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SOME COMMUTATIVITY THEOREMS OF PRIME RINGS WITH GENERALIZED (σ, τ)-DERIVATION

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Abstract. In this paper, we extend some well known results concerning generalized derivations of prime rings to a generalized (σ, τ)-derivation.

1. Introduction

Let R will be an associative ring with center Z, σ, τ two mappings from R into itself. For any $x, y \in R$, we write $[x, y]$ and $[x, y]_{\sigma,\tau}$ for $xy - yx$ and $x\sigma(y) - \tau(y)x$ respectively. We set $C_{\sigma,\tau} = \{c \in R \mid c\sigma(x) = \tau(x)c$ for all $x \in R\}$ and call (σ, τ)-center of R. Recall that a ring R is prime if $xRy = 0$ implies $x = 0$ or $y = 0$. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. For a fixed $a \in R$, the mapping $I_a : R \to R$ given by $I_a(x) = [a, x]$ is a derivation which is said to be an inner derivation.

Recently, in [7], Bresar defined the following notation. An additive mapping $f : R \to R$ is called a generalized derivation if there exists a derivation $d : R \to R$ such that

$$f(xy) = f(x)y + xd(y) \quad \text{for all } x, y \in R.$$

Basic examples are derivations and generalized inner derivations (i.e., maps of type $x \to ax + xb$ for some $a, b \in R$). One may observe that the concept of generalized derivations includes the concept of derivations and of the left multipliers (i.e., $f(xy) = f(x)y$ for all $x, y \in R$). Hence it should be interesting to extend some results concerning these notions to generalized derivations.

Inspired by the definition (σ, τ)-derivation, the notion of generalized derivation was extended as follows: Let σ, τ be two automorphisms of R. An additive mapping $f : R \to R$ is called a generalized (σ, τ)-derivation on R if there exists a (σ, τ)-derivation $d : R \to R$ such that

$$f(xy) = f(x)\sigma(y) + \tau(x)d(y) \quad \text{for all } x, y \in R.$$

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Of course a generalized $(1,1)$-derivation is a generalized derivation on R, where 1 is the identity mapping on R.

Let S be a nonempty subset of R. A mapping F from R to R is called commutativity preserving on a subset S of R if $[x,y]=0$ implies $[F(x),F(y)]=0$ for all $x,y\in S$. The mapping F is called strong commutativity preserving (scp) on S if $[F(x),F(y)]=[x,y]$ for all $x,y\in S$. There is also a growing literature strong commutativity preserving (scp) maps and derivations (for reference see [6], [5], [8], [10], [11], etc.). In [2], the authors explored the commutativity of the ring R satisfying one of the following conditions: (i) $[d(x),F(y)]=0$, (ii) $[d(x),F(y)]=\pm[x,y]$, (iii) $d(x)F(y)\pm xy\in Z$ and also they proved (vi) $F(xy)\pm xy\in Z$, (vii) $F(xy)\pm yx\in Z$ and (viii) $F(x)F(y)\pm xy\in Z$ for all $x,y\in R$ in some appropriate subset of the ring R in [3]. The major purpose of this paper is to prove these theorems for a generalized (σ,τ)-derivation of R.

Throughout the paper, we denote a generalized (σ,τ)-derivation $f:R\to R$ determined by a (σ,τ)-derivation d of R with (f,d) and make some extensive use of the basic commutator identities:

$$\begin{aligned}
[x,yz]&=y[x,z]+[x,y]z,\\
[xy,z]&=[x,z]y+x[y,z],\\
[xy,z]_{\sigma,\tau}&=x[y,z]_{\sigma,\tau}+[x,\tau(z)]y=x[y,\sigma(z)]+[x,z]_{\sigma,\tau}y,\\
[x,yz]_{\sigma,\tau}&=\tau(y)[x,z]_{\sigma,\tau}+[x,y]_{\sigma,\tau}\sigma(z).
\end{aligned}$$

2. Results

Lemma 1. *Let (f,d) be a generalized (σ,τ)-derivation of a prime ring R with* char$R\neq 2$. *If $af(x)=0$ for all $x\in R$, then $a=0$ or $d=0$.*

Proof. Replacing x by xy in the hypothesis, we have

$$af(xy)=af(x)\sigma(y)+a\tau(x)d(y)=0\quad\text{for all } x,y\in R.$$

By the hyphothesis, the first term is zero in this equation. Hence we obtain that $aRd(y)=0$ for all $y\in R$. By the primeness of R, we have $a=0$ or $d=0$. □

The following theorems are motivated from [2].

