Bull. Korean Math. Soc. **48** (2011), No. 4, pp. 779–789
DOI 10.4134/BKMS.2011.48.4.779

ON THE LEBESGUE SPACE OF VECTOR MEASURES

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Abstract. In this paper we study the Banach space $L^1(G)$ of real valued measurable functions which are integrable with respect to a vector measure G in the sense of D. R. Lewis. First, we investigate conditions for a scalarly integrable function f which guarantee $f \in L^1(G)$. Next, we give a sufficient condition for a sequence to converge in $L^1(G)$. Moreover, for two vector measures F and G with values in the same Banach space, when F can be written as the integral of a function $f \in L^1(G)$, we show that certain properties of G are inherited to F; for instance, relative compactness or convexity of the range of vector measure. Finally, we give some examples of $L^1(G)$ related to the approximation property.

1. Introduction

Integration of real valued measurable functions with respect to Banach space valued countably additive vector measures was introduced and studied by Lewis in [7] and [8]. The Banach space $L^1(G)$ of real valued measurable functions integrable with respect to a Banach space valued countably additive vector measure G was studied from the aspect of Banach lattice by Curbera in [1], [2] and [3]. In particular, Curbera characterized $L^1(G)$ as order continuous Banach lattices with weak order unit.

In this paper we focus on different aspects of $L^1(G)$. Let X be a Banach space and G be a X-valued countably additive vector measure. After fixing notation and definitions in Preliminaries, we study in Section 3 conditions for a scalarly integrable function f which guarantee $f \in L^1(G)$.

In Section 4 we consider the convergence in $L^1(G)$ and for an integrable function f the induced vector measure F given by the integration

$$F(E) = \int_E f dG.$$

We give a sufficient condition for a sequence which guarantees the convergence in $L^1(G)$. Then this result improves Lewis's results. And, as a consequence

Received December 11, 2009; Revised April 28, 2010.
2010 *Mathematics Subject Classification.* Primary 28B10.
Key words and phrases. Lebesgue space of vector measure, convergence in $L^1(G)$, the range of vector measures, Lyapunov convexity theorem, the approximation property.

of the Orlicz-Pettis theorem, the induced vector measure F turns out to be a countably additive vector measure. Now, assuming that F is induced in the above fashion, we show that if G has a relatively compact range, then so does F. In the same vein we prove that if G satisfies the Lyapunov convexity theorem, then so does F.

In the final section we consider the approximation property for $L^1(G)$. We give some examples; we illustrate these examples using Szakowski's counterexample of a Banach lattice which lacks in the compact approximation property.

2. Preliminaries

Throughout this paper by X and Y we denote real Banach spaces. By B_X we mean the closed unit ball of X and X^* is the dual of X. For a measurable space (Ω, Σ) and a countably additive vector measure $G : \Sigma \to X$ we define its semivariation $\|G\|$ by

$$\|G\|(E) = \sup_{x^* \in B_{X^*}} |x^*G|(E).$$

Here $|x^*G|$ is the variation of the signed measure x^*G. It is well-known that $\|G\|(\Omega) < \infty$, $\|G\|$ is monotone and subadditive; refer to [5]. Let's denote by $\mathrm{ca}(\Sigma)$ the set of signed measures $\lambda : \Sigma \to \mathbb{R}$. Nikodym's convergence theorem states that if (μ_n) is a sequence from $\mathrm{ca}(\Sigma)$ for which

$$\lim_{n\to\infty} \mu_n(E) = \mu(E)$$

exists for each $E \in \Sigma$, then $\mu \in \mathrm{ca}(\Sigma)$; refer to [4]. Recall that $\mathrm{ca}(\Sigma)$ is a Banach space if we define $\|\lambda\| = |\lambda|(\Omega)$. By $\lambda \in \mathrm{ca}^+(\Sigma)$, we mean that $\lambda : \Sigma \to [0, \infty)$ is a finite countably additive measure. $G : \Sigma \to X$ is a weakly countably additive vector measure if for each $x^* \in X^*$, x^*G is a signed measure. Orlicz-Pettis theorem states that a weakly countably additive vector measure on Σ is countably additive; refer to [5, Corollary I.4.4]. If $G : \Sigma \to X$ is a countably additive vector measure, then $\{G(E) : E \in \Sigma\}$ is relatively weakly compact; refer to [5, Corollary I.3.7].

