정규성을 허용하는 특별한 부호화 행렬의 구성

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Constructions of the special sign pattern matrices that allow normality

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요 약

행렬들 중 그것의 성분으로 부호인 + 와 0 만을 갖는 행렬을 우리는 비음인 부호화 행렬이라 한다 . 또한 비음인 부호화 행렬 A가 그것과 같은 부호를 갖는 실수 정규행렬 B 가 존재하면 정규성을 허용한다고 한다. 본 논문은 참고문헌[1] 에서 밝힌 형태와 다른 특별한 형태를 조사했고, 실수 행렬 중 비음인 정규행렬을 구성하는 흥미로운 방법을 제공했다

ABSTRACT

By a nonnegative sign pattern we mean a matrix whose entries are from the set $\{+,0\}$. A nonnegative sign pattern A is said to allow normality if there is a normal matrix B whose entries have signs indicated by A. In this paper we investigated some nonnegative normal pattern that is different to the pattern in [1]. Some interesting constructions of nonnegative integer normal matrices are provided.

키워드
sign pattern matrix, allow, normality

Ⅰ. Introduction

A matrix whose entries consist of the symbols $+,-$, and 0 is called a sign pattern matrix. For a real matrix B, by sgn$B$ we mean the sign pattern matrix in which each positive (respectively,negative,zero) entry is replaced by + (respectively, $-,0$). For each $n\times n$ sign pattern matrix A, there is a natural class of real matrices whose entries have the signs indicated by A.

If $A=(a_{ij})$ is an $n\times n$ sign pattern matrix, then the sign pattern class of A is defined by

$$Q(A)=\mathrm{B}\in \mathrm{M_n(R)} \mid sgn\mathrm{B} = \mathrm{A}.$$

Recall that a real $n\times n$ matrix B is said to be normal if $BB^T=B^TB$.

Analogously, a square sign pattern matrix A allow normality if there is a normal matrix $B\in Q(A)$. To avoid repetition, we often use "sign pattern" or just "pattern" to mean sign pattern matrix.

Recall that a matrix $A\in M_n$ is said to be reducible if either

(a) $n=1$ and $A=0$ or

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(b) $n \geq 2$, there is a permutation matrix, and there is some integer r with $1 \leq r \leq n-1$ such that (1).

$$P^T AP = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix} \quad (1)$$

where $B \in M_r$, $D \in M_{n-r}$, $C \in M_{r,n-r}$, and $0 \in M_{n-r,r}$ is a zero matrix. A matrix $A \in M_n$ is said to be irreducible if it is not reducible.

A cell is a matrix with exactly one nonzero entry and it equals 1. If the nonzero entry of a cell is in the (i,j) location, we denote the cell by E_{ij}.

The main purpose of this paper is to investigate some sign patterns of nonnegative normal matrices. Such matrices have recently been examined in [1],[2], and [3]. We devote Section 2 to consider nonnegative patterns that allow integer normality and to sturdy of basic properties of nonnegative normal matrices. In section 3 we give an interesting open problem for the irreducible normal pattern.

II. Results for the integer normality

Trivially, any symmetric matrix is normal, so we can concentrate on nonsymmetric normal matrix. Since a matrix is normal if and only if it is permutation similar to a direct sum of irreducible normal matrices, we can thus focus our study on irreducible normal matrices. In this paper, we use the notation N is the set of nonnegative sign pattern that allow normality.

Lemma 2.1 The set of $n \times n$ nonsymmetric irreducible patterns in N is closed under

(i) permutation similarity, and

(ii) transposition.

Lemma 2.2 Let A be symmetric. Then the form is (2).

$$\begin{bmatrix} B & C \\ C^T & A \end{bmatrix} \in M_n(R) \quad (2)$$

is normal if and only if B is normal and (3).

$$(B - B^T)C = 0. \quad (3)$$

Lemma 2.3 (Theorem 3.5, [1]) Let A be irreducible. Then the form is (4).

$$\begin{bmatrix} A & J_1 \\ J_2 & J_3 \end{bmatrix} \in N \quad (4)$$

where J_1, J_2, and J_3 are all $+$ patterns of appropriate size.

Theorem 2.4 Let A be irreducible. If A allows an integer normal matrix, then the form is (5)

$$M = \begin{bmatrix} A & J_1 \\ J_2 & J_3 \end{bmatrix} \quad (5)$$

allows also where J_1, J_2, and J_3 are all $+$ patterns of appropriate size.

