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A NOTE ON GENERALIZED LICHNEROWICZ-OBATA THEOREMS FOR RIEMANNIAN FOLIATIONS

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Abstract. It was obtained in [5] generalized Lichnerowicz and Obata theorems for Riemannian foliations, which reduce to the results on Riemannian manifolds for the point foliations. Recently in [3], they studied a generalized Obata theorem for Riemannian foliations admitting transversal conformal fields. Each transversal conformal field is a λ-automorphism with $\lambda = 1 - \frac{2}{q}$ in the sense of [8]. In the present paper, we extend certain results established in [3] and study Riemannian foliations admitting λ-automorphisms with $\lambda \geq 1 - \frac{2}{q}$.

1. Introduction

The study of relationships between the eigenvalues of the Laplacian and geometric quantities on a Riemannian manifold is one of interesting topics in Riemannian geometry. Among all the eigenvalues, the smallest positive eigenvalue plays the most important role. For example, the standard Lichnerowicz comparison theorem ([6]) says that if the Ricci operator ρ of a closed connected Riemannian manifold M of dimension $m \geq 2$ satisfies $\rho \geq c(m-1)\mathrm{id}$ for some $c > 0$, where id denotes the identity operator, then the smallest positive eigenvalue α of the Laplacian Δ acting on functions satisfies $\alpha \geq cm$. The Obata theorem ([7]) says that equality occurs if and only if the M is isometric to the standard m-sphere with constant sectional curvature c. In particular, when M is Einstein with constant scalar curvature σ, the four following conditions are equivalent to each other.

(C1) M is isometric to the standard m-sphere with radius $\frac{1}{\sqrt{c}}$ in the $(m+1)$-dimensional Euclidean space,

(C2) M admits a non-homothetic conformal vector field,

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(C3) M admits a non constant function f satisfying $\nabla\nabla f + cfg = 0$,
(C4) M admits a non constant function f satisfying $\Delta f = cmf$.

Here ∇ denotes the Levi-Civita connection of g. Lee and Richardson ([5]) generalized these results to the context of Riemannian foliations. That is,

Theorem 1.1. *Let $(M, g, \mathcal{F})$ be a closed connected Riemannian manifold with Riemannian foliation $\mathcal{F}$ of codimension $q \geq 2$. If the transversal Ricci operator ρ_D of satisfies $\rho_D \geq c(q-1)\mathrm{id}$ for some $c > 0$, then the smallest positive eigenvalue α_B of the basic Laplacian Δ_B acting on basic functions satisfies $\alpha_B \geq cq$. In addition, the equality occurs if and only if*

(1) *the leaf space $M/\mathcal{F}$ is isometric to the space of orbits of a discrete subgroup of $O(q)$ acting on the standard q-sphere with constant sectional curvature c,*
(2) *if we choose the metric on M so that the mean curvature form κ is basic, then $\mathcal{F}$ is harmonic,*
(3) *each level set of the α_B eigenfunction is the set of the leaves corresponding to a latitude of the q-sphere, and the volume of this level set is the volume of the maximum leaf times the volume of the latitude.*

Theorem 1.1 provides a relationship between the above conditions (C1) and (C4). Related to the condition (C2), Jung and Jung ([3]) recently obtained a generalized Obata theorem (Theorem 5.9) for Riemannian foliations admitting transversal conformal fields.

Theorem 1.2. *Let $(M, g, \mathcal{F})$ be as in Theorem 1.1. Let κ be basic such that $\delta_B\kappa = 0$ and the transversal scalar curvature σ_D be non-zero constant. Assume that $\rho_D \geq \frac{\sigma_D}{q}\mathrm{id}$. If M admits a transversal non-Killing conformal field, then $\mathcal{F}$ is transversally isometric to the action of a finite subgroup of $O(q)$ acting on the standard q-sphere with constant sectional curvature $c = \frac{\sigma_D}{q(q-1)}$.*

Geometric transversal infinitesimal automorphisms such as transversal Killing, affine, projective, conformal, Jacobi fields have been attacked by many peoples. These are all examples of λ-automorphisms in the sense of [8]. For instance, each transversal Killing field is a λ-automorphism for all $\lambda \in \mathbb{R}$. Each transversal conformal and projective field is a $(1-\frac{2}{q})$-automorphism and $(-\frac{2}{q+1})$-automorphism respectively.