For two vector measures $F : \Sigma \to X$ and $G : \Sigma \to Y$ we say that F is absolutely continuous with respect to G, in symbol $F \ll G$, if given $\varepsilon > 0$ there is $\delta > 0$ such that $\|F(E)\| < \varepsilon$ whenever $E \in \Sigma$ and $\|G\|(E) < \delta$. If $F : \Sigma \to X$ is a countably additive vector measure and $\lambda \in \mathrm{ca}(\Sigma)^+$, then it is known that $F \ll \lambda$ if and only if $F(E) = 0$ whenever $\lambda(E) = 0$. Vitali-Hahn-Saks theorem states that if μ is a finitely additive nonnegative real-valued measure on Σ and (F_n) is a sequence of X-valued μ-continuous vector measures on Σ such that $\lim_{n\to\infty} F_n(E)$ exists for each $E \in \Sigma$, then

$$\lim_{\mu(E)\to 0} F_n(E) = 0$$

uniformly in n; refer to [5]. A Rybakov control measure for G is a measure of the form $|x^*G|$ such that $G \ll |x^*G|$. If $G : \Sigma \to X$ is a countably additive vector measure, then, according to the famous theorem of Rybakov, G has a

Rybakov control measure. Moreover, if $|x_0^*G|$ is a Rybakov control measure for G and $x^* \in X^*$, then all $|\{\alpha x^* + (1-\alpha)x_0^*\}G|$ are Rybakov control measures for G except for countably many $\alpha \in \mathbb{R}$; refer to [13].

Following D. R. Lewis [7] we define the Lebesgue space $L^1(G)$ by the set of measurable functions $f : \Omega \to \mathbb{R}$ such that

(i) $\int_\Omega |f| d|x^*G| < \infty$ for each $x^* \in X^*$ (scalarly integrable); and

(ii) for each $E \in \Sigma$ there is a vector in X, denoted by $\int_E f dG$, satisfying

$$x^* \int_E f dG = \int_E f dx^*G$$

for all $x^* \in X^*$. We regard two functions f and g in $L^1(G)$ equal if there is $E \in \Sigma$ such that $\|G\|(E) = 0$ and $f = g$ on E^c. We endow each $f \in L^1(G)$ with its norm

$$\|f\|_{L^1(G)} = \sup_{\|x^*\| \le 1} \int_\Omega |f| d|x^*G|.$$

It is well known that $\|f\|_{L^1(G)} < \infty$ for all $f \in L^1(G)$. For a quicker way of checking this observe that $\|f\|_{L^1(G)} \le 2\|f\|$ where

$$\|f\| = \sup_{E \in \Sigma} \Big\| \int_E f dG \Big\|.$$

And by the Orlicz-Pettis theorem the vector measure $E \to \int_E f dG$ is countably additive, hence it is bounded.

In [1] Curbera shows that $L^1(G)$ is an order continuous Banach lattice with weak order unit over (Ω, Σ, μ) for any Rybakov control measure μ for G. In particular, from the order continuity it follows that the simple functions are dense in $L^1(G)$.

3. Scalarly integrable functions

In Preliminaries, we observed that $\|f\|_{L^1(G)} < \infty$ if $f \in L^1(G)$. Stefansson proved that $\|f\|_{L^1(G)} < \infty$ if f is scalarly integrable only; refer to [14]. We reprove the same result independently. We start this section by showing the following definition.

