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COMMON FIXED POINT OF WEAKLY COMPATIBLE MAPS WITHOUT CONTINUITY IN MENGER SPACES

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Abstract. In this paper we prove common fixed point of weakly compatible maps without continuity in Menger spaces. We show that continuity of any mapping is not required for the existence of fixed point. We improve some earlier results.

1. Introduction and Preliminaries

Sessa [16] generalized the notion of commuting maps given by Jungck [5] and introduced weakly commuting mappings. Further, Jungck [6] introduced more generalized commutativity called compatibility. In 1998, Jungck and Rhoades [7] introduced the notion of weakly compatible maps and showed that compatible maps are weakly compatible but converse need not true.

Menger [9] introduced the notion of probabilistic metric space, which is generalization of metric space and study of these spaces was expanded rapidly with pioneering work of Schewizer and Sklar [1], [2]. The existence of fixed points for compatible mappings on probabilistic metric space is shown by Mishra [21].

Recently, fixed point theorems in Menger spaces have been proved by many authors including Bylka [4], Pathak, Kang and Baek [8], Stojakovic [10], [11], [12], Hadzic [13], Dedeic and Sarapa [14], Rashwan and Hedar [15], Radu [22], [23].

Sehgal and Bharucha-Reid [24], Cho, Murthy and Stojakovic [25], Singh and Jain [3], Sharma and Bagwan [17], Sharma and Deshpande [18], Sharma, Pathak and Tiwari [19], Sharma, Deshpande and Tiwari [20].

Now we begin with some definitions:

Let R denote the set of reals and R^+ the non-negative reals. A mapping $F : R \to R^+$ is called a distribution function if it is non- decreasing and left continuous with $\inf F = 0$ and $\sup F = 1$. We will denote by L the set of all distribution functions.

A probabilistic metric space is a pair (X, F), where X is non empty set and F is a mapping from $X \times X$ to L. For $(p, q) \in X \times X$, the distribution

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function $F(p,q)$ is denoted by Fp,q. The function Fp,q are assumed to satisfy the following conditions:

(P_1) $Fp,q(x) = 1$ for every $x > 0$ if and only if $p = q$,
(P_2) $Fp,q(0) = 0$ for every $p,q \in X$,
(P_3) $Fp,q(x) = Fq,p(x)$ for every $p,q \in X$,
(P_4) if $Fp,q(x) = 1$ and $Fq,r(y) = 1$ then $Fp,r(x+y) = 1$ for every $p,q,r \in X$ and $x,y > 0$. In metric space (X,d) the metric d induces a mapping.

$F : X \times X \to L$ such that $F(p,q)(x) = Fp,q(x) = H(x - d(p,q))$ for every $p,q \in X$ and $x \in R$, where H is a distributive function defined by

$$H(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0. \end{cases}$$

Definition 1. A function $t : [0,1] \times [0,1] \to [0,1]$ is called a T-norm if it satisfies the following conditions:

(t_1) $t(a,1) = a$ for every $a \in [0,1]$ and $t(0,0) = 0$,
(t_2) $t(a,b) = t(b,a)$ for every $a,b \in [0,1]$,
(t_3) If $c \geq a$ and $d \geq b$ then $t(c,d) \geq t(a,b)$, for every $a,b,c \in [0,1]$,
(t_4) $t(t(a,b),c) = t(a,t(b,c))$ for every $a,b,c \in [0,1]$.

Definition 2. A Menger space is a triple (X,F,t), where (X,F) is a PM-space and t is a T-norm with the following condition:

(P_5) $Fp,r(x+y) \geq t(Fp,q(x), Fq,r(y))$ for every $p,q,r \in X$ and $x,y \in R^+$.

An important T-norm is the T-norm $t(a,b) = min\{a,b\}$ for all $a,b \in [0,1]$ and this is the unique T-norm such that $t(a,a) \geq a$ for every $a \in [0,1]$. Indeed if it satisfies this condition, we have

$$\begin{aligned} min\{a,b\} &\leq t(min\{a,b\}, min\{a,b\}) \\ &\leq t(a,b) \\ &\leq t(min\{a,b\}, 1) \\ &= min\{a,b\} \end{aligned}$$

Therefore $t = min$.

In the sequel, we need the following definitions.

Definition 3. ([22], [23]) Let (X,F,t) be a Menger space with continuous T-norm t. A sequence $\{x_n\}$ of points in X is said to be convergent to a point $x \in X$ if for every $\epsilon > 0$

$$\lim_{n\to\infty} Fx_n, \quad x(\epsilon) = 1.$$

Definition 4. ([22], [23]) Let (X,F,t) be a Menger space with continuous T-norm t. A sequence $\{x_n\}$ of points in X is said to be Cauchy sequence if for every $\epsilon > 0$ and $\lambda > 0$, there exists an integer $N = N(\epsilon,\lambda) > 0$ such that $Fx_n, x_m(\epsilon) > 1 - \lambda$ for all $m,n \in N$.

