RELAXED PROXIMAL POINT ALGORITHMS BASED ON $A−$MAXIMAL RELAXED MONOTONICITY FRAMEWORKS WITH APPLICATIONS

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Abstract. Based on the $A−$maximal $(m)−$relaxed monotonicity frameworks, the approximation solvability of a general class of variational inclusion problems using the relaxed proximal point algorithm is explored, while generalizing most of the investigations, especially of Xu (2002) on strong convergence of modified version of the relaxed proximal point algorithm, Eckstein and Bertsekas (1992) on weak convergence using the relaxed proximal point algorithm to the context of the Douglas-Rachford splitting method, and Rockafellar (1976) on weak as well as strong convergence results on proximal point algorithms in real Hilbert space settings. Furthermore, the main result has been applied to the context of the $H−$maximal monotonicity frameworks for solving a general class of variational inclusion problems. It seems the obtained results can be used to generalize the Yosida approximation that, in turn, can be applied to first-order evolution inclusions, and can also be applied to Douglas-Rachford splitting methods for finding the zero of the sum of two $A−$maximal (m)-relaxed monotone mappings.

1. Introduction

Let X be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and with the norm $\| \cdot \|$ on X. We consider the inclusion problem: find a solution to

$$0 \in M(x), \tag{1}$$

where $M : X \to 2^X$ is a set-valued mapping on X.

Recently, Xu [12] considered the following algorithm to the context of solving the variational inclusion problem (1):

$$x^{k+1} = \alpha_k x^0 + (1 - \alpha_k)P_k(x^k) + \epsilon_k \, \forall k \geq 0, \tag{2}$$

where $P_k = (I + \rho_k M)^{-1}$ is the classical resolvent, $\{\alpha_k\}$ and $\{\rho_k\}$ are sequences of real numbers, and $\{\epsilon_k\}$ is the sequence of errors chosen appropriately. Unlike the findings of Rockafellar [7] on weak as well as strong convergence (limited

to restricted sense), and the work of Eckstein and Bertsekas [4] on weak convergence of the relaxed proximal point algorithm, Xu [12] achieved the strong convergence of the algorithm.

Rockafellar ([7], Theorem 1) investigated the general weak convergence of the proximal point algorithm to the context of solving (1), by showing for M maximal monotone, that the sequence $\{x^k\}$ generated for an initial point x^0 by the proximal point algorithm

$$x^{k+1} \approx P_k(x^k) \tag{3}$$

converges weakly to a solution of (1), provided the approximation is made sufficiently accurate as the iteration proceeds, where $P_k = (I + \rho_k M)^{-1}$ is the resolvent operator of M for a sequence $\{\rho_k\}$ of positive real numbers, that is bounded away from zero. We observe from (3) that x^{k+1} is an approximate solution to inclusion problem

$$0 \in M(x) + \rho_k^{-1}(x - x^k). \tag{4}$$

This work was further studied by several researchers, including Eckstein and Bertsekas [4] who relaxed the proximal point algorithm used in [7], widely cited in literature, and applied to the approximation solvability of (1). They also applied their obtained results to the Douglas-Rachford splitting method for finding a zero of the sum of two monotone mappings, while this turned out to be a specialized case of the proximal point algorithm. We observe that most the of variational problems, including minimization or maximization of functions, variational inequality problems, quasivariational inequality problems, decision and management sciences, and engineering sciences can be unified into form (1). The notion of the general maximal monotonicity has played a crucially significant role by providing a powerful tool to develop new proximal point algorithms in exploring and studying convex programming as well as variational inequalities. For more details, we refer the reader to [1-15].

