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CONJUGACY CLASSES OF AUTOMORPHISMS p-GROUPS

Rubén A. Hidalgo

Abstract. In this paper we provide examples of pairs of conformally non-equivalent, but topologically equivalent, p-groups $H_1, H_2 < \mathrm{Aut}(S)$, where S is a closed Riemann surface of genus $g \geq 2$, so that S/H_j has genus zero and all its cone points are of order equal to p.

1. Introduction

We denote by $\mathrm{Aut}(S)$ the group of conformal automorphisms of a Riemann surface S. If S_1 and S_2 are Riemann surfaces, then we say that $H_1 < \mathrm{Aut}(S_1)$ and $H_2 < \mathrm{Aut}(S_2)$ are *topologically equivalent* (respectively, *conformally equivalent*) if there is an orientation preserving homeomorphism (respectively, a conformal homeomorphism) $f : S_1 \to S_2$ so that $H_2 = fH_1f^{-1}$. In this paper, we assume $S_1 = S_2$. Sources for the matter of characterization of topological conjugacy for surface automorphisms by certain purely algebraic data are J. Nielsen [14], W. J. Harvey [12], and J. Gilmann [9]. In general, it is not hard to construct an example of a closed Riemann surface S, of genus $g \geq 2$, and pairs $H_1, H_2 < \mathrm{Aut}(S)$, $H_1 \cong H_2$, so that H_1 and H_2 are topologically non-equivalent.

If $H < \mathrm{Aut}(S)$, where S is a closed Riemann surface of genus $g \geq 2$, and $R = S/H$, then denote by $\mathcal{M}_g(H)$ the locus in $\mathcal{M}_g$ (the moduli space of genus g) consisting of points parametrizing Riemann surfaces S' which admit a group H' of conformal automorphisms topologically equivalent to H. One has, of course, $\mathcal{M}_g(H) = \mathcal{M}_g(H')$. It is well known that $\mathcal{M}_g(H)$ is an irreducible subvariety of $\mathcal{M}_g$ of dimension $3g'-3+r$; where g' is the genus of R and r is the number of points over which the natural projection $S \to R$ is ramified ([5], [6]). Moreover, $\mathcal{M}_g(H)$ fails to be normal if and only if there is a Riemann surface S' of genus g admitting two groups $H_1, H_2 < \mathrm{Aut}(S')$ which are topologically equivalent to H, but not conformally equivalent to each other ([5], [6]).

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From now on, let S be a closed Riemann surface of genus $g \geq 2$, let p be a prime, let G be a finite p-group, and let $H_1, H_2 < \mathrm{Aut}(S)$ be so that $H_1 \cong H_2 \cong G$.

If $G \cong \mathbb{Z}_p$ and S/H_j (for $j = 1, 2$) has genus zero, then H_1 and H_2 are conformally equivalent. This fact is consequence of a classical Castelnuovo-Severi theorem [3] in the case that $g > (p-1)^2$ and, for the general case, this was proved by González-Diez in [4]. An alternative proof of this result was also obtained by Gromadzki [10]. If we drop the condition for S/H_j to be of genus zero, then this may be false [6]; but if $p \geq 2\gamma + 1$, where γ is the genus of S/H_j, then H_1 and H_2 are conformally equivalent [11] (in fact, the condition $p \geq 2\gamma + 1$ asserts that H_j is a p-Sylow subgroup of $\mathrm{Aut}(S)$).

If $p = 2$ and $G \cong \mathbb{Z}_8$, then in [7] there is a construction of (a 1-dimensional family) a closed Riemann surface S of genus $g = 9$ and $H_1, H_2 < \mathrm{Aut}(S)$ so that $H_j \cong \mathbb{Z}_8$ are topologically equivalent but not conformally equivalent. In this example, the quotient orbifolds S/H_j has signature $(0; 4, 4, 8, 8)$. A generalization has been provided in [2], where $G \cong \mathbb{Z}_{2^{n+1}}$ and $n \geq 2$, so that S/H_j (for $j = 1, 2$) has signature $(0; 2^n, 2^n, 2^{n+1}, 2^{n+1})$. Note that in these examples, there are cone points of order different from the prime $p = 2$.

If $G \cong \mathbb{Z}_p^n$, where $n \geq 2$ is an integer, then in [8] it is proved that if S/H_j (for $j = 1, 2$) has signature $(0; p, \overset{n+1}{\dots}, p)$, then H_1 and H_2 are conformally equivalent.

This makes us to wonder if the above is true without this relation between the exponent in the order of the group G and the number of ramification points in S/H_j which, in the above, are respectively n and $n+1$. The following provides counter-examples to such an expectation.

Theorem 1. (1) *Let $n \geq 3$ be an integer. Then there exists a prime p_n so that, for every prime $p \geq p_n$, there exists a closed Riemann surface S, of genus $g \geq 2$, and subgroups $H_1, H_2 < \mathrm{Aut}(S)$, $H_1 \cong H_2 \cong \mathbb{Z}_p^{n-1}$, with S/H_j of signature of the form $(0; p, \dots, p)$, which are topologically equivalent but not conformally equivalent.*

(2) *There exists a closed Riemann surface S, of genus $g = 5$, and subgroups $H_1, H_2 < \mathrm{Aut}(S)$, $H_1 \cong H_2 \cong \mathbb{Z}_2^2$, with S/H_j of signature of the form $(0; 2, 2, 2, 2, 2)$, which are topologically equivalent but not conformally equivalent.*

2. Proof of Theorem 1

Let us consider an integer $n \geq 3$ and p a prime (if $p = 2$, then we assume that $n \geq 4$). Let $\lambda_1, \dots, \lambda_{n-2} \in \mathbb{C} - \{0, 1\}$ be so that $\lambda_i \neq \lambda_j$, for $i \neq j$.