Definition 3.1. Let $f : \Omega \to \mathbb{R}$ be measurable with respect to Σ. We define a space $\mathcal{M}(f)$ of set functions as follows. For each $k \in \mathbb{N}$ write $\Omega_k = \{\omega \in \Omega : |f(\omega)| > \frac{1}{k}\}$ and put $\Sigma_k = \{\Omega_k \cap E : E \in \Sigma\}$. We let $\mathcal{M}(f)$ be the set of all set functions $\lambda : \bigcup_{k=1}^\infty \Sigma_k \to \mathbb{R}$ such that $\lambda \upharpoonright \Sigma_k \in \mathrm{ca}(\Sigma_k)$ for all $k \in \mathbb{N}$ and

$$\sup_{k \ge 1} \int_{\Omega_k} |f| d|\lambda| < \infty.$$

Observe that in the above integral we adapted the convention of writing λ in place of $\lambda \upharpoonright \Sigma_k$. We endow each $\lambda \in \mathcal{M}(f)$ with its norm

$$\|\lambda\| = \sup_{k \ge 1} \int_{\Omega_k} |f| d|\lambda|.$$

It is easy to check that $\|\cdot\|$ is a norm.

On our way to the proof of completeness of $\mathcal{M}(f)$ we will need the following type of Fatou's lemma.

Lemma 3.2. *If $\mu, \mu_n \in \mathrm{ca}^+(\Sigma)$ for $n = 1, 2, \ldots$ and $\mu(E) = \lim_{n\to\infty} \mu_n(E)$ for each $E \in \Sigma$, then for each measurable $g : \Omega \to [0, \infty)$, we have $\int_\Omega g d\mu \leq \liminf_{n\to\infty} \int_\Omega g d\mu_n$.*

Proof. We observe that for each simple φ, $\int_\Omega \varphi d\mu = \lim_{n\to\infty} \int_\Omega \varphi d\mu_n$.

We recall that $\int_\Omega g d\mu = \sup_{0\leq\varphi\leq g} \int_\Omega \varphi d\mu$ where φ's are simple measurable functions. Since μ_n is a nonnegative measure, for each $0 \leq \varphi \leq g$ and $n \in \mathbb{N}$, we have

$$\int_\Omega \varphi d\mu_n \leq \int_\Omega g d\mu_n.$$

Hence $\int_\Omega \varphi d\mu \leq \liminf_{n\to\infty} \int_\Omega g d\mu_n$ and $\int_\Omega \; g d\mu \leq \liminf_{n\to\infty} \int_\Omega g d\mu_n$. □

Theorem 3.3. *The space $\mathcal{M}(f)$ is a Banach space.*

Proof. Let (λ_n) be a Cauchy sequence in $\mathcal{M}(f)$. Then given $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that if $n, m \geq n_0$, then $\sup_k \int_{\Omega_k} |f| d|\lambda_n - \lambda_m| < \varepsilon$. For each k

$$\int_{\Omega_k} |f| d|\lambda_n - \lambda_m| \geq \frac{1}{k} |\lambda_n - \lambda_m|(\Omega_k),$$

hence $(\lambda_n \restriction \Sigma_k)$ is a Cauchy sequence in $\mathrm{ca}(\Sigma_k)$. Since $\mathrm{ca}(\Sigma_k)$ is a Banach space, we obtain λ^k in $\mathrm{ca}(\Sigma_k)$ such that $\lambda_n \restriction \Sigma_k$ converges to λ^k in $\mathrm{ca}(\Sigma_k)$, hence $\lambda^k(E) = \lim_{n\to\infty} \lambda_n(E)$ for all $E \in \Sigma_k$. Then we can define $\lambda : \bigcup_{k=1}^\infty \Sigma_k \to \mathbb{R}$ by putting $\lambda = \lambda^k$ on Σ_k. Since any Cauchy sequence is bounded, there is $M > 0$ such that for all n, $\sup_k \int_{\Omega_k} |f| d|\lambda_n| \leq M$. Now we check that $\lambda_n \to \lambda$ in $\mathcal{M}(f)$. Since for each k and a fixed $n \geq n_0$, $\lim_{m\to\infty} |\lambda_n - \lambda_m|(E) = |\lambda_n - \lambda|(E)$ for all $E \in \Sigma_k$, we have, by virtue of Lemma 3.2, that $\int_{\Omega_k} |f| d|\lambda_n - \lambda| \leq \liminf_m \int_{\Omega_k} |f| d|\lambda_n - \lambda_m| \leq \varepsilon$ for all k. Hence if $n \geq n_0$, then $\sup_k \int_{\Omega_k} |f| d|\lambda_n - \lambda| \leq \varepsilon$. So, $\sup_k \int_{\Omega_k} |f| d|\lambda| \leq \int_{\Omega_k} |f| d|\lambda - \lambda_{n_0}| + \int_{\Omega_k} |f| d|\lambda_{n_0}| < \varepsilon + M$. That is, $\lambda \in \mathcal{M}(f)$ and $\lambda_n \to \lambda$ in $\mathcal{M}(f)$. □