Proof. It suffices to consider the case where J_3 is the 1×1 pattern $(+)$. Let $B \in Q(A)$ be integer normal, ρ be the Perron root of B, and $v > 0$ be the Perron vector. Since B is normal, we have $B^T v = \rho v$. Since $v = (v_1, v_2, \cdots, v_n)$ be the Perron vector and each component is positive, $\sum_{i=1}^{n} v_i = 1$ and each v_i is a rational number. Therefore, there exists a positive integer k such that the components of kv are all integer and $B^T kv = \rho kv$. So $(B - B^T)kv = 0$, and hence, by Lemma 2.2 and 2.3,

$$\begin{bmatrix} B & kv \\ (kv)^T & 1 \end{bmatrix} \in Q(M) \quad (6)$$

is integer normal.

Corollary 2.5 If $n \geq 3$ and the $n \times n$ nonnegative pattern A has only one zero entry, then A allow the integer normality

Proof. We know that

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \in Q(\begin{bmatrix} 0 & + & + \\ + & + & + \\ + & + & + \end{bmatrix})$$

and

$$\begin{bmatrix} 1 & 0 & 68 \\ 60 & 12 & 45 \\ 32 & 75 & 76 \end{bmatrix} \in Q(\begin{bmatrix} + & 0 & + \\ + & + & + \\ + & + & + \end{bmatrix}).$$

The proof follows from theorem 2.4 and permutation similarity.

Theorem 2.5 Let B be an $n \times n$ irreducible per-symmetric matrix. A is normal if and only if BB^T is per-symmetric

Proof. Let $B=[b_{ij}]$ be an normal per-symmetric matrix, $BB^T=[p_{ij}]$ and $B^TB=[q_{ij}]$. Then for each i,j, $p_{ij}=q_{ij}$, $p_{ij}=R_i \cdot R_j = R_j \cdot R_i = p_{ji}$ and $q_{ij}=C_i \cdot C_j = C_j \cdot C_i = q_{ji}$. Since B is per-symmetric, That is,

for each i,j $b_{ij}=b_{(n+1-j)(n+1-i)}$ and

$$\begin{aligned} p_{ij} = R_i \cdot R_j &= (b_{i1}, b_{i2}, \cdots, b_{\in}) \cdot (b_{j1}, b_{j2}, \cdots, b_{jn}) \\ &= (b_{n(n+1-i)}, b_{(n-1)(n+1-i)}, \cdots, b_{1(n+1-i)}) \cdot \\ &\quad (b_{n(n+1-j)}, b_{(n-1)(n+1-j)}, \cdots, b_{1(n+1-j)}) \\ &= C_{(n+1-i)} \cdot C_{(n+1-j)} = q_{(n+1-i)(n+1-j)} \\ &= p_{(n+1-i)(n+1-j)}. \end{aligned}$$

Therefore, BB^T is per-symmetric.

Conversely, let BB^T be per-symmetric. Then

$$\begin{aligned} p_{ij} &= p_{(n+1-i)(n+1-j)} \\ &= R_{(n+1-i)} \cdot R_{(n+1-j)} \\ &= (b_{(n+1-i)1}, b_{(n+1-i)2}, \cdots, b_{(n+1-i)n}) \\ &\quad \cdot (b_{(n+1-j)1}, b_{(n+1-j)2}, \cdots, b_{(n+1-j)n}) \\ &= C_i \cdot C_j \quad (By\ the\ per-symmetry\ of\ B) \\ &= q_{ij}. \end{aligned}$$

Therefore, for each i,j, $BB^T = B^TB$.

Corollary 2.6 Let A be an $n \times n$ per-symmetric matrix. The followings are equivalent.

(1) A is normal
(2) ARA is symmetric
(3) AA^T is per-symmetric

Proof. (1) $\Rightarrow$ (2) ; Since A is per-symmetric, $A = RA^TR$, $A^T = RAR$.

and

$$\begin{aligned} (ARA)^T &= A^TRA^T \\ &= RAA^T = RA^TA = ARA. \end{aligned}$$

Therefore, ARA is symmetric.

