Commun. Korean Math. Soc. **26** (2011), No. 3, pp. 483–498
DOI 10.4134/CKMS.2011.26.3.483

APPROXIMATION OF NEAREST COMMON FIXED POINTS OF ASYMPTOTICALLY *I*-NONEXPANSIVE MAPPINGS IN BANACH SPACES

Yeol Je Cho, Nawab Hussain, and Hemant Kumar Pathak

Abstract. In this paper, we introduce a new class of uniformly pointwise R-subweakly commuting self-mappings and prove several common fixed point theorems and best approximation results for uniformly pointwise R-subweakly commuting asymptotically I-nonexpansive mappings in normed linear spaces. We also establish some results concerning strong convergence of nearest common fixed points of asymptotically I-nonexpansive mappings in reflexive Banach spaces with a uniformly Gâteaux differentiable norm. Our results unify and generalize various known results given by some authors to a more general class of noncommuting mappings.

1. Introduction and preliminaries

We first introduce some definitions for our main results in this paper.

Let M be a subset of a normed linear space $(X, \|\cdot\|)$. The set

$$P_M(u) = \{x \in M : \|x - u\| = \operatorname{dist}(u, M)\}$$

is called the set of best approximants to $u \in X$ out of M, where

$$\operatorname{dist}(u, M) = \inf\{\|y - u\| : y \in M\}.$$

We shall use $\mathbb{N}$ to denote the set of positive integers, $cl(S)$ to denote the closure of a set S and $wcl(S)$ to denote the weak closure of a set S. Let $I : M \to M$ be a mapping. A mapping $T : M \to M$ is called an I-contraction if there exists $0 \le k < 1$ such that

$$\|Tx - Ty\| \le k \, \|Ix - Iy\|, \quad \forall x, y \in M.$$

Received April 10, 2010.
2010 *Mathematics Subject Classification.* 47H10, 54H25.
Key words and phrases. uniformly pointwise R-subweakly commuting mappings, uniformly R-subweakly commuting mappings, asymptotically I-nonexpansive mappings, Banach operator pair, strong convergence, Gâteaux differentiable norm, uniform normal structure.
This work was supported by the Korea Research Foundation Grant funded by the Korean Government (KRF-2008-313-C00050).

If $k = 1$, then T is said to be I-nonexpansive. The mapping T is said to be asymptotically I-nonexpansive if there exists a sequence $\{k_n\}$ of real numbers with $k_n \geq 1$ and $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|Ix - Iy\|, \quad \forall x, y \in M,\ n \geq 1.$$

The mapping T is said to be uniformly asymptotically regular ([4, 8]) on M if, for each $\eta > 0$, there exists $N(\eta) = N$ such that

$$\|T^n x - T^{n+1} x\| < \eta, \quad \forall n \geq N,\ x \in M.$$

The set of fixed points of T (resp., I) is denoted by $F(T)$ (resp., $F(I)$). A point $x \in M$ is a coincidence point (resp., common fixed point) of two mappings I and T if $Ix = Tx$ (resp., $x = Ix = Tx$). The set of coincidence points of I and T is denoted by $C(I,T)$. Let I and T be self-mappings of a metric space (X,d). The mappings I and T are commuting if

$$ITx = TIx, \quad \forall x \in X.$$

Sessa [35] defined the concept of weakly commuting mappings, i.e., the self-mappings I and T on X are said to be weakly commuting if

$$d(ITx, TIx) \leq d(Tx, Ix), \quad \forall x \in X,$$

and, as a generalization of commuting mappings, Jungck [18] defined I and T to be compatible on X if

$$\lim_{n\to\infty} d(ITx_n, TIx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

It is easy to show that commuting mappings are weakly commuting and weakly commuting mappings are compatible. We can find some examples to show these implications (see [18, 19, 35]).

Definition 1.1 ([19]). *I and T are said to be weakly compatible on X if they commute at their coincidence points, i.e., if $Iu = Tu$ for some $u \in X$, then $ITu = TIu$.*

Definition 1.2 ([25]). *I and T are said to be R-weakly commuting on X if there exists $R > 0$ such that*

$$d(ITx, TIx) \leq R\, d(Tx, Ix), \quad \forall x \in X.$$

Definition 1.3 ([26]). *I and T are pointwise R-weakly commuting on X if for given $x \in X$, there exists $R > 0$ such that*

$$d(ITx, TIx) \leq R\, d(Tx, Ix).$$

It was proved in [26] that pointwise R-weak commutativity is equivalent to commutativity at coincidence points, i.e., I and T are pointwise R-weakly commuting if and only if they are weakly compatible.

The ordered pair (T, I) of two self-mappings on a metric space (X, d) is called a Banach operator pair if the set $F(I)$ is T-invariant, namely, $T(F(I)) \subseteq F(I)$. Obviously, commuting pair (T, I) is a Banach operator pair, but not conversely in general. See, for details, [1, 6, 13, 15, 28].

Ćirić [9] introduced and studied self-mappings on metric space X satisfying

$$d(Tx, Ty) \leq \lambda \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$$

for all $x, y \in X$, where $0 < \lambda < 1$. Further, this investigation was developed by Hussain and Jungck [16], Jungck and Hussain [20], O'Regan and Hussain [24] and many other mathematicians.

A set M is said to be q-starshaped with $q \in M$ if the segment

$$[q, x] = \{(1-k)q + kx : 0 \leq k \leq 1\}$$

joining q to x is contained in M for all $x \in M$. Suppose that M is q-starshaped with $q \in F(I)$ and is both T and I-invariant. Then the mappings T and I are said to be:

(1) C_q-commuting ([3, 17]) if $ITx = TIx$ for all $x \in C_q(I, T)$, where $C_q(I, T) = \cup\{C(I, T_k) : 0 \leq k \leq 1\}$ and $T_k x = (1-k)q + kTx$,

(2) pointwise R-subweakly commuting ([24]) if for given $x \in M$, there exists a real number $R > 0$ such that

$$\|ITx - TIx\| \leq R \cdot \operatorname{dist}(Ix, [q, Tx]),$$

(3) R-subweakly commuting on M (see [16, 17]) if there exists a real number $R > 0$ such that

$$\|ITx - TIx\| \leq R \cdot \operatorname{dist}(Ix, [q, Tx]), \quad \forall x \in M,$$

(4) uniformly R-subweakly commuting on $M \setminus \{q\}$ (see [4]) if there exists a real number $R > 0$ such that

$$\|IT^n x - T^n Ix\| \leq R \cdot \operatorname{dist}(Ix, [q, T^n x]), \quad \forall x \in M \setminus \{q\},\ n \in \mathbb{N}.$$

Note that C_q-commuting mappings are pointwise R-subweakly commuting and pointwise R-subweakly commuting mappings are weakly compatible, but not conversely in general and R-subweakly commuting mappings are C_q-commuting, but the converse does not hold in general (see, for examples, [3, 20]).