Corollary 3.4. *If $\int_\Omega |f| d|x^*G| < \infty$ for all $x^* \in X^*$, then*

$$\sup_{\|x^*\|\leq 1} \int_\Omega |f| d|x^*G| < \infty.$$

Proof. By the assumption, for all $x^* \in X^*$, we have

$$\sup_{k\geq 1} \int_{\Omega_k} |f| d|x^*G| = \int_\Omega |f| d|x^*G| < \infty,$$

hence $x^*G \in \mathcal{M}(f)$. So we can define an operator $T : X^* \to \mathcal{M}(f)$ by $T(x^*) = x^*G$. Clearly T is a linear operator. It is enough to show that T is bounded.

Suppose $x_n^* \to x^*$ in X^* and $Tx_n^* \to \lambda$ in $\mathcal{M}(f)$. Then $\sup_k \int_{\Omega_k} |f|d|x_n^*G - \lambda| \to 0$. So for each k, $|x_n^*G - \lambda|(\Omega_k) \to 0$ and

$$\lambda(E) = \lim_{n\to\infty} x_n^*G(E)$$

for all $E \in \Sigma_k$. On the other hand $x^*G(E) = \lim_{n\to\infty} x_n^*G(E)$ for all $E \in \Sigma_k$. Hence we obtain $\lambda = x^*G$ on Σ_k for each k. So $\lambda = x^*G$ on $\bigcup_k \Sigma_k$. By the Closed Graph Theorem T is bounded. □

The following corollary is just Proposition 2 in [14]. We provide our proof.

Corollary 3.5. *Let $\mu \in \mathrm{ca}^+(\Sigma)$ such that $G \ll \mu$. Suppose that $\int_\Omega |f|d|x^*G| < \infty$ for all $x^* \in X^*$. Define $S : X^* \to L^1(\mu)$ by $Sx^* = f \cdot \frac{dx^*G}{d\mu}$. Then S is a bounded linear operator.*

Proof. Since $G \ll \mu$, for each $x^* \in X^*$, we have that $x^*G \ll \mu$, hence the Radon-Nikodym derivative $\frac{dx^*G}{d\mu} \in L^1(\mu)$ and $\frac{d|x^*G|}{d\mu} = |\frac{dx^*G}{d\mu}|$. Thus

$$\begin{aligned}\int_\Omega |Sx^*|d\mu &= \int_\Omega |f|\Big|\frac{dx^*G}{d\mu}\Big|d\mu = \int_\Omega |f|\frac{d|x^*G|}{d\mu}d\mu \\ &= \int_\Omega |f|d|x^*G| < \infty,\end{aligned}$$

which shows that $Sx^* \in L^1(\mu)$. In view of Corollary 3.4 we obtain that

$$\sup_{\|x^*\|\le 1} \int_\Omega |Sx^*|d\mu = \sup_{\|x^*\|\le 1} \int_\Omega |f|d|x^*G| < \infty.$$

This proves our corollary. □

Theorem 3.6. *Suppose that $\int_\Omega |f|d|x^*G| < \infty$ for all $x^* \in X^*$. Let $\mu \in \mathrm{ca}^+(\Sigma)$ such that $G \ll \mu$. Let $S : X^* \to L^1(\mu)$ be the operator given by $Sx^* = f \cdot \frac{dx^*G}{d\mu}$ as in Corollary 3.5. Then the following are equivalent*:

(a) *$f \in L^1(G)$.*

(b) *S is weak*-weak continuous.*

(c) *There is $g \in L^1(G)$ such that $\int_\Omega |f|d|x^*G| \le \int_\Omega |g|d|x^*G|$ for each $x^* \in X^*$.*

Proof. (a) ⇔ (b) It is a known result proved by Stefansson in [14, Theorem 4].