In the present paper we discuss a Riemannian foliation admitting a transversal conformal field from the viewpoint of λ-automorphisms. Then we improve the proof of Theorem 1.2 given in [3] by showing that the foliation is harmonic. When the foliation is transversally Einstein, we prove that a non-homothetic conformal field appeared in (C2) can be replaced by a λ-automorphism with $\lambda \geq 1 - \frac{2}{q}$. Moreover, we generalize the previous Obata's result concerning the conditions (C1) $\sim$ (C4) to transversally Einstein foliations.

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2. Preliminaries

Let $(M, g, \mathcal{F})$ be a m-dimensional closed connected Riemannian manifold with Riemannian foliation $\mathcal{F}$ of codimension $q := m - p \geq 2$ and a bundle-like metric g. It is given by an exact sequence of vector bundles

$$0 \to \mathcal{V} \to TM \xrightarrow{\pi} Q \to 0,$$

where $\mathcal{V}$ is the tangent bundle and Q the normal bundle of $\mathcal{F}$. The metric g determines an orthogonal decomposition $TM = \mathcal{V} \oplus \mathcal{H}$. We identify $\mathcal{H}$ with Q by an isometric splitting

$$(Q, g_Q) \cong (\mathcal{H}, g_{\mathcal{H}}). \tag{2.1}$$

We have an associated exact sequence of Lie algebras

$$0 \to \Gamma(\mathcal{V}) \to V(\mathcal{F}) \xrightarrow{\pi} \bar{V}(\mathcal{F}) \to 0,$$

where $V(\mathcal{F}) := \{Y \in \Gamma(TM) | [V, Y] \in \Gamma(\mathcal{V}) \text{ for all } V \in \Gamma(\mathcal{V})\}$ and $\bar{V}(\mathcal{F}) := \{s \in \Gamma(Q) | s = \pi(Y), Y \in V(\mathcal{F})\}$, called the space of transversal infinitesimal automorphisms of $\mathcal{F}$. Here and hereafter, we denote by $\Gamma(\cdot)$ the space of all smooth sections of a vector bundle $(\cdot)$. The transversal Levi-Civita connection D on Q is a torsion free and metric connection with respect to g_Q (see [9], (5.9)).

Throughout this paper, we use the following notations:

τ : the tension field of $\mathcal{F}$,
$\mathrm{grad}_D\, f$: the transversal gradient of a function $f \in C^\infty(M)$,
$\mathrm{div}_D\, s$: the transversal divergence of $s \in \Gamma(Q)$,
ρ_D : the transversal Ricci operator,
σ_D : the transversal scalar curvature,
Δ : the Laplacian acting on $\Gamma(Q)$,
$\theta(Y)$: the transversal Lie derivative operator for $Y \in V(\mathcal{F})$,
$A_D(Y) := \theta(Y) - D_Y$ for $Y \in V(\mathcal{F})$.

In what follows, we assume that $\tau \in \bar{V}(\mathcal{F})$. This assumption is based on the tenseness theorem ([1]) and a result ([5], Corollary 4.2).

The basic complex $(\Omega_B, d_B := d|_{\Omega_B})$ is a subcomplex of the de Rham complex $(\Omega(M), d)$, where

$$\Omega_B := \{\omega \in \Omega(M) \mid i_V \omega = \theta(V)\omega = 0 \text{ for all } V \in \Gamma(\mathcal{V})\}.$$

Its cohomology

$$H_B := H(\Omega_B, d_B)$$

is called the basic cohomology of $\mathcal{F}$. We often use the following identification by means of (2.1) and g_Q-duality

$$\bar{V}(\mathcal{F}) \cong \Omega_B^1. \tag{2.2}$$

For convenience $\mathcal{F}$ is assumed to be oriented and transversally oriented. The transversal volume form associated to $\mathcal{F}$ is an element $\nu \in \Omega_B^q$ with $d\nu = 0$. The characteristic form is the p-form $\chi_{\mathcal{F}} := *\nu$, expressed in terms of the Hodge star operator $*$. The Riemannian volume form $\mu \in \Omega^m(M)$ is then given by $\mu = \nu \wedge \chi_{\mathcal{F}}$.

