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GPR-SEPARATION AXIOMS

Gnanambal Ilango, Krishnan Balachandran,
and Rayappan Marudhachalam

Abstract. In this paper gpr-open sets are used to define some weak separation axioms and we study some of their basic properties. The implications of these axioms among themselves are also verified.

1. Introduction

In recent years there has been a considerable number of papers considering separation properties, essentiallydefined by replacing open sets by weak forms of open sets. In 1975, Maheswari and Prasad [11] used semi-open sets to define and investigate three new separation axioms, called *semi*-T_2, *semi*-T_1 and *semi*-T_0. In 1987, Bhattacharyya and Lahiri [2] used semi-open sets to define the axiom *semi*-$T_{1/2}$. Jankovic and Reilly [8] and Caldas [3] have also worked on semi separation properties and *semi*-$T_{1/2}$ spaces respectively. In 1990, Kar and Bhattacharyya [9] defined and characterized three new separation axioms called pre-T_0, pre-T_1 and pre-T_2 by using pre-open sets.

In 1999, Gnanambal [6] introduced and investigated the notions of gpr-open sets and pre-regular $T_{1/2}$ spaces in topological spaces. In this paper, we define some weak gpr-separation axioms, namely *gpr*-T_0 , *gpr*-T_1 , *gpr*-T_2 , and *gpr*-T_3 and study further properties of these spaces. The separation axioms R_0 and R_1 were introduced and studied by Shanin [13] and Yang [15] respectively. Later they were rediscovered by Davis [5]. These axioms generalize the separation axioms T_1 and T_2. In this paper we introduce *gpr*-R_0, *gpr*-R_1 and gR_0 spaces.

2. Preliminaries

Throughout this paper (X, τ) represent a nonempty topological space. For a subset A of a space (X, τ), $cl(A)$, $int(A)$, $pcl(A)$, $spcl(A)$ and $gpr\text{-}cl(A)$ denote the closure of A, interior of A, pre closure of A, semi pre closure of A and gpr-closure of A respectively. (X, τ) will be replaced by X if there is no chance

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of confusion.
Let us recall the following definitions which we shall require later.

Definition 2.1. A subset A of a space (X, τ) is called

(i) a *preopen set*[12] if $A \subset int\ cl(A)$ and a *preclosed set* if $cl\ int(A) \subset A$,
(ii) a *semi-open set*[10] if $A \subset cl\ int(A)$ and a *semi-closed set* if $int\ cl(A) \subset A$,
(iii) a *semi-preopen set*[1] if $A \subset cl\ int\ cl(A)$ and a *semi-preclosed set* if $int\ cl\ int(A) \subset A$,
(iv) a *regular open set*[14] if $A = int\ cl(A)$ and a *regular closed set* if $A = cl\ int(A)$.

The pre closure of A (denoted by $pcl(A)$) is the intersection of all pre closed sets that contain A.The semi pre closure of A and generalized pre regular closure of A (denoted by $spcl(A)$ and $gpr\text{-}cl(A)$ respectively) are analogously defined.

Definition 2.2. A subset A of a space (X, τ) is called

(i) a *generalized semi-preclosed set* (briefly gsp-closed)[6] if $spcl(A) \subset U$ whenever $A \subset U$ and U is open,
(ii) a *generalized pre-regular closed set* (briefly gpr-closed)[6] if $pcl(A) \subset U$ whenever $A \subset U$ and U is regular open.

Definition 2.3. ([6]) A space (X, τ) is called a pre-regular $T_{1/2}$ space if every generalized pre-regular closed set is preclosed.

Definition 2.4. ([7]) For a subset A of (X, τ):

$$\tau_g^* = \{V \subset X / gpr\text{-}cl(X - V) = (X - V)\}.$$

Lemma 2.1. ([7]) *For a space (X, τ) every gpr-closed set is closed if and only if $\tau_g^* = \tau$ holds.*

3. GPR-separation axioms

Definition 3.1. ($gpr\text{-}T_0$ space) A space X is a $gpr\text{-}T_0$ space if for every pair of points 'a' and 'b' there exists a gpr-open set U such that at least one of the following statements is true.

(i) a lies in U and b does not lie in U,
(ii) b lies in U and a does not lie in U.

Definition 3.2. ($gpr\text{-}T_1$ space) A space X is a $gpr\text{-}T_1$ space if to each pair of distinct points 'a', 'b' of X, there exists a pair of gpr-open sets, containing 'a' but not 'b', and the other containing 'b' but not 'a'.