In this communication, we examine the approximation solvability of inclusion problem (1) based on the notion of $A-$maximal $(m)-$relaxed monotone mappings, and derive some auxiliary results involving $A-$maximal $(m)-$relaxed monotone mappings to that setting. The notion of the $A-$maximal monotonicity introduced and studied in [7, 8] is more general than the usual maximal monotonicity, especially it could not be reduced to that context but it seems to be more application-oriented. We present a generalization to a well-cited work (in literature) of Eckstein and Bertsekas ([4], Theorem 3) to the case of $A-$maximal $(m)-$relaxed monotone mappings with some specialized versions. In a way it sounds interesting that we observe that our findings do not reduce to existing results in a trivial manner. We note that our main result on the approximation solvability of (1) differs significantly from that of ([4], Theorem 3) in the sense that M is without the monotonicity assumptions, and $A-$maximal

relaxed monotonicity is applied instead of just maximal monotonicity. Moreover, we achieve a strong convergence of the generalized proximal point algorithm, unlike some investigations limited to weak convergence. We present a significant application of the main result on the $A-$maximal $(m)-$relaxed monotonicity to the case of the $H-$maximal monotonicity [5]. While our result on $A-$maximal $(m)-$relaxed monotonicity, Theorem 3.2, we observe does not reduce to the case of [4], there exists a tremendous amount of research work on new developments and applications of proximal point algorithms in literature to the context of approximate solutions of variational inclusion problems of the form (1) in different space settings.

2. Preliminaries

In this section, first we introduce the notion of the $A-$maximal $(m)-$relaxed monotonicity, and then we derive some basic properties along with some auxiliary results for the problem on hand.

Let X be a real Hilbert space with the norm $\|\cdot\|$ and with the inner product $\langle\cdot,\cdot\rangle$. Let $M : X \to 2^X$ be a set-valued mapping on X and we will denote the graph of M, the set $\{(x,y)|y \in M(x)\}$, by M as well. This is equivalent to stating that an operator is any subset M of $X \times X$ and $M(x) = \{y|(x,y) \in M\}$.

The domain of a mapping M, denoted $D(M)$, is its projection onto the first coordinate, is equivalent to

$$D(M) = \{x \in X|\exists y \in X : (x,y) \in M\} = \{x \in X : M(x) \neq \emptyset\}.$$

M has a full domain if $D(M) = X$ and the range of M can be defined as its projection onto the second coordinate, that is.

$$R(M) = \{y \in X|\exists x \in X : (x,y) \in M\}.$$

The inverse M^{-1} of M is $\{(y,x)|(x,y) \in M\}$. For any real number c and mapping M, we define cM by $\{(x,cy)|(x,y) \in M\}$. For any two mappings M and S, we set

$$M + S = \{(x,y+z)|(x,y) \in M, (x,z) \in S\}.$$

Definition 2.1. Let X be a real Hilbert space, and let $M : X \to 2^X$ be a multivalued mapping and $A : X \to X$ be a single-valued mapping on X. The map M is said to be:

(i) Monotone if

$$\langle u^* - v^*, u - v\rangle \geq 0 \,\forall\, (u,u^*), (v,v^*) \in M.$$

(ii) Strongly monotone if there exists a positive constant r such that

$$\langle u^* - v^*, u - v\rangle \geq r\|u - v\|^2 \,\forall\, (u,u^*), (v,v^*) \in M.$$

(iii) Relaxed monotone if there exists a positive constant m such that

$$\langle u^* - v^*, u - v\rangle \geq -m\|u - v\|^2 \,\forall\, (u,u^*), (v,v^*) \in M.$$

(iv) Expansive if there exists a positive constant r such that
$$\|u^* - v^*\| \geq r\|u - v\| \,\forall\, (u, u^*), (v, v^*) \in M.$$

(v) Cocoercive if there exists a positive constant c such that
$$\langle u^* - v^*, u - v \rangle \geq c\|u^* - v^*\|^2 \,\forall\, (u, u^*), (v, v^*) \in M.$$

(vi) Monotone with respect to A if
$$\langle u^* - v^*, A(u) - A(v) \rangle \geq 0 \,\forall\, (u, u^*), (v, v^*) \in M.$$

(vii) Strongly monotone with respect to A if there exists a positive constant r such that
$$\langle u^* - v^*, A(u) - A(v) \rangle \geq r\|u - v\|^2 \,\forall\, (u, u^*), (v, v^*) \in M.$$

