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ON RADICALLY-SYMMETRIC IDEALS

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Abstract. A ring R is called symmetric, if $abc = 0$ implies $acb = 0$ for $a, b, c \in R$. An ideal I of a ring R is called symmetric (resp. radically-symmetric) if R/I (resp. $R/\sqrt{I}$) is a symmetric ring. We first show that symmetric ideals and ideals which have the insertion of factors property are radically-symmetric. We next show that if R is a semicommutative ring, then $T_n(R)$ and $R[x]/(x^n)$ are radically-symmetric, where (x^n) is the ideal of $R[x]$ generated by x^n. Also we give some examples of radically-symmetric ideals which are not symmetric. Connections between symmetric ideals of R and related ideals of some ring extensions are also shown. In particular we show that if R is a symmetric (or semicommutative) (α, δ)-compatible ring, then $R[x; \alpha, \delta]$ is a radically-symmetric ring. As a corollary we obtain a generalization of [13].

0. Introduction

Throughout this paper R denotes an associative ring with identity and $R[x; \alpha, \delta]$ will stands for the Ore extension of R, where α is an endomorphism and δ an α-derivation of R, that is, δ is an additive map such that $\delta(ab) = \delta(a)b + \alpha(a)\delta(b)$ for all $a, b \in R$. Recall from [14] that an ideal I of a ring R has the *insertion of factors property* (or simply, IFP) if $ab \in I$ implies $aRb \subseteq I$ for $a, b \in R$ (H. E. Bell in 1970 introduced this notion for $I = 0$). Observe that every *completely semiprime* ideal I (i.e., $a^2 \in I$ implies $a \in I$) of R has the IFP [13, Lemma 3.2(a)]. If $I = 0$ has the IFP, then R has the IFP (i.e., *semicommutative*). A ring R is called *reduced* if it has no non-zero nilpotent element. By [5], reduced rings have the IFP. If R has the IFP, then it is *Abelian* (i.e., all idempotents are central).

Liang, Wang and Liu [13] introduced weakly semicommutative rings which are a generalization of semicommutative rings. A ring R is called *weakly semicommutative* if for any $a, b \in R, ab = 0$ implies arb is a nilpotent element for any $r \in R$.

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According to Hong et al. [6], for an endomorphism α of a ring R, a *α-ideal* I (i.e., $\alpha(I) \subseteq I$) is called to be *α-rigid* if $a\alpha(a) \in I$ implies $a \in I$ for $a \in R$. They studied connections between α-rigid ideals of R and related ideals of some ring extensions.

Recall from [3], that an ideal I is called *α-compatible* if for each $a, b \in R$, $ab \in I \Leftrightarrow a\alpha(b) \in I$. Moreover, I is said to be *δ-compatible* if for each $a, b \in R$, $ab \in I \Rightarrow a\delta(b) \in I$. If I is both α-compatible and δ-compatible, it called a *(α, δ)-compatible ideal.* If $I = 0$ is a (α, δ)-compatible ideal, we say that R is a (α, δ)-compatible ring. The definition is quite natural, in the light of its similarity with the notion of α-rigid ideals, in [3], the author show that I is a α-rigid ideal if and only if I is α-compatible and completely semiprime.

Following Lambek [12], an ideal I of a ring R is *symmetric* if $abc \in I$ implies $acb \in I$ for $a, b, c \in R$. A ring R is called *symmetric* if $I = 0$ is a symmetric ideal of R. It is obvious that each ideal of a commutative ring is symmetric. Reduced rings are symmetric by the results of Anderson and Camillo [1], but there are many non-reduced commutative (so symmetric) rings.

Kim and Lee [10] proved that if R is Armendariz, then the ordinary polynomial ring over R is symmetric if and only if R is symmetric. There is an example [8] of symmetric ring R for which the ring of polynomials $R[x]$ is not symmetric.

We say an ideal I of a ring R is *radically-symmetric* if $\sqrt{I}$ is a symmetric ideal of R. If $I = 0$ is a radically-symmetric ideal of R, we say R is a radically-symmetric ring.

In this paper we will show that for each $n \geq 2$, there exists a non-zero radically-symmetric ideal of the $n \times n$ upper triangular matrix ring over the ring of integers $\mathbb{Z}$ that is not symmetric. Also we will show that each ideal of R which has the IFP and each symmetric ideal of R are radically-symmetric. Thus radically-symmetric rings are a generalization of symmetric rings. We next show that if R is a semicommutative ringn, then $T_n(R)$ and $R[x]/(x^n)$ are radically-symmetric, where (x^n) is the ideal generated by x^n.