Clearly (a) implies (c) with $g = f$. It remain to check that (c) implies (a).

Fix $E \in \Sigma$ and consider $\int_E fdG : X^* \to \mathbb{R}$ given by $(\int_E fdG)x^* = \int_E fdx^*G$. Then we have $\int_E fdG \in X^{**}$. We claim that $\int_E fdG \in 2QW$ in X^{**}, where $W = \overline{co}\{\pm \int_A gdG : A \in \Sigma\}$ and $Q : X \to X^{**}$ is the canonical embedding. Since $A \to \int_A gdG$ is countably additive, we have that W is weakly compact. Hence QW is weak* compact in X^{**}. If $\int_E fdG \notin 2QW$, then by the separation theorem we have an $x^* \in X^*$ and $\alpha \in \mathbb{R}$ such that $2x^*(x) \le \alpha < (\int_E fdG)x^*$ for all $x \in W$. Then one has $A \in \Sigma$ such that

$$\int_\Omega |g|d|x^*G| = x^*\Big(\int_A gdG\Big) + x^*\Big(-\int_{\Omega\setminus A} gdG\Big)$$

$$\leq \alpha < x^*\Big(\int_E f dG\Big) \leq \int_\Omega |f| d|x^*G|;$$

a contradiction. □

4. The convergence in $L^1(G)$ and vector measures induced by integration

Lewis proved the following theorem in his classical paper; refer to [7].

Theorem 4.1. *Let (φ_n) be a sequence in $L^1(G)$ which converges pointwise to f on Ω and g be in $L^1(G)$ such that $|\varphi_n| \leq |g|$ for each n. Then we have $f \in L^1(G)$ and*

$$\int_E f dG = \lim_{n\to\infty} \int_E \varphi_n dG$$

uniformly for all $E \in \Sigma$.

By the above theorem, we observe that (φ_n) converges to f in $L^1(G)$ under the hypotheses of Theorem 4.1. That is, Theorem 4.1 gives a sufficient condition which guarantees convergence in $L^1(G)$. The following theorem gives a new sufficient condition which guarantees convergence in $L^1(G)$.

Theorem 4.2. *Suppose that there exists a sequence (φ_n) in $L^1(G)$ for which $(\int_E \varphi_n dG)$ is Cauchy for each $E \in \Sigma$ and (φ_n) is Cauchy in measure with respect to a Rybakov control measure $|x_0^*G|$. Then there exists $f \in L^1(G)$ such that (φ_n) converges to f in $L^1(G)$.*

Proof. First we have that the set function $F : \Sigma \to X$ given by

$$F(E) = \lim_{n\to\infty} \int_E \varphi_n dG$$

defines a countably additive vector measure. Indeed, by the Nikodym convergence theorem, F is a weakly countably additive vector measure. By the Orlicz-Pettis theorem F is a countably additive vector measure. Since $F \ll G \ll |x_0^*G|$, by the Radon-Nikodym theorem there exists a function $f_0 \in L^1(|x_0^*G|)$ such that $x_0^*F(E) = \int_E f_0 d|x_0^*G|$ for all $E \in \Sigma$. Since for each $E \in \Sigma$,