Definition 5. ([22], [23]) A Menger space (X, F, t) with the continuous T-norm t is said to be complete if every Cauchy sequence in X converges to a point in X.

Theorem A. ([1]) *Let t be a T-norm defined by $t(a, b) = min\{a, b\}$. Then the induced Menger space (X, F, t) is complete if a metric space (X, d) is complete.*

Definition 6. ([21]) Self mappings A and S of a Menger space (X, F, t) are called compatible if $FASx_n, SAx_n(x) \to 1$ for all $x > 0$, whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Sx_n \to u$ for some u in X as $n \to \infty$.

Definition 7. ([7]) Two maps A and B are said to be weakly compatible if they commute at coincidence point.

Lemma 1. ([2]) *Let $\{x_n\}$ be a sequence in a Menger space (X, F, t) with continuous t-norm and $t(x, x) \geq x$. Suppose for all $x \in [0, 1]$ there exists $k \in (0, 1)$ such that for all $x > 0$ and $n \in N$*

$$Fx_n, x_{n+1}(kx) \geq Fx_{n-1}, x_n(x).$$

Then $\{x_n\}$ is a Cauchy sequence in X.

Lemma 2. ([21]) *Let (X, F, t) be a Menger space. If there exists $k \in (0, 1)$ such that for $p, q \in X$, $Fp, q(kx) \geq Fp, q(x)$. Then $p = q$.*

Sharma, Deshpande and Tiwari [20] proved the following.

Theorem A. *Let $A, B, S, T, I, J, L, U, P$ and Q be self maps on a Menger space (X, F, t) with $t(a, a) \geq a$ for all $a \in [0, 1]$, satisfying*

(1) $P(X) \subset ABIL(X), Q(X) \subset STJU(X)$,

(2) *there exists $k \in (0, 1)$ such that*

$$\begin{aligned} FPx, Qy(ku) \geq min\{&FABILy, STJUx(u), FPx, STJUx(u),\\ &FQy, ABILy(u), FQy, STJUx(\alpha, FPx,\\ &ABILy((2-\alpha)u)\} \end{aligned}$$

for all $x, y \in X$, $\alpha \in (0, 2)$ and $u > 0$,

(3) *if one of $P(X)$, $ABIL(X)$, $STJU(X)$ or $Q(X)$ is a complete subspace of X then*
  (i) *P and $STJU$ have a coincidence point and*
  (ii) *Q and $ABIL$ have a coincidence point.*
  Further if

(4) $AB = BA$, $AI = IA$, $AL = LA$, $BI = IB$, $BL = LB$, $IL = LI$, $QL = LQ$, $QI = IQ$, $QB = BQ$, $ST = TS$, $SJ = JS$, $SU = US$, $TJ = JT$, $TU = UT$, $JU = UJ$, $PU = UP$, $PJ = JP$, $PT = TP$,

(5) *the pairs $\{P, STJU\}$ and $\{Q, ABIL\}$ are weakly compatible, then A, B, S, T, I, J, L, U, P and Q have a unique point in X.*

In what follows we prove the following:

2. Main Results

Theorem 1. *Let (X, F, t) be a complete Menger space with $t(a, a) \geq a$ for all $a \in [0, 1]$ and let P, S, T and Q be mappings from X into itself such that*

$$PT(X) \cup QS(X) \subset ST(X), \tag{1.1}$$

there exists a constant $k \in (0, 1)$ such that

$$\begin{aligned} &\min\{F^2Px, Qy(ku), [FSx, Px(ku)FTy, Qy(ku)]F^2Ty, Qy(ku)\} \\ &\geq [pFSx, Px(u) + qFSx, Ty(u)]FSx, Qy(2ku), \end{aligned} \tag{1.2}$$

for all $x, y \in X$ and $t > 0$, where $0 < p$, $q > 1$ such that $p + q = 1$, and

$$\text{the pairs } \{P, S\} \text{ and } \{Q, T\} \text{ are weakly compatible and } ST = TS. \tag{1.3}$$

Then P, S, T and Q have a unique common fixed point.

Proof. Let x_0 be an arbitrary point of X. By (1.1), we can construct $\{x_n\}$ in X as follows $PTx_{2n} = STx_{2n+1}$, $QSx_{2n+1} = STx_{2n+2}$, $n = 0, 1, 2, \cdots$.