A natural question for a given class of ring is: How does the given class behave with respect to polynomial extensions? In Section 2, connections between symmetric ideals of R and related ideals of some ring extensions are also shown. In particular we will show that:

(1) If I is a symmetric (α, δ)-compatible ideal of R, then $I[x; \alpha, \delta]$ is a radically-symmetric ideal of $R[x; \alpha, \delta]$.

(2) If I is a (α, δ)-compatible ideal of R and has the IFP, then $I[x; \alpha, \delta]$ is a radically-symmetric ideal of $R[x; \alpha, \delta]$. As a corollary, if R is a symmetric (α, δ)-compatible ring, then $R[x; \alpha, \delta]$ is a radically-symmetric. Also, if R is a semicommutative (α, δ)-compatible ring, then $R[x; \alpha, \delta]$ is a radically-symmetric ring and hence weakly semicommutative ring. As a corollary we obtain a generalization of [13].

1. Examples

Recall that for an ideal I of R, $\sqrt{I}$ equals the intersection of all prime ideals containing I.

Definition 1.1. For an ideal I of a ring R we say I is radically-symmetric if $\sqrt{I}$ is a symmetric ideal of R. If $I = 0$ is a radically-symmetric ideal of R, we say R is a radically-symmetric ring.

Lemma 1.2. *For an ideal I of a ring R, the following statements are equivalent*:

(1) *I is symmetric*;

(2) *For any $a_1, \dots, a_n \in R$, $a_1 \cdots a_n \in I$ implies $a_{i_1} a_{i2} \cdots a_{i_n} \in I$ for each $\{i_1, i_2, \dots, i_n\} = \{1, 2, \dots, n\}$.*

Proof. (1)⇒(2). For $n = 3$ we have $1a_1(a_2a_3) = a_1a_2a_3 \in I$. Hence $a_2a_3a_1 = 1(a_2a_3)a_1 \in I$, since I is symmetric. By a similar argument one can show that $a_{i_1}a_{i_2}a_{i_3} \in I$ for each $\{i_1, i_2, i_3\} = \{1, 2, 3\}$. Now let $a_1 \cdots a_n \in I$. Then $(a_1a_2)a_3 \cdots a_n \in I$. By induction on n, $(a_1a_2)a_{i_3} \cdots a_{i_n} \in I$ for each $\{i_3, \dots, i_n\} = \{3, \dots, n\}$. Since $a_1a_2(a_3 \cdots a_n) \in I$ and I is symmetric, $a_1(a_3 \cdots a_n)a_2 \in I$. Then by the induction hypothesis, $(a_1a_3)a_{i_3} \cdots a_{i_n} \in I$ for each $\{i_3, \dots, i_n\} = \{2, 4, \dots, n\}$. Continuing this process yields $(a_1a_t)a_{i_3} \cdots a_{i_n} \in I$ for each $t = 2, \dots, n$ and $\{i_3, \dots, i_n\} = \{2, \dots, n\} - \{t\}$. Therefore $a_1a_{i2} \cdots a_{i_n} \in I$ for each $\{i_2, \dots, i_n\} = \{2, \dots, n\}$. By a similar argument we can show that $a_{i_1}a_{i2} \cdots a_{i_n} \in I$ for each $\{i_1, i_2, \dots, i_n\} = \{1, 2, \dots, n\}$.

(2)⇒(1). It is clear. □

Definition 1.3. For an ideal I of a ring R we say I has the radically insertion of factors property (or simply, radically IFP) if $\sqrt{I}$ has the IFP. If $I = 0$ has the radically IFP, we say R has the radically IFP.

Clearly, if $I = 0$ has the IFP, then R has the IFP (i.e., R is semicommutative). The following example shows that, there exists a ring R such that all non-zero ideals of R have the IFP but R does not has the IFP.

Example 1.4. Let $R = \left(\begin{smallmatrix} F & F \\ 0 & F \end{smallmatrix}\right)$, where F is a division ring. The only non-zero proper ideals of R are $I_1 = \left(\begin{smallmatrix} F & F \\ 0 & 0 \end{smallmatrix}\right)$, $I_2 = \left(\begin{smallmatrix} 0 & F \\ 0 & F \end{smallmatrix}\right)$ and $I_3 = \left(\begin{smallmatrix} 0 & F \\ 0 & 0 \end{smallmatrix}\right)$. Huh, Lee and Smoktunowicz [8], show that R/I_i is semicommutative for each i, but R isn't semicommutative.

By using Lemma 1.2 we have the following result.