$$x_0^*F(E) = \lim_{n\to\infty} \int_E \varphi_n dx_0^*G,$$

it follows that

$$\lim_{n\to\infty} \int_E \varphi_n h d|x_0^*G| = \int_E f_0 d|x_0^*G|$$

for each $E \in \Sigma$. Here h is the Radon-Nikodym derivative of x_0^*G with respect to $|x_0^*G|$; observe that $|h(\omega)| = 1$ for $|x_0^*G|$-almost all $\omega \in \Omega$. So $\varphi_n h \to f_0$ weakly in $L^1(|x_0^*G|)$. Then $\{\varphi_n h : n \in \mathbb{N}\}$ is relatively weakly compact in $L^1(|x_0^*G|)$, so $\{\varphi_n h : n \in \mathbb{N}\}$ is uniformly integrable in $L^1(|x_0^*G|)$ and the same with (φ_n). In particular, we have

$$\lim_{|x_0^*G|(E)\to 0} \int_E |\varphi_n| d|x_0^*G| = 0$$

uniformly on $n \in \mathbb{N}$. Since (φ_n) is Cauchy in measure with respect to $|x_0^*G|$, by Vitali's convergence theorem, there is $f \in L^1(|x_0^*G|)$ such that $\varphi_n \to f$ in $L^1(|x_0^*G|)$. Hence $\varphi_n \to f$ in measure with respect to $|x_0^*G|$. Now fix $x^* \in X^*$. Since $|x^*G| \ll |x_0^*G|$, we have $\varphi_n \to f$ in $|x^*G|$-measure. Since $x^*F \ll x^*G$, we can check as above that (φ_n) is uniformly integrable in $L^1(|x^*G|)$. Again, by Vitali's convergence theorem, we have $f \in L^1(|x^*G|)$ and

$$\lim_{n\to\infty} \int_\Omega |\varphi_n - f| d|x^*G| = 0.$$

Now let $E \in \Sigma$ and put

$$\int_E f dG = \lim_{n\to\infty} \int_E \varphi_n dG.$$

Then we have $\int_E f dx^*G = x^* \int_E f dG$ for each $x^* \in X^*$ because $\varphi_n \to f$ in $L^1(|x^*G|)$ for each $x^* \in X^*$. Thus we obtain $f \in L^1(G)$.

Finally we show that (φ_n) converges to f in $L^1(G)$. First, for each $m \in \mathbb{N}$, define F_m on Σ by $F_m(E) = \int_E \varphi_m dG$. Then (F_m) is a sequence of X-valued $|x_0^*G|$-continuous vector measures on Σ such that $\lim_{m\to\infty} F_m(E)$ exists for each $E \in \Sigma$. By Vitali-Hahn-Saks theorem for vector measure (see [5, Corollary I.4.10]), we obtain that

$$\lim_{|x_0^*G|(E)\to 0} F_m(E) = 0$$

uniformly in m. Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that $\|F_m(E)\| < \varepsilon$ for all m whenever $|x_0^*G|(E) < \delta$. Put $E_n = \{\omega : |f(\omega) - \varphi_n| \geq \varepsilon\}$. Since φ_n converges to f in measure with respect to $|x_0^*G|$, there exists $N \in \mathbb{N}$ such that $|x_0^*G|(E_n) < \delta$ whenever $n > N$. Now take any $n > N$ and $E \in \Sigma$. Since $|x_0^*G|(E \cap E_n) \leq |x_0^*G|(E_n) < \delta$, we have

$$\|F_m(E \cap E_n)\| < \varepsilon$$

uniformly in m, so $\|F(E \cap E_n)\| \leq \varepsilon$. Then we have

$$\begin{aligned}
\Big| \int_E (\varphi_n - f) dx^*G \Big| &\leq \Big| \int_{E\setminus E_n} (\varphi_n - f) dx^*G \Big| + \Big| \int_{E\cap E_n} (\varphi_n - f) dx^*G \Big| \\
&\leq \varepsilon \|G\|(\Omega) + \Big| \int_{E\cap E_n} \varphi_n dx^*G \Big| + \Big| \int_{E\cap E_n} f dx^*G \Big| \\
&\leq \varepsilon \|G\|(\Omega) + \Big| x^*\big(\int_{E\cap E_n} \varphi_n dG\big) \Big| + \Big| x^*\big(\int_{E\cap E_n} f dG\big) \Big| \\
&\leq \varepsilon \|G\|(\Omega) + \|F_n(E \cap E_n)\| + \|F(E \cap E_n)\| \\
&\leq \varepsilon \|G\|(\Omega) + 2\varepsilon
\end{aligned}$$

for all $x^* \in B_{X^*}$. Hence we obtain that

$$\int_E f dG = \lim_{n\to\infty} \int_E \varphi_n dG$$

uniformly for all $E \in \Sigma$. Thus we conclude that (φ_n) converges to f in $L^1(G)$. $\square$