(viii) Relaxed monotone with respect to A if there exists a positive constant m such that
$$\langle u^* - v^*, A(u) - A(v) \rangle \geq -m\|u - v\|^2 \,\forall\, (u, u^*), (v, v^*) \in M.$$

(ix) Cocoercive with respect to A if there exists a positive constant γ such that
$$\langle u^* - v^*, A(u) - A(v) \rangle \geq \gamma\|u^* - v^*\|^2 \,\forall\, (u, u^*), (v, v^*) \in M.$$

Definition 2.2. Let X be a real Hilbert space, and let $M : X \to 2^X$ be a mapping on X. The map M is said to be:

(i) Nonexpansive if
$$\|u^* - v^*\| \leq \|u - v\| \,\forall\, (u, u^*), (v, v^*) \in graph(M).$$

(ii) Lipschitz continuous if there exists a constant $s \geq 0$ such that
$$\|u^* - v^*\| \leq s\|u - v\| \,\forall\, (u, u^*), (v, v^*) \in M.$$

Definition 2.3. Let X be a real Hilbert space. Let $A : X \to X$ be a single-valued mapping. The map $M : X \to 2^X$ is said to be $A-$maximal $(m)-$relaxed monotone if

(i) M is $(m)-$relaxed monotone, that is,
$$\langle u^* - v^*, u - v \rangle \geq -m\|u - v\|^2 \,\forall\, (u, u^*), (v, v^*) \in M,$$

(ii) $R(A + \rho M) = X$ for $\rho > 0$.

Definition 2.4. Let X be a real Hilbert space. Let $M : X \to 2^X$ be a $A-$maximal $(m)-$relaxed monotone mapping, and let $A : X \to X$ be strongly monotone mapping with a positive constant r. Then the resolvent operator $R_{\rho,m,A}^M : X \to X$ is defined by
$$R_{\rho,m,A}^M(u) = (A + \rho M)^{-1}(u).$$

Proposition 2.1. *Let X be a real Hilbert space. Let $A : X \to X$ be strongly monotone mapping with a positive constant r, and let $M : X \to 2^X$ be an $A-$maximal $(m)-$relaxed monotone mapping. Then the resolvent operator $R^M_{\rho,m,A} = (A + \rho M)^{-1}$ is single-valued for $r - \rho m > 0$.*

Proof. The proof follows from the definition of the resolvent operator. For any $z \in X$, consider $x, y \in (A + \rho M)^{-1}(z)$. Then

$$-A(x) + z \in \rho M(x) \ and \ -A(y) + z \in \rho M(y).$$

Since M is maximal $(m)-$relaxed monotone, it implies

$$-\rho m \le -\langle x - y, A(x) - A(y)\rangle \le -r\|x - y\|^2$$
$$\Rightarrow \quad (r - \rho m)\|x - y\|^2 \le 0$$
$$\Rightarrow \quad x = y \ for \ r - \rho m > 0.$$

$\square$

Theorem 2.1. *Let X be a real Hilbert space, let $A : X \to X$ be strongly monotone (with a constant $r > 0$) and let $M : X \to 2^X$ be $A-$maximal $(m)-$relaxed monotone. Then*

(i) *For $r - \rho_k m > 1$, $R^M_{\rho_k,m,A}$ is $(r - \rho_k m)-$cocoercive, that is,*

$$\langle R^M_{\rho_k m,,A}(u) - R^M_{\rho_k m,,A}(v), u - v\rangle \ge (r - \rho_k m)\|R^M_{\rho_k,m,A}(u) - R^M_{\rho_k,m,A}(v)\|^2.$$

Proof. To prove we use the definition of scaling, addition, and inversion operations. Now we have

$$(x, y) \in M \Leftrightarrow (A(x) + \rho y, x) \in (A + \rho M)^{-1} \ for \ \rho > 0.$$