Definition 3.3. ($gpr\text{-}T_2$ space) A space X is a $gpr\text{-}T_2$ space if for every pair of points 'a' and 'b' there exists disjoint gpr-open sets which separately contain 'a' and 'b'.

Definition 3.4. A subset $N(x)$ of X is said to be a gpr-neighbourhood of a point $x \in N(x)$ if there exists a gpr-open set G such that $x \in G \subset N(x)$.

Definition 3.5. (*gpr*-T_3 space) A space X is a *gpr*-T_3 space or gpr-regular space if for every point 'a' and a gpr-closed set B with $a \notin B$, there exists disjoint gpr-open sets which separately contain 'a' and B.

We have the following implications.

gpr-T_3 space $\underset{\not\leftarrow}{\rightarrow}$ *gpr*-T_2 space $\underset{\not\leftarrow}{\rightarrow}$ *gpr*-T_1 space $\underset{\not\leftarrow}{\rightarrow}$ *gpr*-T_0 space.

Example 3.1. $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ is a *gpr*-T_2 space but not *gpr*-T_3 space.

Definition 3.6. (*gpr*-R_0 space) A space X is said to be a *gpr*-R_0 space iff for each gpr-open set G and $x \in G$, *gpr*-*cl*$\{x\} \subset G$.

Definition 3.7. (*gpr*-R_1 space) A space X is a *gpr*-R_1 space iff for $x, y \in X$ with *gpr*-*cl*$\{x\} \neq$ *gpr*-*cl*$\{y\}$ there exists disjoint gpr-open sets U and V such that *gpr*-*cl*$\{x\} \subset U$ and *gpr*-*cl*$\{y\} \subset V$.

Definition 3.8. (gR_0 space) A space X is said to be a gR_0 space iff for each open set G and $x \in G$, *gpr*-*cl*$\{x\} \subset G$.

We have the following implication.

gpr-R_0 space $\Rightarrow$ gR_0 space.

Theorem 3.1. *If X is any topological space then gpr-closures of distinct points are distinct.*

Proof. Let $x, y \in X$, $x \neq y$. To show that *gpr*-*cl*$\{x\} \neq$ *gpr*-*cl*$\{y\}$. Consider the set $A = X - \{x\}$. It is clear that $cl(A)$ is either A or X. If $cl(A) = A$, then A is closed and hence gpr-closed. Therefore, $\{x\} = X - A$ is a gpr-open set which contains x but not y. So, $y \notin$ *gpr*-*cl*$\{x\}$. But $x \in$ *gpr*-*cl*$\{x\}$. Hence, *gpr*-*cl*$\{x\} \neq$ *gpr*-*cl*$\{y\}$. If $cl(A) = X$, then A is preopen and so $\{x\} = X - A$ is preclosed. Then $\{x\}$ is gpr-closed. Therefore *gpr*-*cl*$\{x\} = \{x\}$. Since $y \notin$ *gpr*-*cl*$\{x\}$ and $y \in$ *gpr*-*cl*$\{y\}$, *gpr*-*cl*$\{x\} \neq$ *gpr*-*cl*$\{y\}$. Hence the proof. □

Theorem 3.2. *Let X be a topological space such that each one point set is gpr-closed. Then X is a gpr-T_1 space.*

Proof. Let x and y be any pair of distinct points in X. By assumption $\{x\}$ and $\{y\}$ are gpr-closed sets. Then $U_1 = X - \{x\}$ is a gpr-open set containing y but not x. $U_2 = X - \{y\}$ is a gpr-open set containing x but not y. Hence X is a *gpr*-T_1 space. □

Lemma 3.3. ([7]) *For an $x \in X$, $x \in gpr\text{-}cl(A)$ if and only if $V \cap A \neq \phi$ for every gpr-open set V containing x.*

Theorem 3.4. *If X is a gpr-T_3 space (gpr-regular) then for a given point $x \in X$ and a gpr-open set U of x, there is a gpr-open set V of x such that $gpr\text{-}cl(V) \subset U$.*

Proof. Let X be gpr-regular. Let $x \in X$ and U be a gpr-open set of x. Let $B = X - U$. Therefore, B is gpr-closed. Since X is a gpr-regular space, there exists disjoint gpr-open sets V and W containing x and B respectively. If $y \in B$ then the set W is a gpr-open set containing y and it is disjoint from V. Therefore, by Lemma 3.3, $y \notin gpr\text{-}cl(V)$. That is, $gpr\text{-}cl(V)$ is disjoint from B and so $gpr\text{-}cl(V) \subset U$. Hence the result. □