Remark 4.3. By the proof of Theorem 4.1, the uniform bounded condition of a sequence (φ_n) in Theorem 4.1 implies that $(\int_E \varphi_n dG)$ is Cauchy for each $E \in \Sigma$. Then Theorem 4.2 improves Lewis's result. Moreover, Theorem 4.2 improves [7, Theorem 2.4].

For the next two theorems let's assume that F and G are two countably additive vector measures where F is given by the integration with respect to G; that is, there is $f \in L^1(G)$ such that

$$F(E) = \int_\Omega f dG$$

for all $E \in \Sigma$.

Theorem 4.4. *If $G(\Sigma)$ is relatively compact, then $F(\Sigma)$ is relatively compact.*

Proof. In case $f = \chi_A$, then $F(\Sigma) = \{G(E \cap A) : E \in \Sigma\} \subset G(\Sigma)$; hence $\overline{F(\Sigma)}$ is compact. If $f = \sum_{i=1}^n a_i \chi_{A_i}$ is a simple function, then $\overline{F(\Sigma)} \subset \sum_{i=1}^n a_i \overline{G(\Sigma)}$ is compact.

Now let $f \in L^1(G)$ and $\varepsilon > 0$. Since simple functions are dense in $L^1(G)$, we find a simple function h such that $\|f - h\|_{L^1(G)} < \varepsilon$. If we write $H(E) = \int_E hdG$, then

$$\|F(E) - H(E)\| = \sup_{\|x^*\| \le 1} \Big| \int_E (f - h) dx^* G \Big| \le \|f - h\|_{L^1(G)} < \varepsilon.$$

Thus $F(\Sigma) \subset H(\Sigma) + \varepsilon B_X$ where $\overline{H(\Sigma)}$ is compact by the previous paragraph. Thus $\overline{F(\Sigma)}$ is compact. $\square$

Theorem 4.5. *If $\{G(E \cap A) : A \in \Sigma\}$ is weakly compact convex for each $E \in \Sigma$, then the same holds for F.*

In order to prove Theorem 4.5 we rely on Knowles's version of the Lyapunov convexity theorem; see [5], [6] and [10].

Theorem 4.6. *Let (Ω, Σ) be a measurable space, X a Banach space and $G : \Sigma \to X$ a countably additive vector measure. Suppose $\lambda \in$ $\mathrm{ca}^+(\Sigma)$ with $G \ll \lambda$. Then the following are equivalent.*

(a) *$\{G(E \cap A) : A \in \Sigma\}$ is weakly compact convex for each $E \in \Sigma$.*

(b) *If $h \neq 0$ in $L^\infty(\lambda)$, then $\int_\Omega ghdG = 0$ for some $g \in L^\infty(\lambda)$ with $gh \neq 0$ in $L^\infty(\lambda)$.*

Proof of Theorem 4.5. First fix a Rabakov control measure $\lambda = |x_0^* G|$ for G where $\|x_0^*\| \le 1$. Observe that F is a countably additive vector measure and $F \ll \lambda$.

Assume that $\{G(E \cap A) : A \in \Sigma\}$ is weakly compact convex for each $E \in \Sigma$. In order to show that $\{F(E \cap A) : A \in \Sigma\}$ is weakly compact convex for each $E \in \Sigma$, we check (b) of Theorem 4.6.