Thus, we begin the proof with:

$$M \ is \ A - maximal \ (m) - relaxed \ monotone$$
$$\Leftrightarrow \langle x^{'} - x, y^{'} - y\rangle \ge -m\|x^{'} - x\|^2 \ \forall \ (x, y), (x^{'}, y^{'}) \in M$$
$$\Leftrightarrow \langle x^{'} - x, \rho_k y^{'} - \rho_k y\rangle \ge -\rho_k m\|x^{'} - x\|^2$$
$$\Leftrightarrow \langle x^{'} - x, A(x^{'}) - A(x) + \rho_k y^{'} - \rho_k y\rangle \ge \langle A(x^{'}) - A(x), x^{'} - x\rangle - \rho_k m\|x^{'} - x\|^2$$
$$\Leftrightarrow \langle A(x^{'}) + \rho_k y^{'} - (A(x) + \rho_k y), x^{'} - x\rangle \ge (r - \rho_k m)\|x^{'} - x\|^2$$
$$\Leftrightarrow (A + \rho_k M)^{-1} \ is \ (r - \rho_k m) - cocoercive$$

$\square$

Definition 2.5. Let X be a real Hilbert space. A map $M : X \to 2^X$ is said to be maximal monotone if

(i) M is monotone, that is,

$$\langle u^* - v^*, u - v\rangle \ge 0 \ \forall \ (u, u^*), (v, v^*) \in M,$$

(ii) $R(I + \rho M) = X$ for $\rho > 0$.

Furthermore, the resolvent operator $J_\rho^M : X \to X$ is defined by

$$J_\rho^M(u) = (I + \rho M)^{-1}(u).$$

Next we include the following examples of an $A-$maximal $(m)-$relaxed monotone mapping on a reflexive Banach space setting.

Example 2.1. Let X be a reflexive Banach space. $A : X \to X^*$ be $(r)-$strongly monotone with constant $r > 0$, and let $f : X \to \Re$ be locally Lipschitz such that the subdifferential $\partial f : X \to 2^{X^*}$ is $(m)-$relaxed monotone. Then

$$\langle u^* - v^*, u - v \rangle \geq (r - m)\|u - v\|^2, \tag{5}$$

where $u^* \in A(u) + \partial f(u)$, $v^* \in A(v) + \partial f(v)$, and $r - m > 0$. Then ∂f turns out to be $A-$maximal $(m)-$relaxed monotone.

3. Generalizations to $A-$maximal relaxed monotonicity

This section deals with generalizations to Rockafellar's theorems ([7 Theorems 1 and 2]) and to Eckstein and Bertsekas ([4], Theorem 3) on the classical resolvent in light of the new framework of the $A-$maximal $(m)-$relaxed monotonicity while solving inclusion problem (1). We start this section with a generalized auxiliary result crucial to the problem on hand.

Theorem 3.1. *Let X be a real Hilbert space, let $A : X \to X$ be strongly monotone mapping with a positive constant r, and let $M : X \to 2^X$ be $A-$maximal $(m)-$relaxed monotone. Then the following statements are equivalent:*

   (i) *An element $u \in X$ is a solution to* (1),
   (ii) *For an $u \in X$, we have*

$$u = R_{\rho,m,A}^M(A(u)),$$

where

$$R_{\rho,m,A}^M(u) = (A + \rho M)^{-1}(u)\, for\, r - \rho m > 0.$$

Proof. It follows from the definition of resolvent operator corresponding to M. $\square$

Theorem 3.2. *Let X be a real Hilbert space, let $A : X \to X$ be strongly monotone (with a constant $r > 0$) and let $M : X \to 2^X$ be $A-$maximal $(m)-$relaxed monotone. Furthermore, we assume $R_{\rho_k,m,A}^M oA$ is $(\lambda)-$cocoercive with respect to $(I - R_{\rho_k,m,A}^M oA)$ for $\lambda > 0$.*