The class of asymptotically nonexpansive mappings was introduced by Goeble and Kirk [11] and, further, studied by various authors (see [7, 23, 33, 34, 37, 38]). Recently, Beg et al. [4] have proved strong convergence of the sequence of almost fixed points $x_n = (1-\mu_n)q + \mu_n T^n x_n$ to the common fixed point of asymptotically I-nonexpansive mapping T using the uniform R-subweak commutativity of $\{I, T\}$.

In this paper, we introduce a more general class of uniformly pointwise R-subweakly commuting self-mappings which properly contains the class of uniformly R-subweakly commuting mappings. For this new class, we establish some common fixed point theorems and approximation results. We also study the strong convergence of nearest common fixed points of asymptotically I-nonexpansive mappings with and without the uniform pointwise R-subweak commutativity of the mappings I and T in a reflexive Banach space with a uniformly Gâteaux differentiable norm. Our results extend and improve the recent results given in [2, 3, 4, 7, 8, 14, 23, 28, 29, 33, 34, 37] to uniformly pointwise R-subweakly commuting asymptotically I-nonexpansive mappings.

2. Common fixed points and approximation results

O'Regan and Hussain [24] coined the idea of more general mappings called pointwise R-subweakly commuting mappings. We begin with the definition of uniformly pointwise R-subweakly commuting mappings.

Definition 2.1. Let M be a q-starshaped subset of a normed linear space X. Let $I, T : M \to M$ be mappings with $q \in F(I)$. Then I and T are said to be uniformly pointwise R-subweakly commuting if for given $x \in M \setminus \{q\}$, there exists a real number $R > 0$ such that

$$\|IT^n x - T^n Ix\| \leq R \cdot \mathrm{dist}(Ix, [q, T^n x]), \quad \forall\, n \in \mathbb{N}.$$

It is clear from Definition 2.1 that uniformly pointwise R-subweakly commuting mappings on M are pointwise R-subweakly commuting, but not conversely in general as the following example shows:

Example 2.2. Let $X = R$ with usual norm and $M = [1, \infty)$. Let $Tx = 2x - 1$, $Ix = x^2$ for all $x \in M$ and let $q = 1$. Then M is q-starshaped with $Iq = q$, $C_q(I, T) = \{1\}$ and $C_q(I, T^2) = [1, 3]$. Note that I and T are pointwise R-subweakly commuting mappings, but not uniformly pointwise R-subweakly commuting because $\|IT^2 x - T^2 Ix\| \neq 0$ for all $x \in (1, 3]$ whereas $(1, 3]$ being subset of $C_q(I, T^2)$ implies that $\mathrm{dist}(Ix, [q, T^2 x]) = 0$ for all $x \in (1, 3]$.

Uniformly R-subweakly commuting mappings are uniformly pointwise R-subweakly commuting, but the converse does not hold in general. To see this, we consider the following example:

Example 2.3. Let $X = R$ with usual norm and $M = [0, \infty)$. Let $Ix = \frac{x}{2}$ if $0 \leq x < 1$, $Ix = x$ if $x \geq 1$ and $Tx = \frac{1}{2}$ if $0 \leq x < 1$, $Tx = x^2$ if $x \geq 1$. Then M is 1-starshaped with $I1 = 1$ and $C_q(I, T) = [1, \infty]$ and $C_q(I, T^n) \subseteq [1, \infty]$ for each $n > 1$. Clearly, I and T are uniformly pointwise R-subweakly commuting, but not R-weakly commuting for all $R > 0$ (see [3]). Thus I and T are neither R-subweakly commuting nor uniformly R-subweakly commuting mappings.

The following result improves and extends Lemma 3.3 in [4]:

Lemma 2.4. *Let I and T be self-mappings on a nonempty q-starshaped subset M of a normed linear space X. Assume that $q \in F(I)$, I is affine, T and I are* *uniformly pointwise R-subweakly commuting mappings satisfying the following condition*:

$$\begin{aligned} &\|T^n x - T^n y\| \\ (2.1) \quad &\le k_n \cdot \max\{\|Ix - Iy\|, \mathrm{dist}(Ix, [T^n x, q]), \mathrm{dist}(Iy, [T^n y, q]), \\ &\qquad \mathrm{dist}(Ix, [T^n y, q]), \mathrm{dist}(Iy, [T^n x, q]\}, \quad \forall x, y \in M,\ n \ge 1, \end{aligned}$$

where $\{k_n\}$ is a sequence of real numbers with $k_n \ge 1$ and $\lim_{n\to\infty} k_n = 1$. For each $n \ge 1$, define a mapping S_n on M by

$$S_n x = (1 - \mu_n) q + \mu_n T^n x, \quad \forall x \in M,$$

where $\mu_n = \frac{\lambda_n}{k_n}$ and $\{\lambda_n\}$ is a sequence of numbers in $(0, 1)$ such that $\lim_{n\to\infty} \lambda_n = 1$. Then, for each $n \ge 1$, S_n and I have exactly one common fixed point $x_n \in M$ such that

$$I x_n = x_n = (1 - \mu_n) q + \mu_n T^n x_n,$$

provided one of the following conditions hold:

(i) *$clT(M) \subset I(M)$ and, for each $n \ge 1$, $clS_n(M)$ is complete,*
(ii) *$wclT(M) \subset I(M)$ and, for each $n \ge 1$, $wclS_n(M)$ is complete.*

Proof. Since I and T are uniformly pointwise R-subweakly commuting and I is affine with $Iq = q$, it follows that, for given $x \in M$,

$$\begin{aligned} \|IS_n x - S_n I x\| &= \|(1 - \mu_n) q + \mu_n I T^n x - (1 - \mu_n) q - \mu_n T^n I x\| \\ &= \mu_n \|I T^n x - T^n I x\| \\ &\le \mu_n R \cdot \mathrm{dist}(Ix, [q, T^n x]) \\ &\le \mu_n R\ \|Ix - S_n x\|. \end{aligned}$$

Hence I and S_n are pointwise $\mu_n R$-weakly commuting for all $n \ge 1$. Also, by (2.1), we have

$$\begin{aligned} &\|S_n x - S_n y\| \\ = \ &\mu_n \|T^n x - T^n y\| \\ \le \ &\lambda_n \cdot \max\{\|Ix - Iy\|, \mathrm{dist}(Ix, [T^n x, q]), \mathrm{dist}(Iy, [T^n y, q]), \\ &\qquad \mathrm{dist}(Ix, [T^n y, q]), \mathrm{dist}(Iy, [T^n x, q])\} \\ \le \ &\lambda_n \max\{\|Ix - Iy\|, \|Ix - S_n x\|, \|Iy - S_n y\|, \\ &\qquad \|Ix - S_n y\|, \|Iy - S_n x\|\}, \quad \forall x, y \in M,\ n \ge 1. \end{aligned}$$

(i) Since M is q-starshaped, $clT(M) \subset I(M)$, I is affine and $Iq = q$ and so, for each $n \ge 1$, $clS_n(M) \subset I(M)$. By Theorem 2.1 in [17, 20], for each $n \ge 1$, there exists $x_n \in M$ such that $x_n = Ix_n = S_n x_n$. Thus, for each $n \ge 1$, $M \cap F(S_n) \cap F(I) \ne \emptyset$.