Theorem 1. *Let (f,d) be a generalized (σ,τ)-derivation of a noncommutative prime ring R with* char$R\neq 2$. *If $[d(x),f(y)]=0$ for all $x,y\in R$, then $d=0$.*

Proof. If $f=0$, there is nothing to prove. So, we have $f\neq 0$. Assume that $[d(x),f(y)]=0$ for all $x,y\in R$. Substitute yz by y obtaining

$$f(y)[d(x),\sigma(z)]+[d(x),\tau(y)]d(z)+\tau(y)[d(x),d(z)]=0 \text{ for all } x,y,z\in R. \tag{2.1}$$

Taking $z\sigma^{-1}(d(x))$ instead of z in (2.1) and using this equation, we have

$$\begin{aligned}
&[d(x),\tau(y)]\tau(z)d(\sigma^{-1}(d(x)))+\tau(y)[d(x),\tau(z)]d(\sigma^{-1}(d(x)))\\
&\qquad+\tau(y)\tau(z)[d(x),d(\sigma^{-1}(d(x)))]=0.
\end{aligned}\tag{2.2}$$

Substituting ry for y in (2.2) and using (2.2), we conclude that

$$[d(x),\tau(r)]\tau(yz)d(\sigma^{-1}(d(x)))=0 \quad \text{for all } x,y,z,r\in R$$

and so,

$$[d(x),\tau(r)]Rd(\sigma^{-1}(d(x)))=0 \quad \text{for all } x,r\in R.$$

By the primeness of R, we obtain that $d(x)\in Z$ or $d(\sigma^{-1}(d(x)))=0$ for each $x\in R$. We put $K=\{x\in R \mid d(x)\in Z\}$ and $L=\{x\in R \mid d(\sigma^{-1}(d(x)))=0\}$. Then it can be easily seen that K and L both are additive subgroups of R and whose union R. Then by Brauer's trick, either $K=R$ or $L=R$. If $K=R$, then $d(R)\subset Z$, and so R is a commutative ring by [4, Lemma 2]. It contradicts our hyphothesis.

Now assume that $L=R$. That is $d(\sigma^{-1}(d(x)))=0$ for all $x\in R$. Hence writing $\sigma^{-1}(d(z))$ instead of z in (2.1) and using $d(\sigma^{-1}(d(z)))=0$, we have $f(y)[d(x),d(z)]=0$ for all $x,y,z\in R$. That is

$$f(y)d(x)d(z)=f(y)d(z)d(x) \quad \text{for all } x,y,z\in R. \tag{2.3}$$

We know that $d(x)f(y)=f(y)d(x)$ for all $x,y\in R$ by the hyphothesis. Using this in (2.3), we get

$$[d(x),d(z)]f(y)=0 \quad \text{for all } x,y,z\in R.$$

Thus we obtain that $[d(x),d(z)]=0$ for all $x,z\in R$ by Lemma 1, and so, $d=0$ by [9, Theorem 2 (a)]. □

In particular if we take $f=d$, then we have the following result which is a generalization of [9, Theorem 2(a)] even without $\sigma d=d\sigma,\tau d=d\tau$ assumption on ring.