Corollary 1.5. *Symmetric ideals have the IFP.*

Proposition 1.6. *Let an ideal I has the IFP. Then I is a radically-symmetric ideal.*

Proof. First we show that $\sqrt{I} = \{a \in R \,|\, a^n \in I \text{ for some } n \geq 1\}$. Clearly $\sqrt{I} \subseteq \{a \in R \,|\, a^n \in I \text{ for some } n \geq 1\}$. Let $a \in \{a \in R \,|\, a^n \in I \text{ for some } n \geq 1\}$. Then $a^n \in I$ for some $n \geq 1$. Hence $ar_1ar_2 \cdots ar_n \in I$ for each $r_1, r_2, \dots, r_n \in$

R, since I has the IFP. Thus $(aR)^n \subseteq I$. If P is a prime ideal of R containing I, then $(aR)^n \subseteq I \subseteq P$ implies $a \in P$. Hence $a \in \sqrt{I}$ and $\sqrt{I} = \{a \in R \,|a^n \in I$ for some $n \geq 1\}$.

Now, let $abc \in \sqrt{I}$. Then $(abc)^n \in I$ for some positive integer n. Since I has the IFP, by a simple computation one can show that $(acb)^{2n} \in I$. Therefore I is a radically-symmetric ideal of R. □

Corollary 1.7. *Let R be a semicommutative ring. Then R is a radically-symmetric ring.*

For a ring R, let $R_n(R)$ be the set of all $n \times n$ upper-triangular matrices with constant main diagonal. Clearly, $R_n(R)$ is a subring of $T_n(R)$, the $n \times n$ upper triangular matrix ring over R. It is well known $R_n(R) \cong R[x]/(x^n)$, where (x^n) is the ideal of $R[x]$ generated by x^n. In the following we will see the converse of Proposition 1.6 is not true.

Example 1.8. Let $J = \left\{ \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ 0 & 0 & 0 & a_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix} |a_{ij} \in 2p\mathbb{Z} \right\}$ be an ideal of $R_4(\mathbb{Z})$, where $p \neq 2$ is a prime number and $\mathbb{Z}$ is the set of integers. Then $\begin{pmatrix} 0 & p & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & p \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 2p \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in J$, but $\begin{pmatrix} 0 & p & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & p \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 3p^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \notin J$. Hence J doesn't has the IFP, but J is radically-symmetric, by Proposition 1.11.

By a similar way as used in Example 1.8, we can construct numerous radically-symmetric ideals of $R_n(\mathbb{Z})$ such that don't have the IFP for $n \geq 4$.

Example 1.9. Let $J = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} |a_{ij} \in 2p\mathbb{Z} \right\}$ be an ideal of $T_3(\mathbb{Z})$, where p is a prime number and $\mathbb{Z}$ is the set of integers. Then $\begin{pmatrix} p & p & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & p \end{pmatrix} = \begin{pmatrix} 0 & 0 & 4p \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in J$, but $\begin{pmatrix} p & p & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & p \end{pmatrix} = \begin{pmatrix} 0 & 0 & 7p \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \notin J$. Hence J doesn't has the IFP, but is radically-symmetric by Proposition 1.15.

Let J be an ideal of $R_n(R)$ and

$$I = \left\{ a \in R| \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \in J \text{ for some } a_{ij} \in R \right\}.$$

Then I is an ideal of R.

Proposition 1.10. *Let J be an ideal of $R_n(R)$ such that $R_n(I) \subseteq J$, where I is the ideal that mentioned above. Let $A = \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \in R_n(R)$ such that $a^k \in I$ for some non-negative integer k. Then $A^{nk} \in J$.*

Proof. We proceed by induction on n. Let $n = 2$. For a positive integer k, $A^k = \begin{pmatrix} a^k & b_{12} \\ 0 & a^k \end{pmatrix}$ and that $A^{2k} = \begin{pmatrix} a^{2k} & a^k b_{12}+b_{12}a^k \\ 0 & a^{2k} \end{pmatrix}$. Hence $A^{2k} \in J$, since $a^{2k}, a^k b_{12} + b_{12}a^k \in I$. Now, let $A = \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \in R_n(R)$ such that $a^k \in I$ for a non-negative integer k. Consider $A^{(n-1)k} = \begin{pmatrix} a^{(n-1)k} & b_{12} & \cdots & b_{1n} \\ 0 & a^{(n-1)k} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a^{(n-1)k} \end{pmatrix}$ and $A^k = \begin{pmatrix} a^k & c_{12} & \cdots & c_{1n} \\ 0 & a^k & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a^k \end{pmatrix}$. By the induction hypothesis all b_{ij}'s, except b_{1n}, are in I. Let $x = a^k b_{1n} + c_{12}b_{2n} + \cdots + c_{1n}a^{(n-1)k}$. Then $A^{nk} = \begin{pmatrix} a^{nk} & y_{12} & \cdots & y_{1n-1} & x \\ 0 & a^{nk} & \cdots & y_{2n-1} & y_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a^{nk} & y_{n-1n} \\ 0 & 0 & \cdots & 0 & a^{nk} \end{pmatrix} \in J$, since a^{nk}, x and all y_{ij}'s are in I. □