(ii) Since M is q-starshaped, $wclT(M) \subset I(M)$, I is affine and $Iq = q$ and so $wclS_n(M) \subset I(M)$ for each $n \ge 1$. By Theorem 2.1 in [17, 20], the conclusion follows. This completes the proof. □

The following result extends the recent results (Theorems 2.2-2.4) due to Al-Thagafi and Shahzad [3] and the corresponding results in Hussain and Rhoades [17] and O'Regan and Hussain [24] to asymptotically I-nonexpansive mappings.

Theorem 2.5. *Let I and T be self-mappings on a q-starshaped subset M of a normed linear space X. Assume that $q \in F(I)$, I is affine and T is uniformly asymptotically regular and asymptotically I-nonexpansive. If T and I are uniformly pointwise R-subweakly commuting on M, then $M \cap F(T) \cap F(I) \neq \emptyset$ provided one of the following conditions holds*:

(i) *$clT(M) \subset I(M)$, T is continuous and $clT(M)$ is compact,*

(ii) *X is complete, $wclT(M) \subset I(M)$, $wclT(M)$ is weakly compact, I is weakly continuous and $I - T$ is demiclosed at 0.*

Proof. (i) Notice that compactness of $clT(M)$ implies that $clS_n(M)$ is compact and hence complete. From Lemma 2.4, for each $n \geq 1$, there exists $x_n \in M$ such that $x_n = Ix_n = (1-\mu_n)q + \mu_n T^n x_n$ and so hence $x_n \in C_q(I, T^n)$. Since $T(M)$ is bounded, it follows that $\|x_n - T^n x_n\| = (1-\mu_n)\|T^n x_n - q\| \to 0$ as $n \to \infty$.

Now, we have

$$\begin{aligned} &\|x_n - Tx_n\| \\ &= \|x_n - T^n x_n\| + \|T^n x_n - T^{n+1}x_n\| + \|T^{n+1}x_n - Tx_n\| \\ &\leq \|x_n - T^n x_n\| + \|T^n x_n - T^{n+1}x_n\| + k_1\|IT^n x_n - Ix_n\|. \end{aligned} \tag{2.2}$$

Since T and I are uniformly pointwise R-subweakly commuting, I commutes with T^n on $C_q(I, T^n)$. Also, since $x_n \in C_q(I, T^n)$, $x_n = Ix_n$ and T is uniformly asymptotically regular, we have, from (2.2),

$$\|x_n - Tx_n\| \leq \|x_n - T^n x_n\| + \|T^n x_n - T^{n+1}x_n\| + k_1\|T^n x_n - x_n\| \to 0$$

as $n \to \infty$. Thus $x_n - Tx_n \to 0$ as $n \to \infty$. Since $clT(M)$ is compact, there exists a subsequence $\{Tx_m\}$ of $\{Tx_n\}$ such that $Tx_m \to x_0$ as $m \to \infty$. By the continuity of T, we have $Tx_0 = x_0$. Since $T(M) \subset I(M)$, it follows that $x_0 = Tx_0 = Iy$ for some $y \in M$. Moreover, we have

$$\|Tx_m - Ty\| \leq k_1\|Ix_m - Iy\| = k_1\|x_m - x_0\|.$$

Taking the limit as $m \to \infty$, we get $Tx_0 = Ty$. Thus $Tx_0 = Iy = Ty = x_0$. Since I and T are uniformly pointwise R-subweakly commuting on M and $y \in C(I, T)$,

$$\| Tx_0 - Ix_0 \| = \| TIy - ITy \| = 0.$$

Therefore, we have $x_0 \in F(T) \cap F(I)$ and so $M \cap F(T) \cap F(I) \neq \emptyset$.

(ii) The weak compactness of $wclT(M)$ implies that $wclS_n(M)$ is weakly compact and hence complete due to completeness of X (see [3, 20]). From Lemma 2.4, for each $n \geq 1$, there exists $x_n \in M$ such that $x_n = Ix_n = (1-\mu_n)q + \mu_n T^n x$. As in (i), it follow that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$. The weak compactness of $wclT(M)$ implies that there is a subsequence $\{x_m\}$ of $\{x_n\}$ converging weakly to $y \in M$ as $m \to \infty$. By the weak continuity of I, we have $Iy = y$. Also, we have $Ix_m - Tx_m = x_m - Tx_m \to 0$ as $m \to \infty$. If $I - T$ is demiclosed at 0, then $Iy = Ty$. Thus $M \cap F(T) \cap F(I) \neq \emptyset$. This completes the proof. $\square$

Corollary 2.6 ([4]). *Let I and T be self-mappings on a q-starshaped subset M of a normed linear space X. Assume that $clT(M) \subset I(M)$, $q \in F(I)$, I is affine, T is continuous, uniformly asymptotically regular and asymptotically I-nonexpansive. If $clT(M)$ is compact, T and I are uniformly R-subweakly commuting on M, then $M \cap F(T) \cap F(I) \neq \emptyset$.*

Remark 1. Notice that the conditions of the continuity and linearity of I are not needed in Theorem 3.4 of Beg et al. [4].

Corollary 2.7 ([3]). *Let I and T be self-mappings on a q-starshaped subset M of a normed linear space X. Assume that T and I are C_q-commuting on M, $q \in F(I)$, I is affine and T is I-nonexpansive. Then $M \cap F(T) \cap F(I) \neq \emptyset$ provided one of the following conditions holds*:

(i) *$clT(M) \subset I(M)$, T is continuous and $cl(T(M))$ is compact,*
(ii) *X is complete, I is weakly continuous, $wcl(T(M))$ is weakly compact, $wclT(M) \subset I(M)$ and either $I - T$ is demiclosed at 0 or X satisfies Opial's condition.*

Corollary 2.8 ([2, 21]). *Let I and T be self-mappings on a q-starshaped subset M of a normed linear space X. Assume that T and I are commuting on M, $q \in F(I)$, I is affine and T is I-nonexpansive. Then $F(T) \cap F(I) \neq \emptyset$ provided one of the conditions in corollary 2.7 holds.*

The following result extends Theorem 3.2 in Al-Thagafi [2], Theorems 3.1-3.3 in [3], Theorem 7 of Jungck and Sessa [21], the main results in Pathak et al. [27] and Singh [36] to the asymptotically I-nonexpansive mapping T.