Corollary 1. *Let d be a (σ,τ)-derivation of a noncommutative prime ring R with* $\mathrm{char}R\neq 2$. *If $[d(x),d(y)]=0$ for all $x,y\in R$, then $d=0$.*

Theorem 2. *Let (f,d) be a generalized (σ,τ)-derivation of a noncommutative prime ring R with* $\mathrm{char}R\neq 2$. *If $[d(x),f(y)]=\pm[x,y]_{\sigma,\tau}$ for all $x,y\in R$, then $d=0$.*

Proof. If $f=0$, then $[x,y]_{\sigma,\tau}=0$ for all $x,y\in R$. Replacing x by xz, we get $x[z,\sigma(y)]=0$ for all $x,y,z\in R$. Thus we obtain that R is a commutative ring by the primeness of R and so, it contradicts our hypothesis. Hence we suppose that $f\neq 0$.

Assume that $[d(x),f(y)]=\pm[x,y]_{\sigma,\tau}$ for all $x,y\in R$. Replacing y by yz in the hypothesis, we have

$$\begin{aligned} &f(y)[d(x),\sigma(z)]+[d(x),\tau(y)]d(z)+\tau(y)[d(x),d(z)]\\ &=\pm\,\tau(y)[x,z]_{\sigma,\tau} \quad \text{for all } x,y,z\in R. \end{aligned} \tag{2.4}$$

Substitute $z\sigma^{-1}(d(x))$ instead of z in the above relation, we obtain gives

$$\begin{aligned} & [d(x),\tau(y)]\tau(z)d(\sigma^{-1}(d(x))) + \tau(y)[d(x),\tau(z)]d(\sigma^{-1}(d(x))) \\ & + \tau(y)\tau(z)[d(x), d(\sigma^{-1}(d(x)))] \\ = & \pm \tau(y)\tau(z)[x,\sigma^{-1}(d(x))]_{\sigma,\tau} \quad \text{for all } x,y,z \in R. \end{aligned}$$

Taking yr instead of y in this equation, we get

$$[d(x),\tau(y)]\tau(rz)d(\sigma^{-1}(d(x))) = 0 \quad \text{for all } x,y,z,r \in R.$$

Since R is a prime ring, we obtain that

$$d(x) \in Z \text{ or } d(\sigma^{-1}(d(x))) = 0 \quad \text{for each } x \in R.$$

By a standart argument one of these must hold for all $x \in R$. Since R is noncommutative the first possibility gives $d = 0$ by [4, Lemma 2]. Hence we assume that $d(\sigma^{-1}(d(x))) = 0$ for all $x \in R$. Writing $\sigma^{-1}(d(x))$ instead of x in (2.4) and using this, we get $\tau(y)[\sigma^{-1}(d(x)), z]_{\sigma,\tau} = 0$ for all $x, y, z \in R$, and so $[\sigma^{-1}(d(x)), z]_{\sigma,\tau} = 0$ for all $x, z \in R$. Hence

$$\begin{aligned} 0 = [\sigma^{-1}(d(xy)), z]_{\sigma,\tau} &= [\sigma^{-1}(d(x))y + \sigma^{-1}(\tau(x))\sigma^{-1}(d(y)), z]_{\sigma,\tau} \\ &= [\sigma^{-1}(d(x)), z]_{\sigma,\tau} y + \sigma^{-1}(d(x))[y,\sigma(z)] \\ &\quad + \sigma^{-1}(\tau(x))[\sigma^{-1}(d(y)), z]_{\sigma,\tau} + [\sigma^{-1}(\tau(x)),\tau(z)]\sigma^{-1}(d(y)). \end{aligned}$$

Thus we obtain that

$$\sigma^{-1}(d(x))[y,\sigma(z)] + [\sigma^{-1}(\tau(x)),\tau(z)]\sigma^{-1}(d(y)) = 0 \text{ for all } x,y,z \in R.$$

Replacing z by $\sigma^{-1}(y)$ in this equation, we conclude that

$$[\sigma^{-1}(\tau(x)),\tau(\sigma^{-1}(y))]\sigma^{-1}(d(y)) = 0 \quad \text{for all } x,y \in R.$$

Again replacing x by xz yields that

$$[\sigma^{-1}(\tau(x)),\tau(\sigma^{-1}(y))]R\sigma^{-1}(d(y)) = 0 \quad \text{for all } x,y \in R.$$

