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BOUNDED, COMPACT AND SCHATTEN CLASS WEIGHTED COMPOSITION OPERATORS BETWEEN WEIGHTED BERGMAN SPACES

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Abstract. An analytic self-map ϕ of the open unit disk $\mathbb{D}$ in the complex plane and an analytic map ψ on $\mathbb{D}$ induce the so-called weighted composition operator $C_{\phi,\psi} : H(\mathbb{D}) \to H(\mathbb{D}),\ f \mapsto \psi(f \circ \phi)$, where $H(\mathbb{D})$ denotes the set of all analytic functions on $\mathbb{D}$. We study when such an operator acting between different weighted Bergman spaces is bounded, compact and Schatten class.

1. Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the space of all analytic functions on $\mathbb{D}$ endowed with the compact-open topology *co*. Moreover, let ϕ be an analytic self-map of $\mathbb{D}$. Such a map induces through composition the classical composition operator

$$C_\phi : H(\mathbb{D}) \to H(\mathbb{D}),\ f \mapsto f \circ \phi.$$

If we multiply with a map $\psi \in H(\mathbb{D})$, we obtain the weighted composition operator

$$C_{\phi,\psi} : H(\mathbb{D}) \to H(\mathbb{D}),\ f \mapsto \psi(f \circ \phi).$$

(Weighted) composition operators acting on various spaces of analytic functions have been studied by many authors, since this kind of operator appears naturally in a variety of problems, such as e.g. in the study of commutants of multiplication operators or the study of dynamical systems, see the excellent monographs [5] and [15]. For a deep insight in the recent research on (weighted) composition operators we refer the reader to the following sample of papers as well as the references therein: [1], [2], [3], [4], [10], [12], [11], [13], [14].

Let us now explain the setting in which we are interested. We say that a function $v : \mathbb{D} \to (0,\infty)$ is a *weight* if it is bounded and continuous. For a

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weight v we consider the space

$$A_{v,2} := \left\{ f \in H(\mathbb{D});\ \|f\|_{v,2} := \left(\int_{\mathbb{D}} |f(z)|^2 v(z)\, dA(z) \right)^{\frac{1}{2}} < \infty \right\},$$

where $dA(z)$ is the normalized area measure such that area of $\mathbb{D}$ is 1. Endowed with norm $\|\cdot\|_{v,2}$ this is a Hilbert space. Thus, $A_{1,2}$ denotes the usual Bergman space. An introduction to Bergman spaces is given in [9] and [7].

In [16] we characterized the boundedness of weighted composition operators acting between weighted Bergman spaces generated by weights given as the absolute value of holomorphic functions using a method due to Čučković and Zhao [6]. In this paper we treat compactness and Schatten class weighted composition operators in the same setting.

2. Preliminaries

This section is devoted to the collection of basic facts and auxiliary results. For $a, z \in \mathbb{D}$ let $\sigma_a(z)$ be the Möbius transformation of $\mathbb{D}$ which interchanges 0 and a, that is

$$\sigma_a(z) = \frac{a - z}{1 - \overline{a}z}.$$

Obviously

$$\sigma_a'(z) = -\frac{1 - |a|^2}{(1 - \overline{a}z)^2} \quad \text{for every } z \in \mathbb{D}.$$

Moreover, consider the Bergman kernel $K_a(z) = \frac{1}{(1-\overline{a}z)^2}$ as well as the normalized Bergman kernel $k_a(z) = -\sigma_a'(z) = \frac{1-|a|^2}{(1-\overline{a}z)^2} = (1 - |a|^2)K_a(z)$ in $A_{1,2}$ so that $\|k_a\|_{1,2} = 1$. For an analytic self-map ϕ of $\mathbb{D}$ and weights v, w on $\mathbb{D}$ we define the weighted (ϕ, v)-Berezin transform of w as follows

$$[B_{\phi,v}(w)](a) = \int_{\mathbb{D}} |\sigma_a'(\phi(z))|^2 \frac{w(z)}{v(\phi(z))}\, dA(z).$$

It turned out that the Carleson measure is a very useful tool when studying (weighted) composition operators on weighted Bergman spaces, see [6] and [16]. Recall that a positive Borel measure μ on $\mathbb{D}$ is said to be a *Carleson measure* on the Bergman space if there is a constant $C > 0$ such that, for any $f \in A_{1,2}$

$$\int_{\mathbb{D}} |f(z)|^2\, d\mu(z) \le C\|f\|_{1,2}^2.$$

For an arc I in the unit circle $\partial\mathbb{D}$ let $S(I)$ be the Carleson square defined by

$$S(I) = \left\{ z \in \mathbb{D};\ 1 - |I| \le |z| < 1,\ \frac{z}{|z|} \in I \right\}.$$

The following result is well-known. In its present form it is taken from [6, Theorem A] and [8].