Theorem 2.9. *Let M be subset of a normed linear space X and $I, T : X \to X$ be mappings such that $u \in F(I) \cap F(T)$ for some $u \in X$ and $T(\partial M \cap M) \subseteq M$. Suppose that $P_M(u)$ is q-starshaped, $q \in F(I)$, I is affine on $P_M(u)$,*

$$\|Tx - Tu\| \leq \|Ix - Iu\|, \quad \forall x \in P_M(u)$$

and $I(P_M(u)) = P_M(u)$. If T and I are uniformly pointwise R-subweakly commuting on $P_M(u)$ and T is uniformly asymptotically regular and asymptotically I-nonexpansive, then $P_M(u) \cap F(I) \cap F(T) \neq \emptyset$ provided one of the following conditions is satisfied:

(i) *$P_M(u)$ is closed, T is continuous on $P_M(u)$ and $cl(T(P_M(u)))$ is compact,*
(ii) *X is complete, $P_M(u)$ is weakly closed, $wcl(T(P_M(u)))$ is weakly compact, I is weakly continuous and $I - T$ is demiclosed at 0.*

Proof. Let $x \in P_M(u)$. Then $x \in \partial M \cap M$. Since $T(\partial M \cap M) \subseteq M$, Tx must be in M. Also, since $Ix \in P_M(u)$, $u \in F(I) \cap F(T)$ and I, T satisfy the condition

$\|Tx - Tu\| \le \|Ix - Iu\|$, we have

$$\|Tx - u\| = \|Tx - Tu\| \le \|Ix - Iu\| = \|Ix - u\| = \operatorname{dist}(u, M).$$

Thus $Tx \in P_M(u)$ and so $cl(T(P_M(u))) \subseteq I(P_M(u)) = P_M(u)$ if $P_M(u)$ is closed and $wcl(T(P_M(u))) \subseteq I(P_M(u)) = P_M(u)$ whenever $P_M(u)$ is weakly closed. Therefore, the result now follows from Theorem 2.5. This completes the proof. $\square$

We denote by $\Im_0$ the class of closed convex subsets of X containing 0. For any $M \in \Im_0$, we define

$$M_u = \{x \in M : \|x\| \le 2\|u\|\}, \quad C_M^I(u) = \{x \in M : Ix \in P_M(u)\}.$$

It is clear that $P_M(u) \subset M_u \in \Im_0$ (see [2, 16, 17]).

The following result extends Theorems 4.1 and 4.2 in [2, 3] and the corresponding results in [16, 17].

Theorem 2.10. *Let I, T be self-mappings of a normed linear space X with $u \in F(I) \cap F(T)$ and $M \in \Im_0$ such that $T(M_u\ ) \subseteq I(M) \subseteq M$. Suppose that $\|Ix - u\| = \|x - u\|$ for all $x \in M_u$, T is continuous on M_u and T satisfies the condition $\|Tx - u\| \le \|Ix - u\|$ for all $x \in M_u$. If one of the following two conditions is satisfied*:

(a) *$cl(I(M_u))$ is compact,*

(b) *$cl(T(M_u))$ is compact.*

Then we have the following:

(1) *$P_M(u)$ is nonempty, closed and convex,*

(2) *$T(P_M(u)) \subseteq I(P_M(u)) \subseteq P_M(u)$ provided that $\|Ix - u\| = \|x - u\|$ for all $x \in C_M^I(u)$ and $I(P_M(u))$ is closed,*

(3) *$P_M(u) \cap F(I) \cap F(T) \neq \emptyset$ provided that $\|Ix - u\| = \|x - u\|$ for all $x \in C_M^I(u)$, $I(P_M(u))$ is closed, I is affine with $q \in F(I)$, I and T are uniformly pointwise R-subweakly commuting and T is uniformly asymptotically regular and asymptotically I-nonexpansive on $P_M(u)$.*

Proof. (1) (a) We will follow the arguments used in [20, 24]. We may assume that $u \notin M$. If $x \in M \setminus M_u$, then $\|x\| > 2\|u\|$. Note that

$$\|x - u\| \ge \|x\| - \|u\| > \|u\| \ge \operatorname{dist}(u, M).$$

Thus $\operatorname{dist}(u, M_u) = \operatorname{dist}(u,M) \le \|u\|$. Also, $\|z - u\| = \operatorname{dist}(u, clI(M_u))$ for some $z \in clI(M_u)$. This implies that

$$\operatorname{dist}(u, M_u) \le \operatorname{dist}(u, clI(M_u)) \le \operatorname{dist}(u, I(M_u)) \le \|Ix - u\| \le \|x - u\|$$

for all $x \in M_u$. Hence $\|z - u\| = \operatorname{dist}(u, M)$ and so $P_M(u)$ is nonempty. Moreover, it is closed and convex.

(b) The proof is exactly as in Theorem 2.6(i)(b) (see [24]).

(2) Let $z \in P_M(u)$. Then $\|Iz - u\| = \|Iz - Iu\| \le \|z - u\| = \operatorname{dist}(u, M)$. This implies that $Iz \in P_M(u)$ and so $I(P_M(u)) \subseteq P_M(u)$. Let $y \in T(P_M(u))$.

Since $T(M_u) \subseteq I(M)$ and $P_M(u) \subseteq M_u$, there exist $z \in P_M(u)$ and $x_0 \in M$ such that $y = Tz = Ix_0$. Further, we have

$$\|Ix_0 - u\| = \|Tz - Tu\| \leq \|Iz - Iu\| = \|Iz - u\| \leq \|z - u\| = \operatorname{dist}(u, M).$$

Thus $x_0 \in C_M^I(u) = P_M(u)$ and so (2) holds.

(3) By (2), the compactness of $cl(I(M_u))$ (resp., $cl(T(M_u))$) implies that $clT(P_M(u))$ is compact. The conclusion now follows if we apply Theorem 2.5(i) to $P_M(u)$. □

Remark 2. As an application of Theorem 2.5(2), we can prove similarly Theorems 4.3-4.4 in [3] for the uniformly pointwise R-subweakly commuting and asymptotically I-nonexpansive mapping T.