Let $h \neq 0$ in $L^\infty(\lambda)$. In view of Theorem 3.6 we have $fh \in L^1(G)$. In case $fh = 0$ λ-a.e. we may take $g = \chi_\Omega$; then $gh \neq 0$ in $L^\infty(\lambda)$ and

$$\int_\Omega ghdF = \int_\Omega fghdG = 0.$$

Now assume that $fh \neq 0$ in $L^\infty(\lambda)$. Then there exists $E \in \Sigma$ such that $fh\chi_E \in L^\infty(\lambda)$ and $fh\chi_E \neq 0$ in $L^\infty(\lambda)$. We apply Theorem 4.6 to G to obtain $g_1 \in L^\infty(\lambda)$ such that $fh\chi_E g_1 \neq 0$ in $L^\infty(\lambda)$ but

$$\int_\Omega fh\chi_E g_1 dG = 0.$$

Put $g = \chi_E g_1$. Then $gh \neq 0$ in $L^\infty(\lambda)$ and

$$\int_\Omega ghdF = \int_\Omega fghdG = \int_\Omega fh\chi_E g_1 dG = 0.$$

This proves Theorem 4.5. □

5. Approximation property of $L^1(G)$

In this section, we consider the approximation property of $L^1(G)$. We ask some questions: Does $L^1(G)$ have the approximation property in general? And if G is a X-valued countably additive vector measure, then are X and $L^1(G)$ the same from the aspect of the approximation property? By illustrating some examples, we answer these questions.

Definition 5.1. A Banach space X is said to have the *approximation property* if for every compact subset K of X and $\epsilon > 0$, there is a finite rank operator T on X such that $\|Tx - x\| < \epsilon$ for all $x \in K$.

Since $L^1(\mu)$ has the approximation property whenever μ is a nonnegative real valued measure, $L^1(G)$ which is order isomorphic to $L^1(\mu)$ has the approximation property. But $L^1(G)$ does not have the approximation property in general as the following example shows. For this example we need following facts.

Theorem 5.2 ([1, Theorem 8])**.** *Let X be an order continuous Banach lattice with weak order unit. Then there exists a countably additive vector measure G such that X is order isometric to $L^1(G)$.*

Theorem 5.3 ([11, Theorem 2.4.15])**.** *If X is a reflexive Banach lattice, then X has the order continuous norm.*

Example 5.4. *There exists a countably additive vector measure G such that $L^1(G)$ does not have the approximation property.* Thanks to Szankowski there exists a uniformly convex Banach lattice E with weak order unit which fails to have the approximation property; refer to [15] or [10, Theorem 1.g.2]. Since every uniformly convex Banach space is reflexive [9, Proposition 1.e.3], E is a reflexive Banach lattice. Hence, by Theorem 5.3, E has order continuous norm.

By Theorem 5.2, there exists a countably additive vector measure G such that E is order isometric to $L^1(G)$. Since E fails to have the approximation property, $L^1(G)$ does not have the approximation property.

The above example gives natural questions: if G is a X-valued vector measure and $L^1(G)$ has the approximation property, then does X have the approximation?, or if X has the approximation property, then does $L^1(G)$ have the approximation property? Unfortunately, we don't know much about this: we can give a negative answer for the former and we do not know the answer for the latter. We need the following theorem; refer to [12].

Theorem 5.5. *Let X be an infinite dimensional Banach space. Then there exists an X-valued countably additive vector measure G with finite variation which satisfy $L^1(|G|) = L^1(G)$.*

Example 5.6. *There exists an X-valued countably additive vector measure G such that X does not have the approximation property even though $L^1(G)$ has the approximation property.* In Example 5.4, there exists a uniformly convex Banach lattice X with weak order unit which fails to have the approximation property. By Theorem 5.5, there exists an X-valued countably additive vector measure G with finite variation which satisfy $L^1(|G|) = L^1(G)$. Since $L^1(|G|)$ has the approximation property, $L^1(G)$ has the approximation property.

Now we give the following problem.

Problem. *If G is an X-valued countably additive vector measure and X has the approximation property, then does $L^1(G)$ have the approximation property?*

If the above problem had an affirmative answer, then we would have a sufficient condition for a Banach space which guarantees the approximation property for $L^1(G)$ which is not isomorphic to $L^1(\mu)$.

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