Theorem 1 ([6, Theorem A]). *Let μ be a positive Borel measure on $\mathbb{D}$. Then the following statements are equivalent.*

(i) *There is a constant $C_1 > 0$ such that for any $f \in A_{1,2}$ we have*

$$\int_{\mathbb{D}} |f(z)|^2 \, d\mu(z) \leq C_1 \|f\|_{1,2}^2.$$

(ii) *There is a constant $C_2 > 0$ such that, for any arc $I \in \partial\mathbb{D}$,*

$$\mu(S(I)) \leq C_2 |I|^2.$$

(iii) *There is a constant $C_3 > 0$ such that, for every $a \in \mathbb{D}$,*

$$\int_{\mathbb{D}} |\sigma_a'(z)|^2 \, d\mu(z) \leq C_3.$$

The study of the compactness of the operator $C_{\phi,\psi}$ requires the following proposition which can be found in the book of Cowen and MacCluer, see [5].

Proposition 2 (Cowen-MacCluer, [5, Proposition 3.11]). *The operator $C_{\phi,\psi} : A_{v,2} \to A_{w,2}$ is compact if and only if for every bounded sequence $(f_n)_{n\in\mathbb{N}}$ in $A_{v,2}$ such that $f_n \to 0$ uniformly on the compact subsets of $\mathbb{D}$, then $C_{\phi,\psi} f_n \to 0$ in $A_{w,2}$.*

In the sequel we consider the following class of weights. Let ν be a holomorphic function on $\mathbb{D}$, non-vanishing, strictly positive on $[0, 1[$ and satisfying $\lim_{r\to 1} \nu(r) = 0$. Then we define the weight v by

$$v(z) := \nu(z)\overline{\nu(z)} = |\nu(z)|^2 \tag{2.1}$$

for every $z \in \mathbb{D}$.

Next, we give some illustrating examples of weights of this type:

(i) Consider $\nu(z) = (1 - z)^{\alpha}$, $\alpha \geq 1$. Then the corresponding weight is given by $v(z) = (1 - z - \overline{z} + |z|^2)^{\alpha}$.

(ii) Select $\nu(z) = e^{-\frac{1}{(1-z)}}$. Then we obtain the weight $v(z) = e^{\frac{-2+z+\overline{z}}{(1-z-\overline{z}+|z|^2)}}$.

(iii) Choose $\nu(z) = \sin(1 - z)$ and the corresponding weight is given by $v(z) = \sin(1 - z)\sin(1 - \overline{z}) = \frac{1}{2}\left(\cos(-z + \overline{z}) - \cos(2 - z - \overline{z})\right)$.

(iv) Let $\nu(z) = \frac{1}{1-\log(1-z)}$ for every $z \in \mathbb{D}$. Hence we obtain the weight $v(z) = \frac{1}{1-\log(1-z)-\log(1-\overline{z})+\log(1-z)\log(1-\overline{z})}$ for every $z \in \mathbb{D}$.

3. Boundedness

In this section, for a better understanding of the paper we repeat results we got in [16] concerning the boundedness of $C_{\phi,\psi} : A_{v,2} \to A_{w,2}$. In fact, the following result corresponds to the results obtained in [6]. Actually, the idea to use Carleson measures is due to [6].