By the primeness of R and $\sigma,\tau \in \mathrm{Aut}R$, we have $y \in Z$ or $d(y) = 0$ for each $y \in R$. We set $K = \{y \in R \mid y \in Z\}$ and $L = \{y \in R \mid d(y) = 0\}$. Clearly each of K and L is an additive subgroup of R. Morever, R is the set-theoretic union of K and L. But a group can not be the set-theoretic union of two proper subgroups, hence $K = R$ or $L = R$. If $L = R$, then $d(R) = 0$, and so R is a commutative ring by [4, Lemma 2]. Hence R is a commutative ring for any cases. □

Corollary 2. *Let d be a (σ,τ)-derivation of a noncommutative prime ring R with* $\mathrm{char}R \neq 2$. *If $[d(x), d(y)] = \pm[x,y]_{\sigma,\tau}$ for all $x,y \in R$, then $d = 0$.*

The following theorems are motivated from [2].

Theorem 3. *Let (f,d) be a generalized (σ,τ)-derivation of a prime ring R with* $\mathrm{char}R \neq 2$. *If $d(x)f(y) - x\sigma(y) \in C_{\sigma,\tau}$ for all $x,y \in R$, and if $d \neq 0$, then R is a commutative ring.*

Proof. If $f = 0$, then $x\sigma(y) \in C_{\sigma,\tau}$ for all $x, y \in R$. In particular, $[x\sigma(y), y]_{\sigma,\tau} = 0$ for all $x, y \in R$ and hence $[x, y]_{\sigma,\tau}\sigma(y) = 0$ for all $x, y \in R$. Replacing x by xz, we get $[x, \tau(y)]z\sigma(y) = 0$ for all $x, y, z \in R$. Hence it follows then $[x, \tau(y)]R\sigma(y) = 0$ for all $x, y \in R$. Thus the primeness of R forces that for each $y \in R$, either $y \in Z$ or $y = 0$. But $y = 0$ also implies that $y \in Z$. Hence in both cases we find that R is a commutative ring.

Hence, onward we assume that $f \neq 0$. Suppose that $d(x)f(y) - x\sigma(y) \in C_{\sigma,\tau}$ for all $x, y \in R$. Replacing y by yz, obtaining

$$(d(x)f(y) - x\sigma(y))\sigma(z) + d(x)\tau(y)d(z) \in C_{\sigma,\tau} \quad \text{for all } x, y, z \in R.$$

Then it is clear that $[(d(x)f(y) - x\sigma(y))\sigma(z) + d(x)\tau(y)d(z), z]_{\sigma,\tau} = 0$ for all $x, y, z \in R$ by the definition of $C_{\sigma,\tau}$. Hence

$$[d(x)\tau(y)d(z), z]_{\sigma,\tau} = 0 \quad \text{for all } x, y, z \in R. \tag{2.5}$$

That is $d(x)[\tau(y)d(z), z]_{\sigma,\tau} + [d(x), \tau(z)]\tau(y)d(z) = 0$ for all $x, y, z \in R$. Taking $\tau^{-1}(d(t))y$ instead of y in this gives

$$[d(x), \tau(z)]d(t)\tau(y)d(z) = 0 \quad \text{for all } x, y, z, t \in R.$$

By the primeness of R and $\tau \in \mathrm{Aut}R$, we have either $[d(x), \tau(z)]d(t) = 0$ or $d(z) = 0$ for each $z \in R$. We set $K = \{z \in R \mid [d(x), \tau(z)]d(t) = 0$ for all $x, t \in R\}$ and $L = \{z \in R \mid d(z) = 0\}$. Then it can be easily seen that K and L both are additive subgroups of R and whose union R. Then by Brauer's trick, either $K = R$ or $L = R$. If $L = R$, then $d(R) = 0$, and so R is a commutative ring by [4, Lemma 2].