We use the same notation g_Q for the local scalar product on tensors of Q and $|\cdot|^2 := g_Q(\cdot,\cdot)$. Let $\langle\ ,\ \rangle$ be the global scalar product defined by

$$\langle \phi, \psi \rangle := \int_M g_Q(\phi,\psi)\mu. \tag{2.3}$$

It is well-known that on Ω_B

$$\langle \phi, \psi \rangle = \int_M \phi \wedge \bar{*}\psi \wedge \chi_{\mathcal{F}},$$

where $\bar{*}$ is the star operator associated to the holonomy-invariant metric g_Q. Then the formal adjoint δ_B of d_B on Ω_B^r with respect to $\langle\ ,\ \rangle$ is given by the formula

$$\delta_B = (-1)^{r(q+1)+1}\bar{*}d_T\bar{*} = \delta_T + i_\tau, \tag{2.4}$$

where

$$d_T := d_B - \kappa\wedge, \quad \delta_T := (-1)^{r(q+1)+1}\bar{*}d_B\bar{*}. \tag{2.5}$$

Here κ is the mean curvature 1-form of $\mathcal{F}$ which is the g_Q-dual to τ. It should be noted that d_T and δ_T are formal adjoint with respect to $\langle\ ,\ \rangle$. The basic Laplacian acting on Ω_B is defined by

$$\Delta_B := d_B\delta_B + \delta_B d_B. \tag{2.6}$$

3. Main results

For our purpose we start with recalling the notion of λ-automorphisms discussed in [8]. $s \in \bar{V}(\mathcal{F})$ is called a λ-automorphism for $\lambda \in \mathbb{R}$ if it satisfies

$$\Delta s - D_\tau s - \rho_D(s) - \lambda \mathrm{grad}_D \mathrm{div}_D\ s = 0, \tag{3.1}$$

or equivalently the g_Q-dual ω satisfies

$$-\mathrm{tr}D^2\omega - \rho_D(\omega) + \lambda d_B\delta_T\omega = 0. \tag{3.2}$$

As examples, if s is a transversal conformal field, that is,

$$\theta(s)g_Q = 2f_s g_Q, \tag{3.3}$$

then

$$\mathrm{div}_D\ s = qf_s, \quad \Delta s = D_\tau s + \rho_D(s) + (1-\frac{2}{q})\mathrm{grad}_D\mathrm{div}_D\ s. \tag{3.4}$$

In particular, if f_s is constant, s is called a transversal homothetic field. A transversal conformal field is a $(1-\frac{2}{q})$-automorphism. The converse is not true unless $\mathcal{F}$ is harmonic ($\kappa = 0$) ([8]). When we refer in the sequel to a transversal conformal field s, we always mean by f_s the function appearing in (3.3).

Lemma 3.1. *Let $(M, g, \mathcal{F})$ be a closed connected Riemannian manifold with Riemannian foliation $\mathcal{F}$ of codimension $q \geq 2$. Suppose that κ is basic and the transversal scalar curvature σ_D is constant. If M admits a transversal conformal field s, then it holds*

$$\Delta_B f_s = \frac{\sigma_D}{q-1} f_s + \tau(f_s). \tag{3.7}$$

Proof. Let s be a λ-automorphism and ω be its g_Q-dual. From the Weitzenböck formula

$$d_B \delta_T \omega + \delta_T d_B \omega = -\mathrm{tr} D^2 \omega + \rho_D(\omega),$$

we see that (3.2) is equivalent to the formula

$$(1+\lambda) d_B \delta_T \omega + \delta_T d_B \omega = 2\rho_D(\omega). \tag{3.8}$$