Theorem 3.5. *If a space X is both gpr-R_1 and gpr-T_0, then it is a gpr-T_2 space.*

Proof. Given X is both *gpr*-R_1 and *gpr*-T_0. Let $x \neq y$. Since X is *gpr*-T_0, there exists a gpr-open set Ucontaining x such that $y \notin U$. This implies $y \notin gpr\text{-}cl\{x\}$. Therefore $gpr\text{-}cl\{x\} \neq gpr\text{-}cl\{y\}$. Since X is *gpr*-R_1, there exists disjoint gpr-open sets U and V such that $x \in U$ and $y \in V$ and $U \cap V = \phi$. Hence X is *gpr*-T_2. □

Theorem 3.6. *If $GPRO(X)$ is open under arbitrary union for a topological space X, then each of the following are equivalent.*

(i) *X is gpr-T_0,*
(ii) *Each one point set is gpr-closed in X,*
(iii) *Each subset of X is the intersection of all gpr-open sets containing it,*
(iv) *The intersection of all gpr-open sets containing the point $x \in X$ is the set $\{x\}$.*

Proof. $(i) \Rightarrow (ii)$: Let X be *gpr*-T_0. Let $x \in X$. Then for any $y \in X$, $y \neq x$ there exists a gpr-open set G_y containing y but not x. Therefore $y \in G_y \subset \{x\}^c$. Now varying y over $\{x\}^c$, we get $\{x\}^c = \cup\{G_y : y \in \{x\}^c\}$. $\{x\}^c$ is a union of gpr-open sets, and so $\{x\}^c$ is gpr-open. That is $\{x\}$ is gpr-closed in X. $(ii) \Rightarrow (iii)$: Let us assume that each one point set is gpr-closed in X. If $A \subset X$, then for each point $y \notin A$, there exists a set $\{y\}^c$ such that $A \subset \{y\}^c$ and each of these sets $\{y\}^c$ is gpr-open. Therefore, $A = \cap\{\{y\}^c : y \in A^c\}$. Thus the intersection of all gpr-open sets containing A is the set A itself.

$(iii) \Rightarrow (iv)$: Obvious.

$(iv) \Rightarrow (i)$: Let us assume that the intersection of all gpr-open sets containing the point $x \in X$ is the set $\{x\}$. Let $x, y \in X$, $x \neq y$. By hypothesis there exists a gpr-open set G_x such that $x \in G_x$ and $y \notin G_x$. That is, X is a *gpr*-T_0 space. □

For subsequent results we assume that the class $GPRC(X, \tau)$ is closed under arbitrary intersection.

Theorem 3.7. *If X is a pre-regular $T_{1/2}$ space and gpr-closures of distinct points are distinct, then X is gpr-T_0.*

Proof. Let $x, y \in X$ and $x \neq y$. Then, $gpr\text{-}cl\{x\} \neq gpr\text{-}cl\{y\}$. Then there exists a point $z \in X$ such that $z \in gpr\text{-}cl\{x\}$ or $gpr\text{-}cl\{y\}$. Let $z \in gpr\text{-}cl\{x\}$ and $z \notin gpr\text{-}cl\{y\}$. Every gpr-open set containing z intersects $\{x\}$. In otherwords,

x belong to every gpr-open set containing z. In particular $x \in X - gpr\text{-}cl\{y\}$, which is gpr-open and does not contain y. So X is a $gpr\text{-}T_0$ space. □

Theorem 3.8. *If a space (X, τ) is pre-regular $T_{1/2}$, then the following are equivalent.*

(i) *X is $gpr\text{-}T_2$,*
(ii) *If $x \in X$, then for each $y \neq x$ there is a gpr-neighbourhood $N(x)$ of x such that $y \notin gpr\text{-}cl(N(x))$.*

Proof. $(i) \Rightarrow (ii)$: Let X be $gpr\text{-}T_2$. Let $x \in X$. Then for each $y \neq x$ there exists disjoint gpr-open sets A and B such that $x \in A$, $y \in B$. Then $x \in A \subset X - B$. That is, $X - B$ is a gpr-neighbourhood of x. Let $N(x) = X - B$. Then $N(x)$ is a gpr-closed set and $y \notin N(x)$. Therefore $y \notin gpr\text{-}cl(N(x))$.