Now let $K = R$. Hence $[d(x), \tau(z)]d(t) = 0$ for all $x, z, t \in R$. Hence $0 = [d(x), \tau(rz)]d(t) = [d(x), \tau(r)]\tau(z)d(t) + \tau(r)[d(x), \tau(z)]d(t)$ for all $x, z, t, r \in R$. Since the second summand is zero, it is clear that

$$[d(x), r]Rd(t) = 0 \quad \text{for all } r, x, t \in R.$$

By the primeness of R, we have $d(R) \subset Z$ or $d(R) = 0$, and so $d(R) \subset Z$. Thus the proof completed by [4, Lemma 2]. $\square$

Corollary 3. *Let d be a nonzero (σ, τ)-derivation of a prime ring R with* $\mathrm{char}R \neq 2$. *If $d(x)d(y) - x\sigma(y) \in C_{\sigma,\tau}$ for all $x, y \in R$, then R is a commutative ring.*

Proceeding on the same lines with necessary variations we can prove the following theorem.

Theorem 4. *Let (f, d) be a generalized (σ, τ)-derivation of a prime ring R with* $\mathrm{char}R \neq 2$. *If $d(x)f(y) + x\sigma(y) \in C_{\sigma,\tau}$ for all $x, y \in R$, and if $d \neq 0$, then R is a commutative ring.*

Corollary 4. *Let d be a nonzero (σ, τ)-derivation of a prime ring R with* $\mathrm{char}R \neq 2$. *If $d(x)d(y) + x\sigma(y) \in C_{\sigma,\tau}$ for all $x, y \in R$, then R is a commutative ring.*

Theorem 5. *Let (f,d) be a generalized (σ,τ)-derivation of a prime ring R with* char$R \neq 2$. *If $f(xy) - x\sigma(y) \in C_{\sigma,\tau}$ for all $x,y \in R$, and if $d \neq 0$, then R is a commutative ring.*

Proof. If $f = 0$, then $x\sigma(y) \in C_{\sigma,\tau}$ for all $x,y \in R$. Using the same arguments in the begining of the proof of Theorem 3, we get the required result.

Assume that $f \neq 0$ and $f(xy) - x\sigma(y) \in C_{\sigma,\tau}$ for all $x,y \in R$. That is

$$f(x)\sigma(y) + \tau(x)d(y) - x\sigma(y) \in C_{\sigma,\tau} \text{ for all } x,y \in R. \tag{2.6}$$

Substitute yz instead of y in (2.6) obtaining

$$(f(x)\sigma(y) + \tau(x)d(y) - x\sigma(y))\sigma(z) + \tau(x)\tau(y)d(z) \in C_{\sigma,\tau} \text{ for all } x,y,z \in R.$$

By the definition of $C_{\sigma,\tau}$, commutting this term with z gives

$$[\tau(x)\tau(y)d(z), z]_{\sigma,\tau} = 0 \text{ for all } x,y,z \in R.$$

Replacing y by ty in this equation and using this, we get

$$\tau([x,z])\tau(ty)d(z) = 0 \text{ for all } x,y,z,t \in R.$$

Since τ is an automorphism of R, we have

$$[x,z]Rd(z) = 0 \text{ for all } x,z \in R.$$

Using the same arguments in the proof of Theorem 2, we find the required result. □

Corollary 5. *Let d be a nonzero (σ,τ)-derivation of a prime ring R with* char$R \neq 2$. *If $d(xy) - x\sigma(y) \in C_{\sigma,\tau}$ for all $x,y \in R$, then R is a commutative ring.*

Theorem 6. *Let (f,d) be a generalized (σ,τ)-derivation of a prime ring R with* char$R \neq 2$. *If $f(xy) + x\sigma(y) \in C_{\sigma,\tau}$ for all $x,y \in R$, and if $d \neq 0$, then R is a commutative ring.*

Proof. If f is a generalized derivation satisfying the property $f(xy) + x\sigma(y) \in C_{\sigma,\tau}$ for all $x,y \in R$, then $(-f)$ satisfies the condition $(-f)(xy) + x\sigma(y) \in C_{\sigma,\tau}$ for all $x,y \in R$, and hence by Theorem 5, R is a commutative ring. □