(2) $\Rightarrow$ (3) ; Since A is per-symmetric and ARA is symmetric,

$$\begin{aligned} &RAA^TR = RARA = R(ARA)^T = RA^TRA^T \\ &= RRAA^T = AA^T = (AA^T)^T. \end{aligned}$$

Therefore, AA^T is per-symmetric.

(3) $\Rightarrow$ (1) ; By the Theorem 2.5, it is clear.

Let AA be an $n \times n$ irreducible per-symmetric $(0,1)$-matrix. By the above Theorem, we know that A is normal if and only if AA^T is per-symmetric and have the following ;

Corollary 2.7 Let A be an $n \times n$ $(0,1)$-matrix in $Cir(n,k)$. Then A is per-symmetric normal

Proof. Let P be the basic circulant matrix of order n. Since A is in $Cir(n,k)$,

$$A = \sum_{r=1}^{k} P^{i_r} \tag{7}$$

where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. Now we show that AA^T is per-symmetric. Since

$(P^{i_r})^T = (P^T)^{i_r} = (P^{-1})^{i_r} = P^{n-i_r}$,

therefore,

$$A^T = \sum_{r=1}^{k} P^{n-i_r}$$

and

$$\begin{aligned} AA^T &= (\sum_{r=1}^{k} P^{i_r})(\sum_{r=1}^{k} P^{n-i_r}) \\ &= (P^{i_1}+P^{i_2}+\cdots+P^{i_k}) \cdot \\ &\quad (P^{n-i_1}+P^{n-i_2}+\cdots+P^{n-i_k}) \\ &= \sum_{r=1}^{k}\sum_{s=1}^{k} P^{i_r}P^{n-i_s} \\ &= \sum_{r=1}^{k}\sum_{s=1}^{k} P^{n+i_r-i_s}. \end{aligned}$$

We know that for each r,s, $P^{n+i_r-i_s}$ is per-symmetric and the sum of per-symmetric matrices is per-symmetric. Hence AA^T is per-symmetric. By the Theorem 2.5, A is normal.

Ⅲ. Results of the irreducible normality

In this section, we concentrate on non-symmetric (0,1)-normal matrices. As usual, J denotes the all 1′s matrix of the appropriate size. We say that the complement of a (0,1)-matrix B is $J-B$, and we denote the complement of B by B^c.

We know that if A is an (0,1)-normal matrix, then the row and column sum vectors are same. Therefore, the following Lemmas are clear.

Lemma 3.1 The set of $n\times n$ non-symmetric irreducible patterns in N is closed under

(i) permutation similarity,
(ii) transposition, and
(iii) complementation

We say two (0,1)-matrices A, B are equivalent if B can be obtained from A via (finitely many) operations in Lemma 3.1. This yields an equivalence relation on the set of $n\times n$ non-symmetric normal (0,1)-matrices.

Lemma 3.2 (Proposition 2.3, [1]) Let B be an $n\times n$ non-symmetric normal (0,1) matrices. Then, the number of 1′s in B is between 3 and n^2-3. If , further, B and B^c are irreducible, then the number of 1′s in B is between N and N^2-n

We note that if B is a reducible normal (0,1) matrix, then B^c is irreducible. Hence, an equivalence class can contain both reducible and irreducible matrices. We call an equivalence class an irreducible equivalence class if all matrices in it are irreducible. In general, it may be of interest to study (0,1)-matrices B and B^c are both irreducuble.

Proposition 3.3 Let A be an non-symmetric irreducible (0,1)-normal matrix. If $a_{ii}=0$ and the i-th row vector is same as the i-th column vector of A, then the matrix $A+E_{ii}$ is also non-symmetric irreducible (0,1)-normal.

Proposition 3.4 Let A be an non-symmetric irreducible (0,1)-normal matrix. If $a_{ij}=0=a_{ji}$ and the i,j-th row and column vectors of A are all same , then the matrix $A+E_{ij}+E_{ji}$ is also non-symmetric irreducible (0,1)-normal.

