is one-to-one and onto). Write $g = \sum_{n=1}^{\infty}[g, e_n]_{\varphi} e_n$ as in [19, Theorem 18]. Then $h_n = [g, e_n]_{\varphi} \in L^{\infty}(G/\varphi(L))$ for every $n \in \mathbb{N}$ and by Bessel's Inequality ([19, Theorem 11])

$$\sum_{n=1}^{\infty} |h_n(\dot{x})|^2 \leq \|f\|_{\varphi}(\dot{x}) < \infty$$

for a.e. $\dot{x} \in G/\varphi(L)$. Also

$$f = U(g) = U(\sum_{n=1}^{\infty} h_n e_n) = \sum_{n=1}^{\infty} h_n U(e_n) = \sum_{n=1}^{\infty} h_n f_n,$$

showing that $\overline{\mathrm{span}}^{\|\cdot\|_{\varphi}}(f_n) = L^2(G)$. This completes the proof. $\square$

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