Proposition 1.11. *Let J be an ideal of $R_n(R)$ such that $R_n(I) \subseteq J$, where I is the ideal that mentioned above. If I has the IFP, then J is a radically-symmetric ideal of $R_n(R)$ for each $n \geq 2$.*

Proof. Let $A = \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix}$, $B = \begin{pmatrix} b & b_{12} & \cdots & b_{1n} \\ 0 & b & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b \end{pmatrix}$ and $C = \begin{pmatrix} c & c_{12} & \cdots & c_{1n} \\ 0 & c & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c \end{pmatrix} \in R_n(R)$ such that $ABC \in \sqrt{J}$. Then $(abc)^k \in I$ for some positive integer k. Since I has the IFP, one can show that $(acb)^{2k} \in I$. Thus $ACB \in \sqrt{J}$, by Proposition 1.10. Therefore J is a radically-symmetric ideal of $R_n(R)$. □

By using Proposition 1.11 we have the following theorem.

Theorem 1.12. *Let R be a semicommutative ring. Then $R_n(R)$ is a radically-symmetric ring.*

Lemma 1.13. *Let* $J = \left\{ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} | a_{ij} \in I_{ij}, 1 \leq i \leq n, i \leq j \leq n \right\}$, *such that $I_{ij} \subseteq I_{is}$ for $1 \leq i \leq n, i \leq j \leq s \leq n$ and $I_{sj} \subseteq I_{ij}$ for $j = 1, \ldots, n$, $1 \leq i \leq s \leq n$ and I_{ij} is an ideal of R for each i, j. Then J is an ideal of $T_n(R)$.*

Proof. It is straightforward. □

Proposition 1.14. *Let J be an ideal of $T_n(R)$ that mentioned in Lemma 1.13. Let* $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \in T_n(R)$ *such that $a_{ii}^k \in I_{ii}$ for some non-negative integer k and $i = 1, \ldots, n$. Then $(A^{2k+1})^{n-1} \in J$.*

Proof. We proceed by induction on n. For $n = 2$, let $A = \left(\begin{smallmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{smallmatrix}\right)$. Since $A^{2k+1} = \begin{pmatrix} a_{11}^{2k+1} & x \\ 0 & a_{22}^{2k+1} \end{pmatrix}$, where $x = \sum a_{11}^i a_{12} a_{22}^j$, $i + j = 2k, i, j \geq 0$, we have $A^{2k+1} \in J$. Now, assume $n \geq 3$ and $A \in T_n(R)$. Consider $(A^{2k+1})^{n-2} =$

$$\begin{pmatrix} a_{11}^{(2k+1)(n-2)} & b_{12} & \cdots & b_{1n} \\ 0 & a_{22}^{(2k+1)(n-2)} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn}^{(2k+1)(n-2)} \end{pmatrix} \text{ and } A^{2k+1} = \begin{pmatrix} a_{11}^{2k+1} & c_{12} & \cdots & c_{1n} \\ 0 & a_{22}^{2k+1} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn}^{2k+1} \end{pmatrix}.$$

By the induction hypothesis all b_{ij}'s, except b_{1n}, are in I. Hence $(1, n)$-entry of $(A^{2k+1})^{n-1}$ is $x = a_{11}^{(2k+1)} b_{1n} + c_{12} b_{2n} + \cdots + c_{1n-1} b_{n-1n} + c_{1n} a_{nn}^{(2k+1)(n-2)} \in I$, since $a_{11}^{(2k+1)}, a_{nn}^{(2k+1)}, b_{2n}, \ldots, b_{n-1n} \in I$. Therefore $(A^{2k+1})^{n-1} \in J$. $\square$

Proposition 1.15. *Let J be an ideal of $T_n(R)$ that mentioned in Lemma 1.13. If each I_{ii}, $1 \leq i \leq n$ has the IFP, then J is radically-symmetric.*

Proof. Let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{nn} \end{pmatrix}$ and $C = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ 0 & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_{nn} \end{pmatrix}$ $\in T_n(R)$ such that $ABC \in \sqrt{J}$. Then $(a_{ii}b_{ii}c_{ii})^k \in I_{ii}$ for a positive integer k and each i. Since I_{ii} has the IFP, one can show that $(a_{ii}c_{ii}b_{ii})^{2k} \in I_{ii}$ for each i. Thus $ACB \in \sqrt{J}$, by Proposition 1.14. Therefore J is a radically-symmetric ideal of $T_n(R)$. $\square$

By using Proposition 1.15 we have the following theorem.