Since σ_D is constant, a transversal conformal field s satisfies

$$\delta_T(\rho_D(\omega)) = -g_Q(S_D, D\omega) = -\sigma_D f_s,$$

where S_D denotes the transversal Ricci tensor. It follows from (3.4) and (3.8) that

$$(q-1)\delta_T d_B f_s = \sigma_D f_s,$$

which follows (3.7) by means of (2.5). □

It is noted that (3.7) was derived by another way via the formula ([3], Proposition 5.5)

$$\theta(s)\sigma_D = 2(q-1)\delta_T d_B f_s - 2\sigma_D f_s.$$

Corollary 3.2. *Let $(M, g, \mathcal{F}), \kappa$ and σ_D be as in Lemma 3.1. Further suppose that $\delta_B \kappa = 0$. If M admits a transversal non-homothetic conformal field s, then σ_D is positive.*

Proof. Observe from $\delta_B \kappa = 0$ that

$$\langle \tau(f_s), f_s \rangle = \frac{1}{2} \langle d_B(f_s^2), \kappa \rangle = 0.$$

Then Lemma 3.1 implies

$$||d_B f_s||^2 = \langle \frac{\sigma_D}{q-1} f_s + \tau(f_s), f_s \rangle = \frac{\sigma_D}{q-1} ||f_s||^2,$$

so that $\sigma_D > 0$. □

It is well-known the transversal Hodge isomorphism

$$H_B^r \simeq \mathcal{H}_B^r,$$

where $\mathcal{H}_B^r := \ker \Delta_B \cap \Omega_B^r$ and the tautness theorem ($[\kappa] = 0$ in H_B^1) under the positivity of the transversal Ricci curvature (see [12], (7.51) and (8.22)). In the same situation we may show a more stronger result, the harmonicity theorem ($\kappa = 0$). Here we give a direct simple proof by using the transversal Weitzenböck formula for $\phi \in \Omega_B^r$

$$\Delta_B \phi = D_{\mathrm{tr}}^* D_{\mathrm{tr}} \phi + F(\phi) + A_\tau \phi, \tag{3.9}$$

where $D_{\mathrm{tr}}^* D_{\mathrm{tr}}$ is a non-negative and formally self-adjoint operator ([3]).

Lemma 3.3. *Let $(M, g, \mathcal{F})$ be as in Lemma 3.1. Suppose that κ is basic such that $\delta_B \kappa = 0$. If $\rho_D > 0$, then $\mathcal{F}$ is harmonic.*

Proof. Since M is closed, $\kappa \in \Omega_B^1$ implies $d\kappa = 0$ (see [12], (7.5)). Thus $\kappa \in \mathcal{H}_B^1$. Note that

$$(A_\tau \kappa)(t) = -\kappa(A_\tau t) = \kappa(D_t \tau) = \frac{1}{2} d|\kappa|^2(t) \tag{3.10}$$

for any $t \in \bar{V}(\mathcal{F})$. Then (3.9) combined with (3.10) yields

$$D_{\mathrm{tr}}^* D_{\mathrm{tr}} \kappa + \rho_D(\kappa) + \frac{1}{2} d|\kappa|^2 = 0.$$

It follows that

$$\begin{aligned} 0 &= \langle D_{\mathrm{tr}}^* D_{\mathrm{tr}} \kappa + \rho_D(\kappa) + \frac{1}{2} d|\kappa|^2, \kappa \rangle \\ &= ||D_{\mathrm{tr}} \kappa||^2 + \langle \rho_D(\kappa), \kappa \rangle. \end{aligned}$$

Therefore the Bochner technique shows $\kappa = 0$. □

Remark. Lemma 3.3 is proved in [2] in a somewhat different way. Its proof is based on the exactness property of κ from the positivity condition of ρ_D.

Now we are in a position to improve Theorem 1.2 in light of Lemma 3.3. It should be noted that the tautness theorem used in the proof of [3] can be replaced by the harmonicity theorem in our proof.