Corollary 6. *Let d be a nonzero (σ,τ)-derivation of a prime ring R with* char$R \neq 2$. *If $d(xy) + x\sigma(y) \in C_{\sigma,\tau}$ for all $x,y \in R$, then R is a commutative ring.*

Theorem 7. *Let (f,d) be a generalized (σ,τ)-derivation of a prime ring R with* char$R \neq 2$. *If $f(xy) - y\sigma(x) \in C_{\sigma,\tau}$ for all $x,y \in R$, and if $d \neq 0$, then R is a commutative ring.*

Proof. If $f = 0$, then $y\sigma(x) \in C_{\sigma,\tau}$ for all $x,y \in R$. Using the same arguments in the begining of the proof of Theorem 3, we get the required result.

Suppose that $f \neq 0$ and $f(xy) - y\sigma(x) \in C_{\sigma,\tau}$ for all $x,y \in R$. That is

$$f(x)\sigma(y) + \tau(x)d(y) - y\sigma(x) \in C_{\sigma,\tau} \quad \text{for all } x,y \in R. \tag{2.7}$$

Replacing x by xy in (2.7) yields that

$$f(xy)\sigma(y) + \tau(x)\tau(y)d(y) - y\sigma(x)\sigma(y) \in C_{\sigma,\tau} \text{ for all } x, y \in R.$$

Then it is clear that $[f(xy)\sigma(y) + \tau(x)\tau(y)d(y) - y\sigma(x)\sigma(y), y]_{\sigma,\tau} = 0$ for all $x, y \in R$ by the definition of $C_{\sigma,\tau}$. Hence

$$[\tau(x)\tau(y)d(y), y]_{\sigma,\tau} = 0 \quad \text{for all } x, y \in R,$$

and so

$$[\tau(x), y]_{\sigma,\tau}\tau(y)d(y) + \tau(x)[\tau(y)d(y), \sigma(y)] = 0 \text{ for all } x, y \in R. \tag{2.8}$$

Substitute xr instead of x in (2.8) and using this, we arrive at

$$\tau([x, y])\tau(ry)d(y) = 0 \quad \text{for all } x, y, r \in R,$$

and so

$$[x, y]R\tau(y)d(y) = 0 \quad \text{for all } x, y \in R.$$

By the primeness of R, we have $y \in Z$ or $\tau(y)d(y) = 0$ for each $y \in R$.

Let $y \in Z$. By the hypothesis, we have

$$[f(x)\sigma(y) + \tau(x)d(y) - y\sigma(x), z]_{\sigma,\tau} = 0 \quad \text{for all } x \in R.$$

Expanding this equation and using $\sigma(y), d(y), y \in Z$, we arrive at

$$[f(x), z]_{\sigma,\tau}\sigma(y) + [\tau(x), z]_{\sigma,\tau}d(y) - y[\sigma(x), z]_{\sigma,\tau} = 0 \quad \text{for all } x, z \in R.$$

Replacing x by xy in this equation and using this, we have

$$[\tau(x), z]_{\sigma,\tau}\tau(y)d(y) = 0 \quad \text{for all } x, z \in R.$$

Hence $[\tau(tx), z]_{\sigma,\tau}\tau(y)d(y) = \tau(t)[\tau(x), z]_{\sigma,\tau}\tau(y)d(y) + \tau([t, z])\tau(x)\tau(y)d(y) = 0$ for all $x, z, t \in R$. Since the first summand is zero, it is clear that

$$[t, z]R\tau(y)d(y) = 0 \quad \text{for all } t, z \in R.$$

Thus we obtain that R is a commutative ring or $\tau(y)d(y) = 0$. The second case using $\sigma(y) \in Z$ and the primeness of R, we have $y = 0$ or $\tau(y)d(y) = 0$, and so $\tau(y)d(y) = 0$. Thus we obtain that for all $y \in R$, $\tau(y)d(y) = 0$ for any cases.