Another improvement of Lemma 3.3 in [4] is given as follows:

Proposition 2.11. *Let I and T be self-mappings on a nonempty subset M of a normed linear space X. Assume that $F(I)$ is q-starshaped, T and I satisfy* (1) *for each $n \geq 1$ and $x, y \in M$, where $\{k_n\}$ is a sequence of real numbers with $k_n \geq 1$ and $\lim_{n\to\infty} k_n = 1$. For each $n \geq 1$, define a mapping S_n on $F(I)$ by*

$$S_n x = (1 - \mu_n)q + \mu_n T^n x, \quad \forall x \in M,$$

where $\{\mu_n\}$ and $\{\lambda_n\}$ are sequences as in Lemma 2.4. Then, for each $n \geq 1$, S_n and I have exactly one common fixed point x_n in M such that $Ix_n = x_n = (1 - \mu_n)q + \mu_n T^n x_n$ provided one of the following conditions hold:

(i) *$clT(F(I)) \subseteq F(I)$ and, for each $n \geq 1$, $clS_n(M)$ is complete,*

(ii) *$wclT(F(I)) \subseteq F(I)$ and, for each $n \geq 1$, $wclS_n(M)$ is complete.*

Proof. $T(F(I)) \subseteq F(I)$ implies that $T^n(F(I)) \subseteq F(I)$ for each $n \geq 1$ and $F(I)$ is q-starshaped. Thus each S_n is a self-mapping on $F(I)$. Also, by (1),

$$\begin{aligned}
\|S_n x - S_n y\| &= \mu_n \|T^n x - T^n y\| \\
&\leq \lambda_n \max\{\|Ix - Iy\|, \operatorname{dist}(Ix, [T^n x, q]), \operatorname{dist}(Iy, [T^n y, q]), \\
&\qquad \operatorname{dist}(Ix, [T^n y, q]), \operatorname{dist}(Iy, [T^n x, q])\} \\
&\leq \lambda_n \max\{\|x - y\|, \|x - S_n x\|, \|y - S_n y\|, \\
&\qquad \|x - S_n y\|, \|y - S_n x\|\}, \quad \forall x, y \in F(I).
\end{aligned}$$

(i) Since $F(I)$ is q-starshaped and $clT(F(I)) \subset F(I)$, for each $n \geq 1$, $clS_n(F(I)) \subset F(I)$. The completeness of $clS_n(M)$ implies that $clS_n(F(I))$ is complete. By Theorem 2.1 in [17, 20], for each $n \geq 1$, there exists $x_n \in F(I)$ such that $x_n = S_n x_n$. Thus, for each $n \geq 1$, $M \cap F(S_n) \cap F(I) \neq \emptyset$.

(ii) As above, for each $n \geq 1$, $wclS_n(F(I)) \subset F(I)$ and $wclS_n(F(I))$ is complete. By Theorem 2.1 in [17, 20], the conclusion follows. This completes the proof. □

Corollary 2.12. *Let I and T be self-mappings on a nonempty subset M of a normed linear space X. Assume that $F(I)$ is q-starshaped, T and I satisfy* (2.1) *for each $n \geq 1$ and $x, y \in M$, where $\{k_n\}$ is a sequence of real numbers* *with* $k_n \geq 1$ *and* $\lim_{n\to\infty} k_n = 1$. *For each* $n \geq 1$, *define a mapping* S_n *on* $F(I)$ *by*

$$S_n x = (1 - \mu_n)q + \mu_n T^n x, \quad \forall x \in M,$$

where $\{\mu_n\}$ *and* $\{\lambda_n\}$ *are sequences as in Lemma 2.4. Then, for each* $n \geq 1$, S_n *and* I *have exactly one common fixed point* x_n *in* M *such that* $Ix_n = x_n = (1 - \mu_n)q + \mu_n T^n x_n$ *provided one of the following conditions hold*:

(i) $F(I)$ *is closed and* (T, I) *is a Banach operator pair and, for each* $n \geq 1$, $cl(S_n(M))$ *is complete,*

(ii) $F(I)$ *is weakly closed and* (T, I) *is a Banach operator pair and, for each* $n \geq 1$, $wcl(S_n(M))$ *is complete.*

Remark 3. By comparing Lemma 3.3 of Beg et al. [4] with the first case of Lemma 2.11 their assumptions "M is closed, $IM = M$, $T(M \setminus \{q\}) \subset I(M) \setminus \{q\}$, T is continuous, I is linear $q \in F(I)$, M is q-starshaped and T and I are uniformly R-subweakly commuting on M" are replaced with "$F(I)$ is q-starshaped, $clT(F(I)) \subseteq F(I)$ and, for each $n \geq 1$, $clT_n(M)$ is complete."

3. Convergence theorems

Definition 3.1. Let M be a nonempty closed subset of a Banach space X, $I, T : M \to M$ be mappings and $C = \{x \in M : f(x) = \min_{z\in M} f(z)\}$. Then I and T are said to satisfy property (S) ([4, 8]) if, for any bounded sequence $\{x_n\}$ in M, $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ implies $C \cap F(I) \cap F(T) \neq \emptyset$.

The normal structure coefficient $N(X)$ ([4, 7]) of a Banach space X is defined by

$$N(X) = \inf\{\tfrac{\operatorname{diam}(M)}{r_C(M)} : M \text{ is nonempty bounded and convex subset of } X \text{ with } \operatorname{diam}(M) > 0\},$$

where $r_C(M) = \inf_{x\in M} \sup_{y\in M} \|x - y\|$ is the Chebyshev radius of M relative to itself and $\operatorname{diam}(M) = \sup_{x,y\in M} \|x - y\|$ is diameter of M. The space X is said to have the uniform normal structure if $N(X) > 1$. A Banach limit LIM is a bounded linear functional on l^∞ such that

$$\liminf_{n\to\infty} t_n \leq LIM t_n \leq \limsup_{n\to\infty} t_n, \quad LIM t_n = LIM t_{n+1}$$

for all bounded sequences $\{t_n\}$ in l^∞. Let $\{x_n\}$ be bounded sequence in X. Then we can define the real-valued continuous convex function f on X by $f(z) = LIM\|x_n - z\|^2$ for all $z \in X$.

The following lemmas are well known.