Theorem 3.4. *Let $(M, g, \mathcal{F})$ be as in Lemma 3.1. Suppose that κ is basic such that $\delta_B \kappa = 0$ and σ_D is constant. Assume that $\rho_D \geq \frac{\sigma_D}{q} \mathrm{id}$. If M admits a transversal non-homothetic conformal field s, then $\mathcal{F}$ is harmonic and transversally isometric to the action of a finite subgroup of $O(q)$ acting on the standard q-sphere with constant sectional curvature $c = \frac{\sigma_D}{q(q-1)}$.*

Proof. Since σ_D is constant, Corollary 3.2 yields $\sigma_D > 0$, and so $\rho_D > 0$. Then $\mathcal{F}$ is harmonic by Lemma 3.3.

In addition, we get from Lemma 3.1

$$\Delta_B f_s = \frac{\sigma_D}{q-1} f_s. \tag{3.11}$$

If we take $c := \frac{\sigma_D}{q(q-1)}$, then $\alpha_B \leq cq$ by means of (3.11). Hence Generalized Lichnerowicz Theorem (the first part of Theorem 1.1) deduces that $\alpha_B = cq$. Therefore the proof is completed by Generalized Obata Theorem (the second part of Theorem 1.1). □

Remarks. (1) Observe that (3.11) means $f_s = -\frac{1}{q} \delta_T \omega$ is an eigenfunction with the eigenvalue α_B. Thus each level set of f_s is the set of the leaves corresponding to a latitude of the q-sphere, and the volume of this level set is the volume of the maximum leaf times the volume of the latitude.

(2) It was proved in [9] that assuming $\sigma_D \le 0$ if $s \in \bar{V}(\mathcal{F})$ is a transversal conformal field which satisfies $\langle B_D^0(s)s, \tau\rangle \ge 0$, where

$$B_D^\lambda(s) := A_D(s) + {}^tA_D(s) + \lambda(\mathrm{div}_D\ s)\mathrm{id},$$

then s is a transversal Killing field.

(3) In [10] the author considered a notion of foliated transformations for manifolds with Riemannian foliations. Each foliated conformal (resp. foliated isometric) transformation is defined by a transformation preserving the leaves such that the induced map on their normal bundles is conformal (resp. isometric). It has been studied the conditions when a foliated conformal transformation is to be foliated isometric in terms of transversal Ricci or scalar curvatures.

(4) If we further add the assumption that f_s is an eigenfunction of Δ_B with some eigenvalue $\tilde{\alpha_B}$ in Theorem 3.4, the harmonicity of $\mathcal{F}$ may be proved by another way. Indeed, Lemma 3.1 in this case becomes

$$\tau(f_s) = (\tilde{\alpha_B} - \frac{\sigma_D}{q-1})f_s \ge (\alpha_B - cq)f_s,$$

so that

$$0 \ge (\alpha_B - cq)||f_s||^2.$$

Thus Generalized Lichnerowicz Theorem implies $\alpha_B = cq$. It follows from Generalized Obata theorem that $\mathcal{F}$ is harmonic.

Related to the Obata's result stated in Section 1, we now consider the particular case where $\mathcal{F}$ is transversally Einstein, i.e., $\rho_D = \frac{\sigma_D}{q}\mathrm{id}$. Note that any λ-automorphism s satisfies (3.8). In this situation, applying δ_T and using σ_D constant give rise to

$$(1+\lambda)\delta_T d_B f_s = 2\frac{\sigma_D}{q} f_s. \tag{3.12}$$

Therefore by a similar argument as in Theorem 3.4 we conclude:

Theorem 3.5. *Let $(M, g, \mathcal{F})$ and κ be as in Theorem 3.4. Further assume that $\mathcal{F}$ is transversally Einstein. If M admits a λ-automorphism s with $\lambda \ge 1 - \frac{2}{q}$ whose transversal divergence is not constant, then $\mathcal{F}$ is harmonic and transversally isometric to the action of a finite subgroup of $O(q)$ acting on the standard q-sphere with constant sectional curvature $c = \frac{2\sigma_D}{(1+\lambda)q^2}$.*