For an arbitrarily chosen initial point x^0, suppose that the sequence $\{x^k\}$ is generated by the proximal point algorithm $(0 \leq \alpha_k \leq 1)$

$$x^{k+1} = (1 - \alpha_k)x^k + \alpha_k y^k \,\forall\, k \geq 0. \tag{6}$$

such that

$$\|y^k - R_{\rho_k,m,A}^M(A(x^k))\| \leq \delta_k \|y^k - x^k\|,$$

where $\delta_k \to 0$ with $\Sigma_{k=0}^\infty \delta_k < \infty$. Suppose that the sequence $\{x^k\}$ is bounded in the sense that there exists at least one solution to (1). Then we have:

(i) *For $\lambda > 0$, we have*

$$\|R^M_{\rho_k,m,A}(A(x^k)) - R^M_{\rho_k,m,A}(A(x^*))\| \le \frac{1}{\lambda+1}\|x^k - x^*\|,$$

(ii) *The sequence $\{x^k\}$ converges linearly to a solution of (1) for $\Sigma_{k=0}^\infty \delta_k < \infty$, $\alpha = \limsup_{k\to\infty}\alpha_k$, $\rho_k \nearrow \rho \le \infty$, $r - \rho_k m > 0$, and $\lambda > 0$.*

Proof. To prove (i), if we use the (λ)$-$cocoercivity of $R^M_{\rho_k,m,A}\mathrm{o}A$ with respect to $(I - R^M_{\rho_k,m,A}\mathrm{o}A)$, then after a compact manipulation, we arrive at

$$\langle R^M_{\rho_k,m,A}(A(x^k)) - R^M_{\rho_k,m,A}(A(x^*)), x^k - x^* \rangle$$
$$\ge (\lambda+1)\|R^M_{\rho_k,m,A}(A(x^k)) - R^M_{\rho_k,m,A}(A(x^*))\|^2.$$

We begin the proof of (ii) as follows: Suppose that x^* is a zero of M. Note that any zero of M is a fixed of $R^M_{\rho_k,A}\mathrm{o}A$ by Theorem 3.1 for all k. We define for all $k \ge 0$

$$z^{k+1} = (1 - \alpha_k)x^k + \alpha_k R^M_{\rho_k,m,A}(A(x^k)).$$

Next, we estimate using the above inequality

$$\|z^{k+1} - x^*\| = \|(1-\alpha_k)(x^k - x^*) + \alpha_k(R^M_{\rho_k,m,A}(A(x^k)) - R^M_{\rho_k,m,A}(A(x^*)))\|$$
$$\le (1-\alpha_k)\|x^k - x^*\| + \alpha_k\|R^M_{\rho_k,m,A}(A(x^k)) - R^M_{\rho_k,m,A}(A(x^*))\|$$
$$\le (1-\alpha_k)\|x^k - x^*\| + \frac{\alpha_k}{\lambda+1}\|x^k - x^*\|$$
$$= [1 - \alpha_k(1 - \frac{1}{\lambda+1})]\|x^k - x^*\|.$$

Thus, we have

$$\|z^{k+1} - x^*\| \le \theta_k\|x^k - x^*\|,$$

where $\theta_k = [1 - \alpha_k(1 - \frac{1}{\lambda+1})] < 1$.

Next we turn our attention to establish linear convergence. Since

$$x^{k+1} - x^k = \alpha_k(y^k - x^k),$$

we have

$$\|x^{k+1} - x^*\| = \|z^{k+1} - x^* + x^{k+1} - z^{k+1}\|$$
$$= \|z^{k+1} - x^* + \alpha_k(y^k - R^M_{\rho_k,A}(A(x^k)))\|$$
$$\le \|z^{k+1} - x^*\| + \alpha_k\|y^k - R^M_{\rho_k,A}(A(x^k))\|$$
$$\le \|z^{k+1} - x^*\| + \alpha_k\delta_k\|y^k - x^k\|$$
$$= \|z^{k+1} - x^*\| + \delta_k\|x^{k+1} - x^k\|$$
$$\le \|z^{k+1} - x^*\| + \delta_k[\|x^{k+1} - x^*\| + \|x^k - x^*\|]$$
$$\le \theta_k\|x^k - x^*\| + \delta_k[\|x^{k+1} - x^*\| + \|x^k - x^*\|].$$

Hence,

$$\|x^{k+1} - x^*\| \leq \frac{\theta_k + \delta_k}{1 - \delta_k}\|x^k - x^*\|, \tag{7}$$

where $\theta_k = [1 - \alpha_k(1 - \frac{1}{\lambda+1})] < 1$.