Theorem 3. *Let v be a weight as defined in (2.1), w an arbitrary weight, ϕ an analytic self-map of $\mathbb{D}$ and $\psi \in H(\mathbb{D})$. Then the weighted composition operator $C_{\phi,\psi} : A_{v,2} \to A_{w,2}$ is bounded if and only if the Berezin transform $B_{\phi,v}(|\psi|^2 w) \in L^{\infty}(\mathbb{D})$.*

Proof. The main idea of this proof is to reduce the problem to the setting of non-weighted Bergman spaces and to use a reformulation of the Carleson measure condition. By definition, the operator $C_{\phi,\psi} : A_{v,2} \to A_{w,2}$ is bounded if and only if there is $C > 0$ such that for every $f \in A_{v,2}$ the following inequality holds:

$$\int_{\mathbb{D}} |f(\phi(z))|^2 |\psi(z)|^2 w(z)\, dA(z) \leq C \int_D |f(z)|^2 v(z)\, dA(z). \tag{3.1}$$

First, note that $f \in A_{v,2}$ if and only if $g := \nu f \in A_{1,2}$, since

$$\begin{aligned} \|f\|_{v,2}^2 &= \int_{\mathbb{D}} f(z)\overline{f(z)} v(z)\, dA(z) \\ &= \int_D \frac{g(z)}{\nu(z)} \frac{\overline{g(z)}}{\overline{\nu(z)}} v(z)\, dA(z) \\ &= \int_{\mathbb{D}} |g(z)|^2\, dA(z) = \|g\|_{1,2}^2. \end{aligned}$$

Thus, (3.1) is equivalent to the following inequality: There is a constant $M > 0$ such that

$$\int_{\mathbb{D}} \frac{|g(\phi(z))|^2}{v(\phi(z))} |\psi(z)|^2 w(z)\, dA(z) \leq M \int_{\mathbb{D}} |g(z)|^2\, dA(z). \tag{3.2}$$

We write $d\lambda_{v,w,\psi}(z) = |\psi(z)|^2 \frac{w(z)}{v(\phi(z))} dA(z)$. Next, let

$$\mu_{v,w,\psi} = \lambda_{v,w,\psi} \circ \phi^{-1} \tag{3.3}$$

be the corresponding pull-back measure induced by the map ϕ. A change of variables $s = \phi(z)$ gives

$$\begin{aligned} \int_{\mathbb{D}} \frac{|g(\phi(z))|^2}{v(\phi(z))} |\psi(z)|^2 w(z)\, dA(z) &= \int_{\mathbb{D}} |g(\phi(z))|^2\, d\lambda_{v,w,\psi}(z) \\ &= \int_{\mathbb{D}} |g(s)|^2\, d\mu_{v,w,\psi}(s). \end{aligned}$$

Thus, (3.1) is equivalent to $\int_{\mathbb{D}} |g(s)|^2\ d\mu_{v,w,\psi}(s) \leq M \int_{\mathbb{D}} |g(s)|^2\ dA(s)$. By Theorem 1 this holds if and only if

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |\sigma_a'(s)|^2\, d\mu_{v,w,\psi}(s) < \infty.$$

If we change the variable back to z, we obtain

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |\sigma_a'(\phi(z))|^2 |\psi(z)|^2 \frac{w(z)}{v(\phi(z))}\, dA(z) < \infty,$$

as desired. □

4. Compactness

In this section we investigate when a weighted composition operator acting between weighted Bergman spaces is compact. To do this we need the following auxiliary result which is due to Čučković and Zhao.

Lemma 4. *Let* $0 < r < 1$ *and* μ *be a Carleson measure for the Bergman space. Put* $N_r^* := \sup_{|a|\geq r} \int_{\mathbb{D}} |\sigma_a'(z)|^2 \, d\mu(z)$. *Then*

$$\|\mu_r\| \leq MN_r^*,$$

where M *is an absolute constant and* $\mu_r = \mu|_{\mathbb{D}\setminus\mathbb{D}_r}$ *with* $\mathbb{D}_r := \{z \in \mathbb{D}; |z| < r\}$.