Theorem 1.16. *If R is a semicommutative ring, then $T_n(R)$ is a radically-symmetric ring for each $n \geq 2$.*

2. Extensions of symmetric ideals

Definition 2.1. For an ideal I of R, we say that I is α-compatible if for each $a, b \in R$, $ab \in I \Leftrightarrow a\alpha(b) \in I$. Moreover, I is said to be δ-compatible if for each $a, b \in R$, $ab \in I \Rightarrow a\delta(b) \in I$. If I is both α-compatible and δ-compatible, we say that I is (α, δ)-compatible. If $I = 0$ is a (α, δ)-compatible ideal, we say R is a (α, δ)-compatible ring.

Note that there exists a ring R for which all non-zero proper ideals are α-compatible but R isn't α-compatible. For example, consider the ring $R = \left(\begin{smallmatrix} F & F \\ 0 & F \end{smallmatrix}\right)$, where F is a field, and the endomorphism α of R is defined by $\alpha(\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)) = \left(\begin{smallmatrix} a & 0 \\ 0 & c \end{smallmatrix}\right)$ for $a, b, c \in F$.

Proposition 2.2 ([3])**.** *Let R be a ring, J an ideal of R and $\alpha : R \to R$ an endomorphism of R. Then the following conditions are equivalent*:

(1) *J is an α-rigid ideal of R*;
(2) *J is α-compatible, semiprime and has the IFP*;
(3) *J is α-compatible and completely semiprime.*

If δ is an α-derivation of R, then the following are equivalent:

(4) *J is an α-rigid δ-ideal of R*;
(5) *J is (α, δ)-compatible, semiprime and has the IFP*;
(6) *J is (α, δ)-compatible and completely semiprime.*

Proposition 2.3. *Let I be a (α, δ)-compatible ideal of R and $a, b \in R$.*

(1) *If $ab \in I$, then $a\alpha^n(b), \alpha^n(a)b \in I$ for every positive integer n. Conversely, if $a\alpha^k(b)$ or $\alpha^k(a)b \in I$ for some positive integer k, then $ab \in I$.*
(2) *If $ab \in I$, then $\alpha^m(a)\delta^n(b), \delta^n(a)\alpha^m(b) \in I$ for each non-negative integers m, n.*

Proof. (1) If $ab \in I$, then $\alpha^n(a)\alpha^n(b) \in I$, since I is α-ideal. Hence $\alpha^n(a)b \in I$, since I is α-compatible. If $\alpha^k(a)b \in I$, then $\alpha^k(a)\alpha^k(b) \in I$, and so $ab \in I$, since I is α-compatible.

(2) It is enough to show that $\delta(a)\alpha(b) \in I$. If $ab \in I$, then by (1) and δ-compatibility of I, $\alpha(a)\delta(b) \in I$. Hence $\delta(a)b = \delta(ab) - \alpha(a)\delta(b) \in I$. Thus $\delta(a)b \in I$ and $\delta(a)\alpha(b) \in I$, since I is α-compatible. □

Lemma 2.4. *Let I be a (α, δ)-compatible ideal of R. If $(ab)^k \in I$ for some $k \geq 0$, then $(a\alpha(b))^k, (a\delta(b))^k \in I$.*

Proof. Since I is α-compatible and $(ab)^k = (ab)\cdots(ab) \in I$ we have $a\alpha(b)\alpha(ab \cdots ab)$=$a\alpha(bab \cdots ab) \in I$. Hence $a\alpha(b)(ab \cdots ab) \in I$, since I is α-compatible. Now, $a\alpha(b)a\alpha(b)\alpha(ab \cdots ab) = a\alpha(b)a\alpha(b \cdots ab) \in I$. Continuing this procedure yields $(a\alpha(b))^k \in I$. Since I is δ-compatible and $(ab)^k = (ab)\cdots(ab) \in I$, we have $a\delta(bab \cdots ab) = a\delta(b)(ab \cdots ab) + a\alpha(b)\delta(ab \cdots ab) \in I$. Since $a\alpha(b)(ab \cdots ab) \in I$ and I is δ-compatible, we have $a\alpha(b)\delta(ab \cdots ab) \in I$. Thus $a\delta(b)(ab \cdots ab) \in I$. Continuing this procedure yields $(a\delta(b))^k \in I$. □