We assume that these values are so that the group of Möbius transformations keeping invariant the set $\{\infty, 0, 1, \lambda_1, \dots, \lambda_{n-2}\}$ is the trivial group.

Let us consider the closed Riemann surface S defined by the following equations

$$S = \left\{ \begin{array}{c} x_1^p + x_p^2 + x_3^p = 0 \\ \lambda_1 x_1^p + x_2^p + x_4^p = 0 \\ \vdots \\ \lambda_{n-2} x_1^p + x_2^p + x_{n+1}^p = 0 \end{array} \right\} \subset \mathbb{P}^n_{\mathbb{C}}.$$

Let $H = \langle a_1, \ldots, a_n \rangle \cong \mathbb{Z}_p^n$, where a_j is defined by multiplication of the coordinate x_j by $e^{2\pi i/p}$. Set $a_{n+1} = (a_1 a_2 \cdots a_n)^{-1}$. Then a_{n+1} is multiplication of the coordinate x_{n+1} by $e^{2\pi i/p}$.

It is well known [8] that $H < \mathrm{Aut}(S)$ with S/H of signature $(0; p, \overset{n+1}{\ldots}, p)$ and that H is the unique subgroup K, up to conjugation in $\mathrm{Aut}(S)$, satisfying that $K \cong \mathbb{Z}_p^n$ and that S/K has such a signature. Moreover, we may identify S/H with the orbifold whose Riemann surface structure is the Riemann sphere and whose cone points (all of them of order p) are given by ∞, 0, 1, λ_1, …, λ_{n-2}. The natural regular branched cover, with H as Deck group, is given by

$$\pi([x_1 : \cdots : x_{n+1}]) = -\left(\frac{x_2}{x_1}\right)^p.$$

The map π sends the set of fixed points of a_1 to ∞; the set of fixed points of a_2 to 0; the set of fixed points of a_3 to 1; and, for $j \in \{4, \ldots, n+1\}$, it sends the set of fixed points of a_j to λ_{j-3}.

It is also known that $\pi : S \to S/H$ is a homology branched cover, that is, if Γ is a Fuchsian group so that $\mathbb{H}^2/\Gamma = S/H$, then $S = \mathbb{H}^2/\Gamma'$, where Γ' denotes the derived subgroup of Γ, and $H = \Gamma/\Gamma'$.

Set $H_1 = \langle a_1, a_2, \ldots, a_{n-1} \rangle \cong \mathbb{Z}_p^{n-1}$ and $H_2 = \langle a_2, a_3, \ldots, a_n \rangle \cong \mathbb{Z}_p^{n-1}$. It is clear that, for $j = 1, 2$, S/H_j has signature $(0; p, \overset{(n-1)p}{\ldots}, p)$.

Let us consider an orientation preserving homeomorphism $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, of order two, so that $f(\infty) = \lambda_{n-3}$, $f(0) = 0$, $f(1) = 1$, $f(\lambda_1) = \lambda_1, \ldots, f(\lambda_{n-4}) = \lambda_{n-4}$, $f(\lambda_{n-2}) = \lambda_{n-2}$.

As $\pi : S \to S/H$ is a homology branched cover, the homeomorphism f lifts to an orientation preserving homeomorphism $\widehat{f} : S \to S$ so that $fHf^{-1} = H$ and, by the property of f at the cone point of S/H, that $fH_1f^{-1} = H_2$, that is, these two groups are topologically equivalent.

2.1. Part (1)

As a consequence of the results in [13], it is possible to find a prime number p_n so that, if $p \geq p_n$, then H is a normal subgroup of $\mathrm{Aut}(S)$. It follows that, in this case, $\mathrm{Aut}(S)/H$ acts as a group of conformal automorphisms of the orbifold S/H. As we have supposed that no Möbius transformation, different from the identity, may keep invariant the set $\{\infty, 0, 1, \lambda_1, \ldots, \lambda_{n-2}\}$, we have that $\mathrm{Aut}(S) = H$. It follows that, for $p \geq p_n$, the groups H_1 and H_2 cannot be conformally equivalent.

2.2. Part (2)

If $p = 2$ and $n = 4$, under the above conditions, we have (see [1]) that S is a closed Riemann surface of genus 5 for which $\mathrm{Aut}(S) = H$. If we set $H_1 = \langle a_1, a_2\rangle \cong \mathbb{Z}_2^2$ and $H_2 = \langle a_1, a_3\rangle \cong \mathbb{Z}_2^2$, then these two groups are topologically equivalent but cannot be conformally equivalent.

3. A final remark

In [13] is proved that, if we fix $\gamma, r \geq 0$ and $s \geq 1$ integers so that $2\gamma-2+r > 0$, then there exists a prime $q = q(\gamma, r, s)$ with the following property: "if $p \geq q$ is a prime, S is a closed Riemann surface of genus $g \geq 2$, $H_1, H_2 < \mathrm{Aut}(S)$, $|H_1| = |H_2| = p^s$, S/H_j has genus γ and exactly r cone points, then $H_1 = H_2$". This property is not in contradiction with Theorem 1. In fact, in our family of examples, the quotient S/H has signature $(0; p, \overset{n+1}{\ldots}, p)$ and the quotients S/H_j have signature $(0; p, \overset{2+(n-1)p}{\ldots}, p)$. We are considering primes $p \geq p_n = q(0, n+1, n)$. On the other hand, the prime $q(0, 2+(n-1)p, n-1)$ is necessarily bigger than p.

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Departamento de Matemática
Universidad Técnica Federico Santa María
Valparaíso, Chile
E-mail address: ruben.hidalgo@usm.cl