Proof. (3.12) implies

$$(1+\lambda)\delta_T d_B f = 2\frac{\sigma_D}{q} f,$$

where $f := \delta_T \omega$. Then

$$\Delta_B f = \frac{2\sigma_D}{(1+\lambda)q} f + \tau(f). \tag{3.13}$$

Since $\mathrm{div}_D\ s$ is not constant and $\delta_B \kappa = 0$, we get $\sigma_D > 0$. Hence $\rho_D > 0$. Thus $\mathcal{F}$ is harmonic by Lemma 3.3. Then (3.13) induces $\alpha_B \le cq$.

On the other hand, we find from $\lambda \geq 1 - \frac{2}{q}$

$$\rho_D = \frac{\sigma_D}{q}\mathrm{id} \geq c(q-1)\mathrm{id},$$

which follows $\alpha_B = cq$. Therefore we complete the proof by virtue of Theorem 1.1. □

Remarks. (1) Theorem 3.5 shows that the constant sectional curvature attains the maximum $c = \frac{\sigma_D}{q(q-1)}$ at which s is a transversal non-homothetic conformal field.

(2) It was obtained in [8] that a λ-automorphism s whose transversal divergence is constant is transversal Killing if and only if $\langle B_D^0(s)s, \tau\rangle \geq 0$.

Corollary 3.6. *Under the same assumptions as in Theorem 3.5, the four following statements are equivalent to each other.*

(F1) *$\mathcal{F}$ is harmonic and transversally isometric to the action of a finite subgroup of $O(q)$ acting on the standard q-sphere with constant sectional curvature c.*

(F2) *M admits a transversal non-homothetic conformal field,*

(F3) *M admits a non-constant basic function f satisfying $DDf + cfg_Q = 0$,*

(F4) *M admits a non-constant basic function f satisfying $\delta_T d_B f = cqf$,*

where $c = \frac{\sigma_D}{q(q-1)}$.

Proof. Generalized Obata Theorem says (F1) $\Leftrightarrow$ (F4). Besides Theorem 3.4 means (F2) $\Rightarrow$ (F1). Furthermore, we easily see that (F3) $\Rightarrow$ (F2). Indeed, if we take $s = Df$, then (F3) implies $\theta(s)g_Q = 2DDf = -2cfg_Q$, so s is a desired one. Therefore it suffices to show (F4) $\Rightarrow$ (F3).

To begin with, we observe that by a similar argument of Theorem 3.4, (F4) yields $0 < c = \frac{\sigma_D}{q(q-1)}$, and so $\mathcal{F}$ is harmonic. Hence

$$cqf = \delta_T d_B f = \Delta_B f. \tag{3.14}$$

Furthermore it holds for any $h \in \Omega_B^0$ that ([5])

$$\frac{1}{2}\Delta_B|d_B h|^2 = g_Q(\Delta_B d_B h, d_B h) - |Dd_B h|^2 - S_D(Dh, Dh), \tag{3.15}$$

where $S_D(s,t) := g_Q(\rho_D(s), t)$ for $s, t \in \bar{V}(\mathcal{F})$. Then (3.14) and (3.15) together with $h = f$ imply

$$\begin{aligned} 0 &= \langle \rho_D(Df), Df\rangle - ||\Delta_B f||^2 + ||DDf||^2 \\ &= \frac{\sigma_D}{q}||Df||^2 - ||cqf||^2 + c^2 q||f||^2 + ||DDf + cfg_Q||^2 \\ &= ||DDf + cfg_Q||^2, \end{aligned}$$

which follows (F3). □

Remarks. (1) We informed later that $(F1) \Leftrightarrow (F3)$ was proved in [4] in a more general setting.

(2) Corollary 3.6 is still true even if we replace the transversally Einstein condition by a weaker condition $S_D \geq c(q-1)g_Q$.

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