$(ii) \Rightarrow (i)$: Let $x, y \in X$, $x \neq y$. Then by hypothesis, there exists a gpr-neighbourhood $N(x)$ of x such that $y \notin gpr\text{-}cl(N(x))$. Then $x \notin X - gpr\text{-}cl(N(x))$, $y \in X - gpr\text{-}cl(N(x))$. And $X - gpr\text{-}cl(N(x))$ is gpr-open. Also there exists a gpr-open set A such that $x \in A \subset N(x)$ and $A \cap (X - gpr\text{-}cl(N(x))) = \phi$. Therefore, X is a $gpr\text{-}T_2$ space. □

Theorem 3.9. *A space X is a $gpr\text{-}R_0$ space iff for each gpr-closed set F and $x \notin F$ there exists a gpr-open set U such that $F \subset U$, $x \notin U$.*

Proof. Let X be a $gpr\text{-}R_0$ space and $F \subset X$ be a gpr-closed set not containing the point $x \in X$. Then $X - F$ is gpr-open and $x \in X - F$. Since X is a $gpr\text{-}R_0$ space, $gpr\text{-}cl\{x\} \subset X - F$. That is $F \subset X - gpr\text{-}cl\{x\}$. Let $U = X - gpr\text{-}cl\{x\}$. Then U is a gpr-open set such that $F \subset U$ and $x \notin U$. Conversely, let $x \in U$ where U is gpr-open in X. Then $X - U$ is a gpr-closed set and $x \notin X - U$. By hypothesis, there is a gpr-open set W such that $X - U \subset W$ and $x \notin W$. Now, $X - W \subset U$ and $x \in X - W$. $X - W$ is gpr-closed and hence $gpr\text{-}cl\{x\} \subset X - W \subset U$. Therefore, X is a $gpr\text{-}R_0$ space. □

Theorem 3.10. *If a space X is both $gpr\text{-}T_0$ and $gpr\text{-}R_0$, then X is a $gpr\text{-}T_1$ space.*

Proof. By hypothesis, the space X is both $gpr\text{-}T_0$ and $gpr\text{-}R_0$. To show that X is a $gpr\text{-}T_1$ space. Let $x, y \in X$ be any pair of distinct points. Since X is $gpr\text{-}T_0$ space there exists a gpr-open set G such that $x \in G$ and $y \notin G$ or there exists a gpr-open set H such that $y \in H$ and $x \notin H$. Suppose $x \in G$ and $y \notin G$. Then $gpr\text{-}cl\{x\} \subset G$ and $y \notin gpr\text{-}cl\{x\}$. Hence, $y \in H = X - gpr\text{-}cl\{x\}$. It is clear that $x \notin X - gpr\text{-}cl\{x\}$. Therefore there exists gpr-open sets G and H containing x and y respectively such that $y \notin G$ and $x \notin H$. Therefore X is a $gpr\text{-}T_1$ space. □

Theorem 3.11. *A space X is a gR_0 space iff for every closed set F and $x \notin F$, there exists a gpr-open set G such that $F \subset G$ and $x \notin G$.*

Proof. Let X be a gR_0 space and $F \subset X$ be a closed set not containing $x \in X$. Then $X - F$ is open and $x \in X - F$. Since X is gR_0, $gpr - cl\{x\} \subset X - F$. Then $F \subset X - gpr\text{-}cl\{x\}$. Let $G = X - gpr\text{-}cl\{x\}$, then G is gpr-open such that $F \subset G$ and $x \notin G$. Conversely, let $x \in G$, where G is open in X. Then $X - G$ is a closed set and $x \notin X - G$. By hypothesis, there exists a gpr-open set H such that $X - G \subset H$ and $x \notin H$. Now, $X - H \subset G$ and $x \in X - H$. $X - H$ is gpr-closed and so $gpr\text{-}cl\{x\} \subset X - H \subset G$. Therefore, X is a gR_0 space. □

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Gnanambal Ilango
Department of Mathematics, Government Arts College, Coimbatore-641 018, India
E-mail address: gnanamilango@yahoo.co.in

Krishnan Balachandran
Department of Mathematics, Bharathiar University, Coimbatore -641 046, India
E-mail address: kbkb1956@yahoo.com

Rayappan Marudhachalam
Department of Mathematics, Kumaraguru College of Technology,
Coimbatore 641 006, India
E-mail address: marudhu14@gmail.com