Lemma 3.2 ([4, 7])**.** *Let* X *be a Banach space with uniformly Gâteaux differentiable norm and* $u \in X$. *Let* $\{x_n\}$ *be bounded sequence in* X. *Then* $f(u) = \inf_{z\in X} f(z)$ *if and only if* $LIM\langle z, J(x_n - u)\rangle = 0$ *for all* $z \in X$, *where* $J : X \to X^*$ *is the normalized duality mapping and* $\langle \cdot, \cdot \rangle$ *denotes the generalized duality pairing.*

Lemma 3.3 ([4, 7])**.** *Let M be a convex subset of a smooth Banach space X, D be a nonempty subset of M and P be a retraction from M onto D. Then P is sunny and nonexpansive if and only if*

$$\langle x - Px, J(z - Px)\rangle \le 0, \quad \forall x \in M, z \in D.$$

In 1967, Browder [5] and Halpern [12] proved strong convergence theorems in the framework of Hilbert spaces with implicit and explicit iteration, respectively. These results have been extended in various directions.

Theorem 3.4 ([30])**.** *Let M be a bounded closed convex subset of a uniformly smooth Banach space X. Let T be a nonexpansive self-mapping on M. Fix $u \in M$ and define a net $\{y_\alpha\}$ in M by*

$$y_\alpha = (1 - \alpha)Ty_\alpha + \alpha u, \quad \forall \alpha \in (0, 1).$$

Then $\{y_\alpha\}$ converges strongly to $Pu \in F(T)$ as $\alpha \to +0$, where P is the unique sunny nonexpansive retraction from M onto $F(T)$.

Further generalizations of the above mentioned results were studied by some authors (see [7, 10, 22, 31, 38] and references cited therein).

Now, we are ready to prove strong convergence to nearest common fixed points of asymptotically I-nonexpansive mappings which are uniformly pointwise R-subweakly commuting.

Theorem 3.5. *Let M be a closed convex subset of a reflexive Banach space X with a uniformly Gâteaux differentiable norm. Let I and T be self-mappings on M such that $clT(M) \subset I(M)$, $q \in F(I)$ and I is affine. Suppose that T is continuous, uniformly asymptotically regular and asymptotically I-nonexpansive with a sequence $\{k_n\}$ satisfying $k_n \ge 1$ and $\lim_{n\to\infty} k_n = 1$. Let $\{\lambda_n\}$ be sequence of real numbers in $(0, 1)$ such that $\lim_{n\to\infty} \lambda_n = 1$ and $\lim_{n\to\infty} \frac{k_n-1}{k_n-\lambda_n} = 0$. If I and T are uniformly pointwise R-subweakly commuting on M, then we have the following*:

(1) *for each $n \ge 1$, there is exactly one x_n in M such that*

$$Ix_n = x_n = (1 - \mu_n)q + \mu_n T^n x_n, \tag{3.1}$$

(2) *if $\{x_n\}$ is bounded and I, T satisfy the property (S), then $\{x_n\}$ converges strongly to $Pq \in F(T) \cap F(I)$, where P is the sunny nonexpansive retraction from M onto $F(T)$.*

Proof. (1) follows from Lemma 2.4.

(2) Since $\{x_n\}$ is bounded, we can define a function $f : M \to R^+$ by $f(z) = LIM\|x_n - z\|^2$ for all $z \in M$. Since f is continuous and convex, $f(z) \to \infty$ as $\|z\| \to \infty$ and X is reflexive, $f(z_0) = \min_{z\in M} f(z)$ for some $z_0 \in M$. Clearly, the set $C = \{x \in M : f(x) = \min_{z\in M} f(z)\}$ is nonempty. Since $\{x_n\}$ is bounded and I, T satisfy property (S), it follows that $C \cap F(I) \cap F(T) \ne \emptyset$.

Suppose that $v \in C \cap F(I) \cap F(T)$, then, by Lemma 3.2, we have $LIM\langle x - v, J(x_n - v)\rangle \le 0$ for all $x \in M$. In particular, we have

$$LIM\langle q - v, J(x_n - v)\rangle \le 0. \tag{3.2}$$

From (3.1), we have

$$x_n - T^n x_n = (1-\mu_n)(q - T^n x_n) = \frac{1-\mu_n}{\mu_n}(q - x_n). \tag{3.3}$$

Now, for any $v \in C \cap F(I) \cap F(T)$, we have

$$\begin{aligned} \langle x_n - T^n x_n, J(x_n - v)\rangle &= \langle x_n - v + T^n v - T^n x_n, J(x_n - v)\rangle \\ &\ge -(k_n - 1)\|x_n - v\|^2 \\ &\ge -(k_n - 1)K^2 \end{aligned}$$

for some $K > 0$. It follows from (3.3) that

$$\langle x_n - q, J(x_n - v)\rangle \le \frac{k_n - 1}{k_n - \lambda_n} K^2$$

and hence

$$LIM\langle x_n - q, J(x_n - v)\rangle \le 0. \tag{3.4}$$

This together with (3.2) implies that

$$LIM\langle x_n - v, J(x_n - v)\rangle = LIM\|x_n - v\|^2 = 0.$$

Thus there is a subsequence $\{x_m\}$ of $\{x_n\}$ which converges strongly to v. Suppose that there is another subsequence $\{x_j\}$ of $\{x_n\}$ which converges strongly to y (say). Since T is continuous and $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$, y is a fixed point of T. It follows from (3.4) that

$$\langle v - q, J(v - y)\rangle \le 0, \quad \langle y - q, J(y - v)\rangle \le 0.$$

Adding these two inequalities, we get $\langle v - y, J(v - y)\rangle = \|v - y\|^2 \le 0$ and thus $v = y$. Consequently, $\{x_n\}$ converges strongly to $v \in F(I) \cap F(T)$. We can define now a mapping P from M onto $F(T)$ by $\lim_{n\to\infty} x_n = Pq$. From (3.4), we have

$$\langle q - Pq, J(v - Pq)\rangle \le 0, \quad \forall q \in M,\ v \in F(T).$$

Thus, by Lemma 3.3, P is the sunny nonexpansive retraction on M. Notice that $x_n = Ix_n$ and $\lim_{n\to\infty} x_n = Pq$ and so, by the same argument as in the proof of Theorem 2.5(i), we obtain $Pq \in F(I)$. This completes the proof. □

Corollary 3.6 ([4]). *Let M be a closed convex subset of a reflexive Banach space X with a uniformly Gâteaux differentiable norm. Let I and T be continuous self-mappings on M such that $I(M) = M$, $clT(M) \subset I(M)$, $q \in F(I)$ and I is affine. Suppose that T is uniformly asymptotically regular and asymptotically I-nonexpansive with a sequence $\{k_n\}$ satisfying $k_n \ge 1$ and $\lim_{n\to\infty} k_n = 1$. Let $\{\lambda\}$ be sequence of real numbers in $(0,1)$ such that $\lim_{n\to\infty}\lambda_n = 1$ and $\lim_{n\to\infty}\frac{k_n-1}{k_n-\lambda_n} = 0$. If I and T are uniformly R-subweakly commuting on M, then we have the following*:

(1) *for each* $n \geq 1$, *there is exactly one* x_n *in* M *such that*

$$Ix_n = x_n = (1 - \mu_n)q + \mu_n T^n x_n,$$

(2) *if* $\{x_n\}$ *is bounded and* I, T *satisfy the property* (S), *then* $\{x_n\}$ *converges strongly to* $Pq \in F(T) \cap F(I)$, *where* P *is the sunny nonexpansive retraction from* M *onto* $F(T)$.