Proposition 3.5 Let B be an $(n-1)\times(n-1)$ non-symmetric irreducible (0,1)-normal matrix. If ,for each $i_1, i_2, \cdots, i_k$−th row and column vectors of B and $1 \le k \le (n-2)$, the row sum vectors of matrix $[r_{i_1}^T, r_{i_2}^T, \cdots, r_{i_k}^T]$ and $[c_{i_1}, c_{i_2}, \cdots, c_{i_k}]$ are same, then the $n\times n$ matrix A as following;

$A=$

$$\begin{bmatrix} * & \vdots & 0 \cdots 0 \; i_{(n-k+1)} \, i_{(n-k+2)} \cdots i_{(n-1)} \; i_n \\ \cdots & \vdots & \cdots\cdots\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ 0 & \vdots & \\ \vdots & \vdots & \\ 0 & \vdots & \\ i_{(n-k+1)} & \vdots & \\ i_{(n-k+2)} & \vdots & PBP^T \\ \vdots & \vdots & \\ i_{(n-1)} & \vdots & \\ i_n & \vdots & \end{bmatrix}$$

where P is a permutation matrix that reordered $i_1, i_2, \cdots, i_k$ − th row and column vectors of B to $i_{n-k+1}, i_{n-k+2}, \cdots, i_{n-1}$ − th row and column vectors, $i_{(\,.\,)}$ are all 1 and the $a_{11} = *$ is 0 or 1, is also non-symmetric irreducible (0,1)-normal.

Proof. By the expansion of Proposition 3.3, it is clear.

In [1], if A is an $n \times n$ (0,1)-normal matrix in irreducible equivalence class, then the number of one entries of A, $|A|$ is possible from n to $n^2 - n$. Since $J - A$ is contained in the A-irreducible equivalence class, we have that the number of one entries of A is possible from n to $[n^2/2]$ up to equivalence.

In fact, if $n = 3$, the possible number $|A|$ is 3 or 4. But there exists the only one irreducible equivalence class with $|A| = 3$. If $n = 4$, the possible number of $|A|$ is from 4 to 8. But there exists the only two irreducible equivalence classes with $|A| = 4$ and 8. If $n = 5$, the possible number of $|A|$ is from 5 to 12. But there does not exist the three irreducible equivalence classes with $|A| = 6, 7,$ and 8. Therefore, we can not confidence that the existence of A-irreducible equivalence class for each possible number of $|A|$.

Now, when $n = 6$, we construct an non-symmetric irreducible (0,1)-normal matrix for each possible number of $|A|$ from 6 to 18 and the integer K of $A(K)$ denote $|A|$ that is the number of non-zero entries in the following ;

$$A(\ 6) = \begin{bmatrix} 0&0&0&0&0&1 \\ 0&0&0&0&1&0 \\ 0&0&0&1&0&0 \\ 0&0&1&0&0&0 \\ 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \end{bmatrix}, \quad A(11) = \begin{bmatrix} 0&0&0&1&1&1 \\ 0&0&0&0&0&1 \\ 0&0&0&0&1&0 \\ 1&0&0&1&0&0 \\ 1&1&0&0&0&0 \\ 1&0&1&0&0&0 \end{bmatrix},$$

$$A(13) = \begin{bmatrix} 0&0&0&0&1&1 \\ 0&1&0&0&1&1 \\ 0&0&0&0&0&1 \\ 0&0&0&0&1&0 \\ 1&1&1&0&0&0 \\ 1&1&0&1&0&0 \end{bmatrix}, \quad A(17) = \begin{bmatrix} 0&0&0&1&1&1 \\ 0&0&0&1&1&0 \\ 0&0&0&0&0&1 \\ 1&1&0&0&1&0 \\ 1&0&1&0&1&1 \\ 1&1&0&1&0&1 \end{bmatrix},$$

$$\begin{aligned} &A(7) = A(6) + E_{33}, \quad A(8) = A(7) + E_{44}, \\ &A(9) = A(7) + E_{14} + E_{41}, \\ &A(10) = A(8) + E_{13} + E_{31}, \\ &A(12) = A(11) + E_{11}, A(14) = A(13) + E_{11}, \\ &A(15) = A(13) + E_{12} + E_{21}, \\ &A(16) = A(15) + E_{11}, \quad A(18) = A(17) + E_{11} \end{aligned}$$

We know that the above each normal matrices is contained in irreducible equivalent classes. Thus we make the following conjecture, the truth of which for $n = 6$ follows from the above statement.

[Conjecture]. Let be an $n \times n$ (0,1)-normal matrix where $n \geq 6$. Then there exists A-irreducible equivalence class for each possible number of $|A|$ from n to $[n^2/2]$

The conjecture seems reasonable because if $n = 3, 4,$ and 5, we can not use Proposition 3.3 and 3.4. But if $n \geq 6$, they are very useful.

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