Finally, all we need is to show the uniqueness of the solution to (1). Assume that x^* is a zero of M. It follows from above arguments that there exists a limit

$$lim_{k\to\infty} inf \|x^k - x^*\| = a^* < \infty, \tag{8}$$

which is nonnegative and finite, and as a result, $\|x^k - x^*\| \to a^*$. Consider x_1^* and x_2^* as two limit points of the sequence $\{x^k\}$. Then we have

$$\|x^k - x_1^*\| = a_1, \ \|x^k - x_2^*\| = a_2$$

and both exist and are finite. If we express

$$\|x^k - x_2^*\|^2 = \|x^k - x_1^*\|^2 + 2\langle x^k - x_1^*, x_1^* - x_2^*\rangle + \|x_1^* - x_2^*\|^2,$$

then it follows that

$$lim_{k\to\infty} \langle x^k - x_1^*, x_1^* - x_2^*\rangle = \frac{1}{2}[a_2^2 - a_1^2 - \|x_1^* - x_2^*\|^2].$$

Since x_1^* is a limit point of $\{x^k\}$, the left hand side limit must tend to zero. Therefore,

$$a_1^2 = a_2^2 - \|x_1^* - x_2^*\|^2.$$

Similarly, we obtain

$$a_2^2 = a_1^2 - \|x_1^* - x_2^*\|^2.$$

This results in $x_1^* = x_2^*$.   □

Note that the proof model used here for the uniqueness of a limit point of the sequence $\{x^k\}$ to the context of solving (1) is based on the method adopted in Martinnet [6], Rockafellar [7] and Eckstein and Beretsekas [4].

Next, we have the following result on the $H-$maximal monotonicity - a generalization to general maximal monotonicity by Fang and Huang [5] in literature.

4. An application to $H-$maximal monotonicity

In this section, we apply Theorem 3.2 to derive the approximation solvability of variational inclusion problems of the form (1) to the case of the $H-$maximal monotonicity frameworks. We observe, unlike Theorem 3.2, that the obtained result on the approximation solvability does not hold for a classical resolvent setting, because the construction breaks down.

Definition 4.1. Let X be a real Hilbert space. Let $H : X \to X$ be a single-valued mapping. The map $M : X \to 2^X$ is said to be $H-$maximal monotone if

   (i) M is monotone,
   (ii) $R(H + \rho M) = X$ for $\rho > 0$.

Definition 4.2. Let X be a real Hilbert space. Let $M : X \to 2^X$ be an $H-$maximal monotone mapping, and let $A : X \to X$ be strongly monotone mapping with a positive constant r. Then the resolvent operator $R_{\rho,H}^M : X \to X$ is defined by

$$J_{\rho,H}^M(u) = (H + \rho M)^{-1}(u)\, for\, r > 0.$$

Theorem 4.1. Let X be a real Hilbert space, let $A : X \to X$ be strongly monotone mapping with a positive constant r, and let $M : X \to 2^X$ be $H-$maximal monotone. Then the following statements are equivalent:

  (i) An element $u \in X$ is a solution to (1),
  (ii) For an $u \in X$, we have

$$u = R_{\rho,H}^M(H(u)),$$

where

$$R_{\rho,H}^M(u) = (H + \rho M)^{-1}(u)\, for\, r > 0.$$

Proof. It follows from the definition of resolvent operator corresponding to M. □

Theorem 4.2. Let X be a real Hilbert space, let $H : X \to X$ be strongly monotone (with a constant $r > 0$) and let $M : X \to 2^X$ be $H-$maximal monotone. Furthermore, we assume $R_{\rho_k,H}^M oH$ is $(\lambda)-$cocoercive with respect to $(I - R_{\rho_k,H}^M oH)$ for $\lambda > 1$.