Lemma 2.5. *Let I be a (α, δ)-compatible ideal of R and has the IFP. Then*

(1) *$\sqrt{I}$ is a (α, δ)-compatible ideal of R and has the IFP.*
(2) *$I[x; \alpha, \delta]$ and $\sqrt{I}[x; \alpha, \delta]$ are ideals of $R[x; \alpha, \delta]$.*

Proof. (1) By the proof of Proposition 1.6, $\sqrt{I} = \{a \in R \,|\, a^n \in I \text{ for some } n \geq 1\}$, hence the result follows from Lemma 2.4 and Proposition 2.3.

(2) It follows from (α, δ)-compatibility of I and $\sqrt{I}$. □

In [6, Example 2], the authors show that there exists a non-zero ideal I of a ring R such that has the IFP but ideal $I[x]$ of $R[x]$ isn't symmetric. In the sequel we will show that if I has the IFP, then $I[x]$ is radically-symmetric and hence has the radically IFP. More generally, we will show that: (1) If I

is a (α, δ)-compatible ideal of R and has the IFP, then the ideal $I[x;\alpha,\delta]$ of $R[x;\alpha,\delta]$ is radically-symmetric and hence has the radically IFP.

For non-empty subsets A, B of R and $r \in R$, put $AB = \{ab|a \in A, b \in B\}$, $A^0 = \{1\}$ and $rA = \{ra|a \in A\}$.

Notation. Let α be an endomorphism, δ an α-derivation of R, $0 \leq i \leq j$ and $a \in R$. Let us write f_i^j for the set of all "words" in α and δ in which there are i factors of α and $j-i$ factors of δ. For instance, $f_j^j(a) = \{\alpha^j(a)\}$, $f_0^j(a) = \{\delta^j(a)\}$ and $f_{j-1}^j(a) = \{\alpha^{j-1}\delta(a), \alpha^{j-2}\delta\alpha(a), \ldots, \delta\alpha^{j-1}(a)\}$.

Lemma 2.6. *Let I be a (α,δ)-compatible ideal of R and has the IFP. Then $\sqrt{I}[x;\alpha,\delta] = \{f \in R[x;\alpha,\delta]\,|f^k \in I[x;\alpha,\delta]$ for some $k \geq 1\}$.*

Proof. Note that $\sqrt{I} = \{a \in R\,|a^n \in I$ for some $n \geq 1\}$, by the proof of Proposition 1.6. Let $f(x) = a_0 + \cdots + a_n x^n \in \{f \in R[x;\alpha,\delta]\,|f^n \in I[x;\alpha,\delta]$ for some $k \geq 1\}$. Then $(f(x))^k \in I[x;\alpha,\delta]$ for some positive integer k and $a_n\alpha^n(a_n)\cdots\alpha^{k(n-1)}(a_n) \in I$, since it is the leading coefficient of $(f(x))^k$. Hence $a_n \in \sqrt{I}$, since $\sqrt{I}$ is α-compatible. Since $\sqrt{I}[x;\alpha,\delta]$ is an ideal of $R[x;\alpha,\delta]$ and $a_n \in \sqrt{I}$, we have $a_n x^n \in \sqrt{I}[x;\alpha,\delta]$. There exists $g(x), h(x) \in R[x;\alpha,\delta]$ such that $f(x)^k = (a_0 + \cdots + a_{n-1}x^{n-1})^k + a_n x^n g(x) + h(x) a_n x^n$. Hence $(a_0 + \cdots + a_{n-1}x^{n-1})^k \in \sqrt{I}[x;\alpha,\delta]$, since $\sqrt{I}[x;\alpha,\delta]$ is an ideal of $R[x;\alpha,\delta]$ and $a_n x^n \in \sqrt{I}[x;\alpha,\delta]$. By using induction on n, we have $a_i \in \sqrt{I}$ for each i. Thus $\{f \in R[x;\alpha,\delta]\,|f^k \in I[x;\alpha,\delta]$ for some $k \geq 1\} \subseteq \sqrt{I}[x;\alpha,\delta]$.