Now let $z_n = STx_n$. Then by (1.2), we have

$$\begin{aligned} &\min\{F^2PTx_{2n}, QSx_{2n+1}(ku), [FSTx_{2n}, PTx_{2n}(ku)FTSx_{2n+1}, \\ &\qquad QSx_{2n+1}(ku)]F^2TSx_{2n+1}, QSx_{2n+1}(ku)\} \\ &\geq [pFSTx_{2n}, PTx_{2n}(u) + qFSTx_{2n}, TSx_{2n+1}(u)]FSTx_{2n}, QSx_{2n+1}(2ku), \end{aligned}$$

then

$$\begin{aligned} &\min\{F^2z_{2n+1}, z_{2n+2}(ku), [Fz_{2n}, z_{2n+2}(ku)Fz_{2n+1}, z_{2n+2}(ku)] \\ &\qquad F^2z_{2n+1}, z_{2n+2}(ku)\} \\ &\geq [pFz_{2n}, z_{2n+1}(u) + qFz_{2n}, z_{2n+1}(u)]Fz_{2n}, z_{2n+2}(2ku). \end{aligned}$$

So,

$$\begin{aligned} &\min\{F^2z_{2n+1}, z_{2n+2}(ku), [Fz_{2n}, z_{2n+1}(ku)Fz_{2n+1}, z_{2n+2}(ku)]\} \\ &\geq [p + q]Fz_{2n}, z_{2n+1}(u)Fz_{2n}, z_{2n+2}(2ku), \end{aligned}$$

and

$$\begin{aligned} &\min\{Fz_{2n+1}, z_{2n+2}(ku), [Fz_{2n}, z_{2n+2}(2ku)\} \\ &\geq [p + q]Fz_{2n}, z_{2n+1}(u)Fz_{2n}, z_{2n+2}(2ku), \end{aligned}$$

Thus it follows it follows that

$$Fz_{2n+1}, z_{2n+2}(ku) \geq Fz_{2n}, z_{2n+1}(u),$$

$0 < k < 1$ and for all $u > 0$. Similarly, we also have

$$Fz_{2n+2}, z_{2n+3}(ku) \geq Fz_{2n+1}, z_{2n+2}(u),$$

$0 < k < 1$ and for all $u > 0$. In general, for $m = 1, 2, \cdots$, we have

$$Fz_{m+2}, z_{m+3}(ku) \geq Fz_{m+1}, z_{m+2}(u),$$

$0 < k < 1$ and for all $u > 0$. Hence by Lemma 1, $\{z_n\}$ is a Cauchy sequence in X. Since X is complete, $\{z_n\}$ converges to a point $z \in X$. Since $\{PTx_{2n}\}$ and $\{QSx_{2n+1}\}$ are subsequences of $\{z_n\}$, $PTx_{2n} \to z$ and $QSx_{2n+1} \to z$ as

$n \to \infty$. Let $y_n = Tx_n$ and $w_n = Sx_n$ for $n = 1, 2, \cdots$. Then, we have $Py_{2n} \to z$, $Sy_{2n} \to z$, $Tw_{2n+1} \to z$ and $Qw_{2n+1} \to z$.

Now taking $x = z$ and $y = w_{2n+1}$ in (1.2), we have

$$\begin{aligned}&\min\{F^2Pz, Qw_{2n+1}(ku), [FSz, Pz(ku)FTw_{2n+1}, Qw_{2n+1}(ku)]\\ &\qquad F^2Tw_{2n+1}, Qw_{2n+1}(ku)\}\\ &\geq [pFSz, Pz(u) + qFSz, Tw_{2n+1}(u)]FSz, Qw_{2n+1}(2ku).\end{aligned}$$

Since the pair $\{P, S\}$ is weakly compatible therefore, P and S commute at their coincidence point, i.e., if $Pz = Sz$ for some $z \in X$, then $PSz = SPz$ and taking the limit $n \to \infty$, we have

$$\begin{aligned}&\min\{F^2Pz, z(ku), [FPz, Pz(ku)Fz, z(ku)]F^2z, z(ku)\}\\ &\geq [pFPz, Pz(u) + qFPz, z(u)]FPz, z(2ku).\end{aligned}$$

$$F^2Pz, z(ku) \geq [p + qFPz, z(u)]FPz, z(2ku).$$

Since Fx, y is non-decreasing for all x, y in X, we have

$$FPz, z(2ku)FPz, z(ku) \geq [p + qFPz, z(u)]FPz, z(2ku).$$

Thus

$$FPz, z(ku) \geq p + qFPz, z(u).$$

So,

$$FPz, z(ku) \geq p/(1-q) = 1,$$

for all $u > 0$, so $Pz = z$. Therefore $Pz = Sz = z$.