Theorem 5. *Let* v *be a weight as defined in* (2.1) *and* w *be an arbitrary weight. For an analytic self-map* ϕ *of* $\mathbb{D}$ *and a map* $\psi \in H(\mathbb{D})$ *the weighted composition operator* $C_{\phi,\psi} : A_{v,2} \to A_{w,2}$ *is compact if and only if*

$$\limsup_{|a|\to 1} [B_{\phi,v}(|\psi|^2 w)](a) = 0.$$

Proof. Let us first assume that $\limsup_{|a|\to 1}[B_{\phi,v}(|\psi|^2 w)](a) = 0$. Let $(f_n)_n \subset A_{v,2}$ be a bounded sequence which converges to 0 uniformly on the compact subsets of $\mathbb{D}$. By Proposition 2 we have to show that

$$\|C_{\phi,\psi} f_n\|_{w,2} \to 0 \text{ if } n \to \infty.$$

Again we use the fact that $f_n \in A_{v,2}$ if and only if $g_n := \nu f_n \in A_{1,2}$. We fix $0 < r < 1$ to obtain

$$\begin{aligned}
\|C_{\phi,\psi} f_n\|_{w,2}^2 &= \int_{\mathbb{D}} |\psi(z)|^2 |f_n(\phi(z)|^2 w(z) \, dA(z) \\
&= \int_{\mathbb{D}\setminus\mathbb{D}_r} \frac{|\psi(z)|^2}{v(\phi(z))} |g_n(\phi(z))|^2 w(z) \, dA(z) \\
&\quad + \int_{\mathbb{D}_r} |\psi(z)|^2 w(z) |f_n(\phi(z))|^2 \, dA(z) \\
&= \int_{\mathbb{D}\setminus\mathbb{D}_r} |g_n(s)|^2 \, d\mu_{v,w,\psi}(s) + \int_{\mathbb{D}_r} |\psi(z)|^2 w(z) |f_n(\phi(z))|^2 \, dA(z) \\
&= I_1 + I_2,
\end{aligned}$$

where $\mu_{v,w,\psi}$ is the pull-back measure we already defined in (3.3). Obviously, we have that

$$I_2 \to 0 \text{ if } n \to \infty.$$

In order to treat the term I_1 we write $\mu_r := \mu_{v,w,\psi}|_{\mathbb{D}\setminus\mathbb{D}_r}$ and apply Theorem 1 as well as Lemma 4 to arrive at

$$I_1 = \int_{\mathbb{D}\setminus\mathbb{D}_r} |g_n(s)|^2 \, d\mu_r(s) \leq K\|\mu_r\| \|g_n\|_{1,2}^2 \leq CKMN_r^*,$$

where C, K, M are absolute constants. Letting $n \to \infty$ and $r \to 1$ we get

$$\begin{aligned}\lim_{n\to\infty} \|C_{\phi,\psi} f_n\|_{w,2}^2 &\leq CKM \lim_{r\to 1} N_r^* \\ &= CKM \limsup_{|a|\to 1} \int_{\mathbb{D}} |\sigma_a'(s)|^2 \, d\mu_{v,w,\psi}(s) \\ &= CKM \limsup_{|a|\to 1} \int_{\mathbb{D}} \frac{|\sigma_a'(\phi(z))|^2 w(z)}{v(\phi(z))} |\psi(z)|^2 \, dA(z) \\ &= CKM \limsup_{|a|\to 1} [B_{\phi,v}(|\psi|^2 w)](a) = 0.\end{aligned}$$

Hence, the operator must be compact.

Conversely, for $a \in \mathbb{D}$ we consider functions $h_a(z) := -\frac{\sigma_a'(z)}{\nu(z)}$ for every $z \in \mathbb{D}$. Obviously, $\|h_a\|_{v,2}^2 = \int_{\mathbb{D}} \frac{|\sigma_a'(z)|^2}{v(z)} v(z) \, dA(z) = 1$ and $h_a \to 0$ uniformly on the compact subsets of $\mathbb{D}$ as $|a| \to 1$. Since the operator is compact we have that

$$\|C_{\phi,\psi} h_a\|_{w,2}^2 = \int_{\mathbb{D}} \frac{|\sigma_a'(\phi(z))|^2}{v(\phi(z))} |\psi(z)|^2 w(z) \, dA(z) \to 0$$

as $|a| \to 1$, and the claim follows. □

5. Schatten class operators

The last section is dedicated to the study of Schatten class weighted composition operators between weighted Bergman spaces. For more information on Schatten class operators we refer the reader e.g. to the excellent monograph [17].