For an arbitrarily chosen initial point x^0, suppose that the sequence $\{x^k\}$ is generated by the proximal point algorithm $(0 \leq \alpha_k \leq 1)$

$$x^{k+1} = (1 - \alpha_k)x^k + \alpha_k y^k \, \forall\, k \geq 0 \tag{9}$$

such that

$$\|y^k - R_{\rho_k,H}^M(H(x^k))\| \leq \delta_k \|y^k - x^k\|,$$

where $\delta_k \to 0$ with $\Sigma_{k=0}^{\infty}\delta_k < \infty$. Suppose that the sequence $\{x^k\}$ is bounded in the sense that there exists at least one solution to (1). Then we have:

  (i) For $\lambda > 0$, we have

$$\|R_{\rho_k,H}^M(H(u)) - R_{\rho_k,H}^M(H(v))\| \leq \frac{1}{\lambda + 1}\|u - v\|,$$

  (ii) The sequence $\{x^k\}$ converges linearly to a solution of (1) for $\Sigma_{k=0}^{\infty}\delta_k < \infty$, $\alpha_k \in [0,1]$, $\rho = \inf \rho_k$, and $\lambda > 0$.

Proof. The proof is analogous to Theorem 3.2, but we include a brief sketch for the sake of the completeness. The proof of (ii) follows from (i)for $\lambda > 0$.

Suppose that x^* is a zero of M. Note that any zero of M is a fixed of $R_{\rho_k,H}^M oH$ by Theorem 4.1 all k. We define for all k

$$z^{k+1} = (1 - \alpha_k)x^k + \alpha_k R_{\rho_k,H}^M(H(x^k)).$$

Next, we estimate using the above inequality

$$\|z^{k+1} - x^*\| = \|(1 - \alpha_k)(x^k - x^*) + \alpha_k(R_{\rho_k,A}^M(H(x^k)) - R_{\rho_k,A}^M(H(x^*)))\|$$
$$\leq (1 - \alpha_k)\|x^k - x^*\| + \alpha_k\|R_{\rho_k,H}^M(H(x^k)) - R_{\rho_k,A}^M(H(x^*))\|$$
$$\leq (1 - \alpha_k)\|x^k - x^*\| + \frac{\alpha_k}{\lambda + 1}\|^k - x^*\|$$
$$= [1 - \alpha_k(1 - \frac{1}{\lambda + 1})]\|x^k - x^*\|.$$

Thus, it follows that

$$\|z^{k+1} - x^*\| \leq \theta_k^*\|x^k - x^*\|,$$

where $\theta_k^* = [1 - \alpha_k(1 - \frac{1}{\lambda+1})] < 1$.

Next we turn our attention to establish linear convergence. Since

$$x^{k+1} - x^k = \alpha_k(y^k - H(x^k)),$$

we have

$$\|x^{k+1} - x^*\| = \|z^{k+1} - x^* + x^{k+1} - z^{k+1}\|$$
$$= \|z^{k+1} - x^* + \alpha_k(y^k - R_{\rho_k,H}^M(H(x^k)))\|$$
$$\leq \|z^{k+1} - x^*\| + \alpha_k\|y^k - R_{\rho_k,A}^M(H(x^k))\|$$
$$\leq \|z^{k+1} - x^*\| + \alpha_k\delta_k\|y^k - x^k\|$$
$$= \|z^{k+1} - x^*\| + \delta_k\|x^{k+1} - x^k\|$$
$$\leq \|z^{k+1} - x^*\| + \delta_k[\|x^{k+1} - x^*\| + \|x^k - x^k\|]$$
$$\leq \theta_{k^*}\|x^k - x^*\| + \delta_k[\|x^{k+1} - x^*\| + \|x^k - x^k\|].$$