Notice that the conditions of the continuity and linearity of I are not needed in Theorem 3.6 of Beg et al. [4].

Corollary 3.7. *Let* M *be a closed convex subset of a reflexive Banach space* X *with a uniformly Gâteaux differentiable norm. Let* I *and* T *be self-mappings on* M *such that* $clT(M) \subset I(M)$, $q \in F(I)$ *and* I *is affine. Suppose that* T *is continuous and asymptotically* I*-nonexpansive with a sequence* $\{k_n\}$ *satisfying* $k_n \geq 1$ *and* $\lim_{n\to\infty} k_n = 1$. *Let* $\{\lambda_n\}$ *be sequence of real numbers in* $(0, 1)$ *such that* $\lim_{n\to\infty} \lambda_n = 1$ *and* $\lim_{n\to\infty} \frac{k_n - 1}{k_n - \lambda_n} = 0$. *If* I *and* T *are uniformly pointwise* R*-subweakly commuting on* M, *then we have the following*:

(1) *for each* $n \geq 1$, *there is exactly one* x_n *in* M *such that*

$$Ix_n = x_n = (1 - \mu_n)q + \mu_n T^n x_n,$$

(2) *if* $\{x_n\}$ *is bounded,* $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ *and* I, T *satisfy the property* (S), *then* $\{x_n\}$ *converges strongly to* $Pq \in F(T) \cap F(I)$, *where* P *is the sunny nonexpansive retraction from* M *onto* $F(T)$.

Corollary 3.8. *Let* M *be a closed convex subset of a reflexive Banach space* X *with a uniformly Gâteaux differentiable norm. Let* I *and* T *be pointwise* R*-subweakly commuting self-mappings on* M *such that* $clT(M) \subset I(M)$, $q \in F(I)$, I *is affine,* T *is continuous and* I*-nonexpansive on* M. *Let* $\{\lambda_n\}$ *be a sequence of real numbers in* $(0, 1)$ *such that* $\lim_{n\to\infty} \lambda_n = 1$. *Then we have the following*:

(1) *for each* $n \geq 1$, *there is exactly one* x_n *in* M *such that*

$$Ix_n = x_n = (1 - \lambda_n)q + \lambda_n Tx_n,$$

(2) *if* $\{x_n\}$ *is bounded and* I, T *satisfy property* (S), *then* $\{x_n\}$ *converges strongly to* $Pq \in F(T) \cap F(I)$, *where* P *is the sunny nonexpansive retraction from* M *onto* $F(T)$.

Proof. (1) For each $n \geq 1$, define a mapping T_n on M by

$$T_n x = (1 - \lambda_n)q + \lambda_n Tx, \quad \forall x \in M.$$

Then, following the proof lines of Lemma 2.4, we get the conclusion.

(2) Since $\{x_n\}$ is bounded, $\lim_{n\to\infty} \lambda_n = 1$ and

$$\|x_n - Tx_n)\| = \|Ix_n - Tx_n)\| \leq (\lambda_n^{-1} - 1)(\|q\| + \|x_n\|) \to 0,$$

the conclusion now follows from Theorem 3.6. □

Corollary 3.9 ([7])**.** *Let* M *be a closed convex bounded subset of a Banach space* X *with a uniformly Gâteaux differentiable norm possessing the uniform normal structure. Let* $T : M \to M$ *be an asymptotically nonexpansive mapping* *with a sequence* $\{k_n\}$ *satisfying* $k_n \geq 1$ *and* $\lim_{n\to\infty} k_n = 1$. *Let* $u \in M$ *be fixed,* $\{\lambda_n\}$ *be sequence of real numbers in* $(0,1)$ *such that* $\lim_{n\to\infty} \lambda_n = 1$ *and* $\lim_{n\to\infty} \frac{k_n-1}{k_n-\lambda_n} = 0$. *Then we have the following*:

(1) *for each* $n \geq 1$, *there is unique* x_n *in* M *such that*

$$x_n = (1-\mu_n)u + \mu_n T^n x_n,$$

(2) *if* $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, *then* $\{x_n\}$ *converges strongly to a fixed point of* T.

Proof. (1) follows from Lemma 2.4.

(2) It is well known that every Banach space with the uniform normal structure is reflexive (see [7]). Since $\{x_n\}$ is bounded, we can define a function $f : M \to R^+$ by $f(z) = LIM\|x_n - z\|^2$ for all $z \in M$. As in the proof of Theorem 3.5(2), the set $C = \{x \in M : f(x) = \min_{z\in M} f(z)\}$ is nonempty. Define the set

$$\omega_w(x) := \{y \in X : y = \text{ weak-}\lim_j T^{n_j} x \text{ for some } n_j \to \infty\}.$$

Using the assumption that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, it is easy to see that $\omega_w(x) \subset C$ for each $x \in C$. By Lemma 2.2 [7], T has a fixed point in C. Thus T satisfies the property (S) and so the result follows from Theorem 3.5. □

Remark 4. (1) Theorem 3.5 improves and extends the results of Beg et al. [4], Cho et al. [8], Lim and Xu [23], Schu [33, 34] to more general classes of mappings.

(2) As an application of Theorem 2.12, the strong convergence results similar to Theorem 3.5 can be proved without any type of commutativity condition on the mappings.

(3) Corollary 3.8 extends the result of Schu [32] to I-nonexpansive mappings.

(4) It is worth to notice that the class of asymptotically I-nonexpansive maps properly contains the class of asymptotically nonexpansive maps (see Examples 1.2 and 1.3 in [37]).

References

[1] F. Akbar and A. R. Khan, *Common fixed point and approximation results for noncommuting maps on locally convex spaces*, Fixed Point Theory Appl. **2009** (2009), Art. ID 207503, 14 pp.

[2] M. A. Al-Thagafi, *Common fixed point and best approximation*, J. Approx. Theory **85** (1996), no. 3, 318–323.

[3] M. A. Al-Thagafi and N. Shahzad, *Noncommuting selfmaps and invariant approximations*, Nonlinear Anal. **64** (2006), no. 12, 2778–2786.