Now, let $f(x) = a_0 + \cdots + a_n x^n \in \sqrt{I}[x;\alpha,\delta]$. Then $a_i^{m_i} \in I$ for some $m_i \geq 1$. Let $k = m_0 + \cdots + m_n + 1$. Then

$$(f(x))^k = \sum (a_0^{i_{01}}(a_1x)^{i_{11}} \cdots (a_nx^n)^{i_{n1}}) \cdots (a_0^{i_{0k}}(a_1x)^{i_{1k}} \cdots (a_nx^n)^{i_{nk}}),$$

where $i_{0r} + i_{1r} + \cdots + i_{nr} = 1$ and $0 \leq i_{rs} \leq 1$ for $r = 1, \ldots, k$. Each coefficient of $(a_0^{i_{01}}(a_1x)^{i_{11}} \cdots (a_nx^n)^{i_{n1}}) \cdots (a_0^{i_{0k}}(a_1x)^{i_{1k}} \cdots (a_nx^n)^{i_{nk}})$ is a sum of such elements $\gamma \in ((f_{r_{01}}^{s_{01}}(a_0))^{i_{01}} \cdots (f_{r_{n1}}^{s_{n1}}(a_n))^{i_{n1}}) \cdots ((f_{r_{0k}}^{s_{0k}}(a_0))^{i_{0k}} \cdots (f_{r_{nk}}^{s_{nk}}(a_n))^{i_{nk}})$. It can be easily checked that there exists $a_t \in \{a_0, \ldots, a_n\}$ such that $i_{t1} + i_{t2} + \cdots + i_{tk} \geq m_t$. Since $a_t^{m_t} \in I$ and I is (α,δ)-compatible and has the IFP, hence by Proposition 2.3, $\gamma \in I$. Thus each coefficient of $(f(x))^k$ belong to I. Therefore $f(x) \in \{f \in R[x;\alpha,\delta]\,|f^k \in I[x;\alpha,\delta]$ for some $k \geq 1\}$. $\square$

Lemma 2.7. *Let I be a (α,δ)-compatible ideal of R and has the IFP and $f(x) = a_0 + \cdots + a_n x^n, g(x) = b_0 + \cdots + b_m x^m \in R[x;\alpha,\delta]$. Then*

(1) *$f(x)g(x) \in \sqrt{I}[x;\alpha,\delta]$ if and only if $a_i b_j \in \sqrt{I}$ for each i, j.*
(2) *$\sqrt{I}[x;\alpha,\delta]$ has the IFP.*

Proof. (1) Note that $f(x)g(x) = \sum_{i=0}^n \sum_{j=0}^m (a_i x^i)(b_j x^j)$. Then $a_n\alpha^n(b_m) \in \sqrt{I}$, since it is the leading coefficient of $f(x)g(x)$. Hence $a_n b_m \in \sqrt{I}$, since $\sqrt{I}$ is α-compatible. Thus $a_n f_i^j(b_m) \subseteq \sqrt{I}$ for each $0 \leq i \leq j$, by Proposition 2.3. Since the coefficient of x^{m+n-1} is $a_n\alpha^n(b_{m-1}) + a_{n-1}\alpha^{n-1}(b_m) + a_n r$, where r is a sum of such elements $\gamma \in f_{n-1}^n(b_m)$ and $a_n r \in \sqrt{I}$, we have $a_n\alpha^n(b_{m-1}) +$

$a_{n-1}\alpha^{n-1}(b_m) \in \sqrt{I}$. Hence $a_n\alpha^n(b_{m-1})b_m + a_{n-1}\alpha^{n-1}(b_m)b_m \in \sqrt{I}$ and that $a_{n-1}\alpha^{n-1}(b_m)b_m \in \sqrt{I}$, since $a_n\alpha^n(b_{m-1})b_m \in \sqrt{I}$. Thus $a_{n-1}b_m \in \sqrt{I}$, by Proposition 2.3 and Lemma 2.5(1). Hence $a_nb_{m-1} \in \sqrt{I}$. Consequently,

$$a_nf_i^j(b_m)\bigcup a_{n-1}f_i^j(b_m)\bigcup a_nf_i^j(b_{m-1}) \subseteq \sqrt{I} \text{ for each } 0 \leq i \leq j.$$

The coefficient of x^{m+n-2} is $a_n\alpha^n(b_{m-2})+a_{n-1}\alpha^{n-1}(b_{m-1})+a_{n-2}\alpha^{n-2}(b_m)+t$, where t is a sum of such elements $\gamma \in \bigcup_{0\leq i\leq j}[a_nf_i^j(b_m)\bigcup a_{n-1}f_i^j(b_m)\bigcup a_nf_i^j(b_{m-1})]$. By a similar way as above, one can show that$a_nb_{m-2}, a_{n-1}b_{m-1}$, $a_{n-2}b_m \in \sqrt{I}$. Continuing this process yields $a_ib_j \in \sqrt{I}$ for each i, j.