Hence,

$$\|x^{k+1} - x^*\| \leq \frac{\theta_k^* + \delta_k}{1 - \delta_k}\|x^k - x^*\|, \tag{10}$$

where $\theta_k^* = [1 - \alpha_k(1 - \frac{1}{\lambda+1})] < 1$. This completes the proof. $\qquad\square$

5. Concluding remark

First of all, although the presence of constant $(r - \rho m) > 0$ appears minimal other than showing the resolvent is single-valued in Proposition 2.1, it is very crucial for the convergence analysis. Because of the linear convergence concerns, we had skip it completely during the proofs of Theorems 3.2 and 4.2 altogether just to avoid the Lipschitz continuity of the mapping A. Moreover, proofs for Theorems 3.2 and 4.2 seem to be more compact and application-oriented based on the strong convergence achieved using the hybrid proximal point algorithm.

It seems the conclusions of Theorem 3.2 can further be applied to Douglas-Rachford splitting methods for finding a zero of the sum of two $A-$maximal relaxed monotone mappings since it turns out that it is a special case of the proximal point algorithm. Among other applications of the Douglas-Rachford splitting, such as the alternating direction method of multipliers for convex

programming decomposition is also specializations of the proximal point algorithm. It has been observed that the relaxation factor α_k can be chosen greater than 1, for example, $\alpha_k = 1.5$ for all k, the convergence rate may increase to a given accuracy about 15% faster than the choice $\alpha_k = 1$ for all k.

References

[1] R. P. Agarwal and R. U. Verma, *General system of $(A, \eta)-$ maximal relaxed monotone variational inclusion problems based on generalized hybrid algorithms*, Communications in Nonlinear Science and Numerical Simulation **15** (2010), 238–251.

[2] R. P. Agarwal and R. U. Verma, *Inexact $A-$ proximal point algorithm and applications to nonlinear variational inclusion problems*, Journal of Optimization Theory and Applications **144(3)** (2010), 431–444.

[3] N. U. Ahmed, *Optimization and identification of Systems Governed by Evolution Equations on Banach Spaces*, Pitman Research Notes in mathematics, Volume **184**, Longman Scientific and Technical, 1988.

[4] J. Eckstein and D. P. Bertsekas, *On the Douglas-Rachford splitting method and the proximal point algorithm for maximal monotone operators*, Mathematical Programming **55** (1992), 293–318, 154–158.

[5] Y. P. Fang and N. J. Huang, *$H-$ monotone operators and system of variational inclusions*, Communications on Applied Nonlinear Analysis **11(1)** (2004), 93–101.

[6] B. Martinet, *Regularisation d'inequations variationelles par approximations successives*, Rev. Francaise Inf. Rech. Oper. **4** (1970), 154–158.

[7] R. T. Rockafellar, *Monotone operators and the proximal point algorithm*, SIAM Journal of Control and Optimization **14** (1976), 877–898.

[8] R. T. Rockaffelar and R. J-B. Wets, *Variational Analysis*, Springer, New York, 2004.

[9] R. U. Verma, *$A-$ monotonicity and its role in nonlinear variational inclusions*, Journal of Optimization Theory and Applications **129(3)** (2006), 457–467.

[10] R. U. Verma, *Approximation solvability of a class of nonlinear setvalued inclusions involving $(A, \eta)-$ monotone mappings*, Journal of Mathematical Analysis and Applications **337** (2008), 969–975.

[11] R. U. Verma, *On general over-relaxed proximal point algorithm and applications*, Optimization, (in press).

[12] H. K. Xu, *Iterative algorithms for nonlinear operators*, Journal of the London Mathematical Society **66(2)** (2002), 240–256.

[13] E. Zeidler, *Nonlinear Functional Analysis and its Applications I*, Springer-Verlag, New York, 1986.

[14] E. Zeidler, *Nonlinear Functional Analysis and its Applications II/B*, Springer-Verlag, New York, 1990.

[15] E. Zeidler, *Nonlinear Functional Analysis and its Applications III*, Springer-Verlag, New York, 1985.

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