Let $\tau(y)d(y) = 0$ for all $y \in R$. Hence

$$\tau(x + y)d(x + y) = \tau(x)d(x) + \tau(x)d(y) + \tau(y)d(y) + \tau(y)d(x) = 0$$

and so

$$\tau(x)d(y) + \tau(y)d(x) = 0 \quad \text{for all } x, y \in R.$$

Taking yt instead of y in this equation, we obtain that

$$\tau(x)d(y)\sigma(t) + \tau([x, y])d(t) = 0.$$

Again writing tz by t yields that $\tau([x, y])Rd(z) = 0$. Since R is a prime ring, we have R is a commutative ring or $d(R) = 0$. In the second case gives R is a commutative ring by [4, Lemma 2]. This complete the proof. $\square$

Corollary 7. *Let d be a nonzero (σ,τ)-derivation of a prime ring R with* $\mathrm{char}R \neq 2$. *If $d(xy) - y\sigma(x) \in C_{\sigma,\tau}$ for all $x, y \in R$, then R is a commutative ring.*

Using similar arguments as above, we can prove the following:

Theorem 8. *Let (f,d) be a generalized (σ,τ)-derivation of a prime ring R with* $\mathrm{char}R \neq 2$. *If $f(xy) + y\sigma(x) \in C_{\sigma,\tau}$ for all $x, y \in R$, and if $d \neq 0$, then R is a commutative ring.*

Corollary 8. *Let d be a nonzero (σ,τ)-derivation of a prime ring R with* $\mathrm{char}R \neq 2$. *If $d(xy) + y\sigma(x) \in C_{\sigma,\tau}$ for all $x, y \in R$, then R is a commutative ring.*

Theorem 9. *Let (f,d) be a generalized (σ,τ)-derivation of a prime ring R with* $\mathrm{char}R \neq 2$. *If $f(x)f(y) - x\sigma(y) \in C_{\sigma,\tau}$ for all $x, y \in R$, and if $d \neq 0$, then R is a commutative ring.*

Proof. If $f = 0$, then $x\sigma(y) \in C_{\sigma,\tau}$ for all $x, y \in R$. Using the same arguments in the begining of the proof of Theorem 3, we get the required result.

Assume that $f \neq 0$. Replacing y by yz in the hypothesis yields that

$$f(x)f(y)\sigma(z) - f(x)\tau(y)d(z) - x\sigma(y)\sigma(z) \in C_{\sigma,\tau} \quad \text{for all } x, y, z \in R. \tag{2.9}$$

By the definition of $C_{\sigma,\tau}$, we have

$$[f(x)\tau(y)d(z), z]_{\sigma,\tau} = 0 \quad \text{for all } x, y, z \in R. \tag{2.10}$$

Writing $\tau^{-1}(f(t))y$ by y in the last relation gives

$$[f(x), \tau(z)]f(t)Rd(z) = 0 \quad \text{for all } x, z, t \in R.$$

By the primeness of R, we have either $[f(x), \tau(z)]f(t) = 0$ or $d(z) = 0$ for each $z \in R$. We set $K = \{z \in R \mid [f(x), \tau(z)]f(t) = 0 \text{ for all } x, t \in R\}$ and $L = \{z \in R \mid d(z) = 0\}$. Then it can be easily seen that K and L both are additive subgroups of R and whose union R. Then by Brauer's trick, either $K = R$ or $L = R$. If $L = R$, then $d(R) = 0$, and so R is a commutative ring by [4, Lemma 2].

Now let assume $K = R$. Thus $[f(x), \tau(z)]f(t) = 0$ for all $x, t \in R$. Hence

$$[f(x), \tau(yz)]f(t) = [f(x), \tau(y)]\tau(z)f(t) + \tau(y)[f(x), \tau(z)]f(t) = 0$$

for all $x, z, t \in R$.