[4] I. Beg, D. R. Sahu, and S. D. Diwan, *Approximation of fixed points of uniformly R-subweakly commuting mappings*, J. Math. Anal. Appl. **324** (2006), no. 2, 1105–1114.

[5] F. E. Browder, *Convergence of approximants to fixed points of nonexpansive non-linear mappings in Banach spaces*, Arch. Rational Mech. Anal. **24** (1967), 82–90.

[6] J. Chen and Z. Li, *Common fixed points for Banach operator pairs in best approximation*, J. Math. Anal. Appl. **336** (2007), no. 2, 1466–1475.

[7] C. E. Chidume, J. Li, and A. Udomene, *Convergence of paths and approximation of fixed points of asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc. **133** (2005), no. 2, 473–480.

[8] Y. J. Cho, D. R. Sahu, and J. S. Jung, *Approximation of fixed points of asymptotically pseudocontractive mappings in Banach spaces*, Southwest J. Pure Appl. Math. **2003** (2003), no. 2, 49–59.

[9] LJ. B. Ćirić, *A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc. **45** (1974), 267–273.

[10] LJ. B. Ćirić, J. S. Ume, and M. S. Khan, *On the convergence of the Ishikawa iterates to a common fixed point of two mappings*, Arch. Math. (Brno) **39** (2003), no. 2, 123–127.

[11] K. Goebel and W. A. Kirk, *A fixed point theorem for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc. **35**(1972), 171–174.

[12] B. Halpern, *Fixed points of nonexpanding maps*, Bull. Amer. Math. Soc. **73** (1967), 957–961.

[13] N. Hussain, *Common fixed points in best approximation for Banach operator pairs with ciric type I-contractions*, J. Math. Anal. Appl. **338** (2008), no. 2, 1351–1363.

[14] N. Hussain, M. Abbas, and J. K. Kim, *Common fixed point and invariant approximation in Menger convex metric spaces*, Bull. Korean Math. Soc. **45** (2008), no. 4, 671–680.

[15] N. Hussain and Y. J. Cho, *Weak contractions, common fixed points and invariant approximations*, J. Inequal. Appl. **2009** (2009), Article ID 390634, 10 pages.

[16] N. Hussain and G. Jungck, *Common fixed point and invariant approximation results for noncommuting generalized (f, g)-nonexpansive maps*, J. Math. Anal. Appl. **321** (2006), no. 2, 851–861.

[17] N. Hussain and B. E. Rhoades, *C_q-commuting maps and invariant approximations*, Fixed Point Theory Appl. **2006** (2006), Art. ID 24543, 9 pp.

[18] G. Jungck, *Compatible mappings and common fixed points*, Internat. J. Math. Math. Sci. **9** (1986), no. 4, 771–779.

[19] ______, *Common fixed points for noncontinuous nonself maps on non metric spaces*, Far East J. Math. Sci. **4** (1996), no. 2, 199–215.

[20] G. Jungck and N. Hussain, *Compatible maps and invariant approximations*, J. Math. Anal. Appl. **325** (2007), no. 2, 1003–1012.

[21] G. Jungck and S. Sessa, *Fixed point theorems in best approximation theory*, Math. Japon. **42** (1995), no. 2, 249–252.

[22] S. H. Khan and N. Hussain, *Convergence theorems for nonself asymptotically nonexpansive mappings*, Comput. Math. Appl. **55** (2008), no. 11, 2544–2553.

[23] T. C. Lim and H. K. Xu, *Fixed point theorems for asymptotically nonexpansive mappings*, Nonlinear Anal. **22** (1994), no. 11, 1345–1355.

[24] D. O'Regan and N. Hussain, *Generalized I-contractions and pointwise R-subweakly commuting maps*, Acta Math. Sin. (Engl. Ser.) **23** (2007), no. 8, 1505–1508.

[25] R. P. Pant, *Common fixed points of noncommuting mappings*, J. Math. Anal. Appl. **188** (1994), no. 2, 436–440.

[26] ______, *Common fixed points of Lipschitz type mapping pairs*, J. Math. Anal. Appl. **240** (1999), no. 1, 280–283.

[27] H. K. Pathak, Y. J. Cho, and S. M. Kang, *An application of fixed point theorems in best approximation theory*, Internat. J. Math. Math. Sci. **21** (1998), no. 3, 467–470.

[28] H. K. Pathak and N. Hussain, *Common fixed points for Banach operator pairs with applications*, Nonlinear Anal. **69** (2008), no. 9, 2788–2802.

[29] N. Petrot, *Approximation of fixed points of uniformly C_q-commuting mappings*, Far East J. Math. Sci. **26** (2007), no. 2, 331–343.

[30] S. Reich, *Strong convergence theorems for resolvents of accretive operators in Banach spaces*, J. Math. Anal. Appl. **75** (1980), no. 1, 287–292.

[31] B. E. Rhoades, *Convergence of an Ishikawa-type iteration scheme for a generalized contraction*, J. Math. Anal. Appl. **185** (1994), no. 2, 350–355.
[32] J. Schu, *Iterative approximation of fixed points of nonexpansive mappings with starshaped domain*, Comment. Math. Univ. Carolin. **31** (1990), no. 2, 277–282.
[33] ______, *Approximation of fixed points of asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc. **112** (1991), no. 1, 143–151.
[34] ______, *Iterative construction of fixed points of asymptotically nonexpansive mappings*, J. Math. Anal. Appl. **158** (1991), no. 2, 407–413.
[35] S. Sessa, *On a weak commutativity condition of mappings in fixed point considerations*, Publ. Inst. Math. (Beograd) (N.S.) **32(46)** (1982), no. 46, 149–153.
[36] S. P. Singh, *An application of a fixed-point theorem to approximation theory*, J. Approx. Theory **25** (1979), no. 1, 89–90.
[37] P. Vijayaraju, *Applications of fixed point theorem to best simultaneous approximations*, Indian J. Pure Appl. Math. **24** (1993), no. 1, 21–26.
[38] H. K. Xu, *Existence and convergence for fixed points of mappings of asymptotically nonexpansive type*, Nonlinear Anal. **16** (1991), no. 12, 1139–1146.

Yeol Je Cho
Department of Mathematics Education and the RINS
Gyeongsang National University
Chinju 660-701, Korea
E-mail address: yjcho@gsnu.ac.kr

Nawab Hussain
Department of Mathematics
King Abdul Aziz University
P.O. Box 80203, Jeddah 21589, Saudi Arabia
E-mail address: nhusain@kau.edu.sa

Hemant Kumar Pathak
School of Studies in Mathematics
Pt. Ravishankar Shukla University
Raipur (C.G.) 492010, India
E-mail address: hkpathak@sify.com