Conversely, suppose that $a_ib_j \in \sqrt{I}$ for each i, j. Since $\sqrt{I}$ is (α, δ)-compatible, $f(x)g(x) \in \sqrt{I}[x; \alpha, \delta]$.

(2) Let $h(x) = c_0 + c_1x + \cdots + c_kx^k \in R[x; \alpha, \delta]$ and $f(x)g(x) \in \sqrt{I}[x; \alpha, \delta]$. Then $a_ib_j \in \sqrt{I}$ for each i, j, by (1). Since $\sqrt{I}$ has the IFP, we have $a_ic_rb_j \in \sqrt{I}$ for each i, j, r. Then $f(x)h(x)g(x) \in \sqrt{I}[x; \alpha, \delta]$, since $\sqrt{I}$ is a (α, δ)-compatible ideal of R. Therefore $\sqrt{I}[x; \alpha, \delta]$ has the IFP. □

Proposition 2.8. *Let I be a (α, δ)-compatible ideal of R and has the IFP. Then* $\sqrt{I[x; \alpha, \delta]} = \sqrt{I}[x; \alpha, \delta] = \{f \in R[x; \alpha, \delta] \,|\, f^k \in I[x; \alpha, \delta]$ *for some* $k \geq 1\}$.

Proof. By Lemma 2.6, it is enugh to show that $\sqrt{I}[x; \alpha, \delta] \subseteq \sqrt{I[x; \alpha, \delta]}$. We show that if Q is a prime ideal of $R[x; \alpha, \delta]$ containing $I[x; \alpha, \delta]$, then $\sqrt{I} \subseteq Q$. Let $a \in \sqrt{I}$. Then $a^k \in I$ for some $k \geq 1$. Hence $ag_1ag_2 \cdots ag_k \in I[x; \alpha, \delta]$ for each $g_1, g_2, \ldots, g_k \in R[x; \alpha, \delta]$, since I is (α, δ)-compatible and has the IFP. Thus $(aR[x; \alpha, \delta])^k \subseteq I[x; \alpha, \delta] \subseteq Q$ implies $a \in Q$. Therefore $\sqrt{I}[x; \alpha, \delta] \subseteq Q$ and $\sqrt{I}[x; \alpha, \delta] \subseteq \sqrt{I[x; \alpha, \delta]}$. □

Theorem 2.9. *Let I be a (α, δ)-compatible ideal of R and has the IFP. Then $I[x; \alpha, \delta]$ is a radically-symmetric ideal of $R[x; \alpha, \delta]$.*

Proof. Let $f(x) = a_0+\cdots+a_nx^n, g(x) = b_0+\cdots+b_mx^m, h(x) = c_0+\cdots+c_kx^k \in R[x; \alpha, \delta]$ and $f(x)g(x)h(x) \in \sqrt{I[x; \alpha, \delta]} = \sqrt{I}[x; \alpha, \delta]$. Then $a_i(g(x)h(x)) \in \sqrt{I}[x; \alpha, \delta]$ for each $i = 0, 1, \ldots, n$, by Lemma 2.7. Hence $a_ib_jc_k \in \sqrt{I}$ for each i, j, k, by Lemma 2.7. Thus $a_ic_kb_j \in \sqrt{I}$ for each i, j, k, by Proposition 1.6. Therefore $f(x)h(x)g(x) \in \sqrt{I}[x; \alpha, \delta]$, since $\sqrt{I}$ is (α, δ)-compatible. □

By using Theorem 2.9 we have the following result:

Corollary 2.10. *Let R be a semicommutative (α, δ)-compatible ring. Then $R[x; \alpha, \delta]$ is a radically-symmetric ring.*

Corollary 2.11. *Let R be a semicommutative ring. Then R is a radically-symmetric ring.*

Lemma 2.12. *Let I be a radically-symmetric ideal of R. Then I has the radically IFP.*

Proof. Let $ab \in \sqrt{I}$. Then $abc \in \sqrt{I}$ for each $c \in R$, since $\sqrt{I}$ is an ideal of R. Hence $acb \in \sqrt{I}$, since I is a radically-symmetric ideal of R. Therefore I has the radically IFP. $\square$

Corollary 2.13 ([13, Theorem 3.1])**.** *Let R be a semicommutative α-compatible ring. Then $R[x;\alpha]$ is a weakly semicommutative ring.*

Proof. It follows from Lemma 2.12 and Corollary 2.10. $\square$

Since symmetric ideals have the IFP, hence we have the following result:

Theorem 2.14. *Let R be a symmetric (α, δ)-compatible ring. Then $R[x;\alpha,\delta]$ is a radically-symmetric ring and hence weakly semicommutative ring.*

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