Since the second summand is zero, it is clear that

$$[f(x), \tau(y)]Rf(t) = 0 \quad \text{for all } x, y, t \in R.$$

By the primeness of R gives $f(R) \subset Z$ or $f(R) = 0$. Hence we have $f(R) \subset Z$ for any cases. Meanwhile according to (2.10), we have

$$f(x)[\tau(y)d(z), z]_{\sigma,\tau} + [f(x), \tau(z)]\tau(y)d(z) = 0,$$

and so

$$f(x)[\tau(y)d(z), z]_{\sigma,\tau} = 0 \quad \text{for all } x, y, z \in R.$$

Using $f(x) \in Z$, we conclude that

$$f(x) = 0 \text{ or } [\tau(y)d(z), z]_{\sigma,\tau} = 0 \quad \text{for all } x, y, z \in R.$$

Since $f \neq 0$, we suppose that $[\tau(y)d(z), z]_{\sigma,\tau} = 0$ for all $y, z \in R$. That is

$$\tau(y)[d(z), z]_{\sigma,\tau} + [\tau(y), \tau(z)]d(z) = 0 \quad \text{for all } y, z \in R.$$

Writing yr by y in the last relation gives

$$\tau([y, z])Rd(z) = 0 \quad \text{for all } y, z \in R.$$

Using the primeness of R, we obtain the following alternative: for all $z \in R$ we have either $z \in Z$ or $d(z) = 0$. By a standart argument one of these must hold for all $z \in R$. Using [4, Lemma 2], we conclude that R is a commutative ring. □

Corollary 9. *Let d be a nonzero (σ,τ)-derivation of a prime ring R with* $\mathrm{char}R \neq 2$. *If $d(x)d(y) - x\sigma(y) \in C_{\sigma,\tau}$ for all $x, y \in R$, then R is a commutative ring.*

Using similar arguments as above, we can prove the following:

Theorem 10. *Let (f, d) be a generalized (σ,τ)-derivation of a prime ring R with* $\mathrm{char}R \neq 2$. *If $f(x)f(y) + x\sigma(y) \in C_{\sigma,\tau}$ for all $x, y \in R$, and if $d \neq 0$, then R is a commutative ring.*

Corollary 10. *Let d be a nonzero (σ,τ)-derivation of a prime ring R with* $\mathrm{char}R \neq 2$. *If $d(x)d(y) + x\sigma(y) \in C_{\sigma,\tau}$ for all $x, y \in R$, then R is a commutative ring.*

Theorem 11. *Let (f, d) and (g, h) be two generalized (σ,τ)-derivations of a prime ring R with* $\mathrm{char}R \neq 2$. *If $f(x)\sigma(y) = \tau(x)g(y)$ for all $x, y \in R$, then R is a commutative ring.*

Proof. If either $f = 0$ or $g = 0$, then we get $\tau(x)g(y) = 0$ for all $x, y \in R$ ($f(x)\sigma(y) = 0$ for all $x, y \in R$). Replacing y by yz (or x by xz), we have $\tau(xy)h(z) = 0$, (or $\tau(x)d(z)\sigma(y) = 0$) for all $x, y, z \in R$. By the primeness of R, we have $h = 0$ (or $d = 0$). Thus R is a commutative ring by [4, Lemma 2]. So we may assume that $f \neq 0$ and $g \neq 0$.

Now let $f(x)\sigma(y) = \tau(x)g(y)$ for all $x, y \in R$. Replacing x by xz, we get $f(xz)\sigma(y) = \tau(xz)g(y)$ for all $x, y, z \in R$. Hence we find that

$$f(x)\sigma(z)\sigma(y) + \tau(x)d(z)\sigma(y) = \tau(x)\tau(z)g(y) \quad \text{for all } x, y, z \in R.$$

Using our hypothesis, the above relation yields that

$$\tau(x)(g(z)\sigma(y) + d(z)\sigma(y) - \tau(z)g(y)) = 0,$$

and so

$$g(z)\sigma(y) + d(z)\sigma(y) - \tau(z)g(y) = 0 \quad \text{for all } y, z \in R. \tag{2.11}$$

Replacing y by yr in (2.11) and using this, we get

$$\tau(zy)h(r) = 0 \quad \text{for all } y, z, r \in R.$$

Since R is a prime ring and $\tau \in \mathrm{Aut}R$, we obtain that $h(R) = 0$. Thus R is a commutative ring by [4, Lemma 2]. $\square$

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