Theorem 6. *Let v be a weight as defined in* (2.1) *and w be an arbitrary weight. Moreover, let the operator $C_{\phi,\psi} : A_{v,2} \to A_{v,2}$ be compact. Then $C_{\phi,\psi} \in S_p$ if and only if $B_{\phi,v}(|\psi|^2 v) \in L^{\frac{p}{2}}(\mathbb{D}, d\lambda)$, where $d\lambda(z) = (1 - |z|^2)^{-2} dA(z)$ is the Möbius invariant measure on $\mathbb{D}$.*

Proof. Taking into account that $f_1, f_2 \in A_{v,2}$ if and only if $g_1 := \nu f_1$, $g_2 := \nu f_2 \in A_{1,2}$, for every $f_1, f_2 \in A_{v,2}$ we get

$$\begin{aligned}\langle C_{\phi,\psi}^* C_{\phi,\psi} f_1, f_2 \rangle &= \langle C_{\phi,\psi} f_1, C_{\phi,\psi} f_2 \rangle \\ &= \int_{\mathbb{D}} f_1(\phi(z)) \overline{f_2(\phi(z))} |\psi(z)|^2 v(z) \, dA(z) \\ &= \int_{\mathbb{D}} g_1(\phi(z)) \overline{g_2(\phi(z))} \frac{|\psi(z)|^2 v(z)}{v(\phi(z))} \, dA(z) \\ &= \int_{\mathbb{D}} g_1(s) \overline{g_2(s)} \, d\mu_{v,v,\psi}(s),\end{aligned}$$

where $\mu_{v,v,\psi}$ is the pull-back measure defined in (3.3). Put $\mu_{v,v,\psi} := \mu$ and

$$T_\mu f(z) := \frac{1}{\nu(z)} \int_{\mathbb{D}} f(s) K_s(z) \nu(s) \, d\mu(s)$$

for every $z \in \mathbb{D}$. Then

$$\begin{aligned} \langle T_\mu f_1, f_2 \rangle &= \int_{\mathbb{D}} \frac{1}{\nu(z)} \int_{\mathbb{D}} f_1(s) K_s(z) \nu(s)\, d\mu(s)\, v(z)\, dA(z) \\ &= \int_{\mathbb{D}} \int_{\mathbb{D}} g_1(s) K_s(z)\, d\mu(s) \overline{g_2(z)}\, dA(z) \\ &= \int_{\mathbb{D}} g_1(s) \overline{\int_{\mathbb{D}} K_z(s) g_2(s)\, dA(z)}\, d\mu(s) \\ &= \int_{\mathbb{D}} g_1(s) \overline{g_2(s)}\, d\mu(s). \end{aligned}$$

Thus, $T_\mu = C^*_{\phi,\psi} C_{\phi,\psi}$. By definition, for $0 < p < \infty$, an operator belongs to the Schatten class S_p if and only if $(T^*T)^{\frac{p}{2}}$ is in the trace class. Hence, by [17] Lemma 1.4.6 on p. 18, $C_{\phi,\psi}$ is in the Schatten class S_p if and only if $T_\mu \in S_{\frac{p}{2}}$. Let $(e_n)_n$ be an orthonormal basis for $A_{1,2}$. Then $(\frac{1}{\nu} e_n)_{n \in \mathbb{N}}$ is an orthonormal basis for $A_{\nu,2}$. Thus, we can reduce the problem to the setting of $A_{1,2}$, since

$$\begin{aligned} \langle T_\mu \frac{e_n}{\nu}, \frac{e_n}{\nu} \rangle &= \int_{\mathbb{D}} \frac{1}{\nu(z)} \int_{\mathbb{D}} \frac{e_n(s)}{\nu(s)} K_s(z) \nu(s)\, d\mu(s) \frac{\overline{e_n(z)}}{\nu(z)} v(z)\, dA(z) \\ &= \int_{\mathbb{D}} \int_{\mathbb{D}} e_n(s) K_s(z)\, d\mu(s) \overline{e_n(z)}\, dA(z). \end{aligned}$$

Now, by Zhu [18] Lemma 1, for $0 < p < \infty$, $T_\mu \in S_{\frac{p}{2}}$ if and only if $B_{\phi,v}(|\psi|^2 v) \in L^{\frac{p}{2}}(\mathbb{D}, d\lambda)$ and the claim follows.

□

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