Now taking $x = y_{2n}$ and $y = z$ in (1.2), we have

$$\begin{aligned}&\min\{F^2Py_{2n}, Qz(ku), [FSy_{2n}, Py_{2n}(ku)FTz, Qz(ku)]F^2Tz, Qz(ku)\}\\ &\geq [pFSy_{2n}, Py_{2n}(u) + qFSy_{2n}, Tz(u)]FSy_{2n}, Qz(2ku).\end{aligned}$$

Again since Q and T are weak compatible therefore, $Tz = Qz$ and Taking the limit $n \to \infty$, we have

$$\begin{aligned}&\min\{F^2z, Qz(ku), [Fz, z(ku)FQz, Qz(ku)]F^2Qz, Qz(ku)\}\\ &\geq [pFz, z(u) + qFz, Qz(u)]Fz, Qz(2ku).\end{aligned}$$

So,

$$F^2z, Qz(ku) \geq [p + qFz, Qz(u)]Fz, Qz(2ku).$$

Since Fx, y is non-decreasing for all x, y in X, we have

$$Fz, Qz(2ku)Fz, Qz(ku) \geq [p + qFz, Qz(u)]Fz, Qz(2ku),$$

then

$$Fz, Qz(ku) \geq p/(1-q) = 1,$$

for all $u > 0$ so, $Qz = z$. Therefore $Tz = Qz = z$. Hence $Pz = Sz = Tz = Qz = z$, i.e., z is a common fixed point of P, S, T and Q. For uniqueness let

v $(v \neq z)$ be another common fixed point of P, Q, S and T. Then using (1.2), we have

$$\min\{F^2Pz, Qv(ku), [FSz, Pz(ku)FTv, Qv(ku)]F^2Tv, Qv(ku)\}$$
$$\geq [pFSz, Pz(u) + qFSz, Tv(u)]FSz, Qv(2ku),$$

So,

$$F^2z, v(ku) \geq [p + qFz, v(u)]Fz, v(2ku),$$

and since Fx, y is non-decreasing for all x, y in X, we have

$$Fz, v(2ku), Fz, v(ku) \geq [p + qFz, v(u)]Fz, v(2ku),$$

$$Fz, v(ku) \geq p + qFz, v(u),$$

which gives

$$Fz, v(ku) \geq p/(1 - q) = 1,$$

for all $u > 0$ so, $z = v$. Hence P, S, T and Q have a unique common fixed point. This completes the proof of the Theorem 1. □

If we put $S = T$ in Theorem 1, we have the following result:

Corollary 1. *Let (X, F, t) be a complete Menger space with $t(a, a) \geq a$ for all $a \in [0, 1]$ and let P, S and Q be maps from X into itself such that*

(1) $P(X) \cup Q(X) \subset S(X)$,
(2) *there exists a constant $k \in (0, 1)$ such that*

$$\min\{F^2Px, Qy(ku), [FSx, Px(ku)FSy, Qy(ku)]F^2Sy, Qy(ku)\}$$
$$\geq [pFSx, Px(u) + qFSx, Sy(u)]FSx, Qy(2ku),$$

for all $x, y \in X$ and $u > 0$, where $0 < p$, $q > 1$ such that $p + q = 1$, and
(3) *the pairs $\{P, S\}$ and $\{Q, S\}$ are weakly compatible. Then P, S and Q have a unique common fixed point.*

If we put $S = T$ and $P = Q$ in Theorem 1, we have the following.

Corollary 2. *Let (X, F, t) be a complete Menger space with $t(a, a) \geq a$ for all $a \in [0, 1]$ and let P and S be maps from X into itself such that*

(1) $P(X) \subset S(X)$,
(2) *there exists a constant $k \in (0, 1)$ such that*

$$\min\{F^2Px, Py(ku), [FSx, Px(ku)FSy, Py(ku)]F^2Sy, Py(ku)\}$$
$$\geq [pFSx, Px(u) + qFSx, Sy(u)]FSx, Py(2ku),$$

for all $x, y \in X$ and $u > 0$, where $0 < p$, $q > 1$ such that $p + q = 1$, and
(3) *the pair $\{P, S\}$ is weakly compatible.*

Then P and S have a unique common fixed point.

If we put $S = T = Ix$ (the identity map on X) in Theorem 1, we have the following.

Corollary 3. *Let (X, F, t) be a complete Menger space with $t(a, a) \geq a$ for all $a \in [0, 1]$ and let P and Q be mappings from X into itself such that*

(1) *there exists a constant* $k \in (0,1)$ *such that*

$$\min\{F^2Px, Qy(ku), [Fx, Px(ku)Fy, Qy(ku)]F^2y, Qy(ku)\}$$
$$\geq [pFx, Px(u) + qFx, y(u)]Fx, Qy(2ku),$$

for all $x, y \in X$ *and* $t > 0$*, where* $0 < p$*,* $q > 1$ *such that* $p + q = 1$*. Then* P *and* Q *have a unique common fixed point. This completes the proof of the Theorem.*

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