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SOME STRONGLY NIL CLEAN MATRICES OVER LOCAL RINGS

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Abstract. An element of a ring is called strongly nil clean provided that it can be written as the sum of an idempotent and a nilpotent element that commute. A ring is strongly nil clean in case each of its elements is strongly nil clean. We investigate, in this article, the strongly nil cleanness of 2×2 matrices over local rings. For commutative local rings, we characterize strongly nil cleanness in terms of solvability of quadratic equations. The strongly nil cleanness of a single triangular matrix is studied as well.

1. Introduction

Throughout, all rings are associative rings with identity. We say that $a \in R$ is strongly clean provided that there exist an idempotent $e \in R$ and a unit $u \in R$ such that $a = e + u$ and $eu = ue$. A ring R is strongly clean in case every element in R is strongly clean. Strong cleanness over commutative local rings was extensively studied by many authors from very different view points (cf. [1-3], [5] and [9-10]). In [6], Diesl introduced the concept of strongly nil cleanness. An element $a \in R$ is strongly nil clean provided that there exist an idempotent $e \in R$ and a nilpotent element $u \in R$ such that $a = e + u$ and $eu = ue$. A ring R is strongly nil clean in case every element in R is strongly clean. As is well known, we have that {strongly nil clean rings} $\subsetneqq$ {strongly π-regular rings} $\subsetneqq$ {strongly clean rings}. The other motivation of studying strongly nil cleanness is derived from Lie algebra. As every square matrix A over local rings admits a diagonal reduction, it follows that if A has a strongly nil clean decomposition, then it satisfies the condition: $A = E + W$, where E is similar to a diagonal matrix, $W \in M_2(R)$ is nilpotent, E and W commute (see Theorem 2.1). It is worth noting that such decomposition over a field is called the Jordan-Chevalley decomposition in Lie theory (cf. [7]). From this, one sees that strongly nil cleanness is also an extension of Jordan-Chevalley decomposition over fields.

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Many elementary properties of strongly nil cleanness were studied by Diesl in [6]. A ring R is local provided that it has only one maximal right ideal. For the ring $T_2(R)$ of all lower triangular matrices over a local ring R, he proved that $T_2(R)$ is strongly nil clean if and only if so is R. However, it is hard to determine when a 2×2 matrix over commutative local rings is strongly nil clean.

It is shown that the ring of all 2×2 matrices over any commutative local ring is not strongly nil clean. Because many authors (Chen-Yang-Zhou, Borooah-Diesl-Dorsey, and Li) gave criteria for the strong cleanness of 2×2 matrix rings over local rings through solutions of quadratic equations or diagonalization, we consider when a single 2×2 matrix over a local ring is strongly nil clean. For commutative local rings, we get criteria on strongly nil cleanness in terms of solvability of quadratic equations. Let R be a commutative local ring. Then $A \in M_2(R)$ is strongly nil clean if and only if A is nilpotent or $I_2 - A$ is nilpotent or the quadratic equation $x^2 - \mathrm{tr}A \cdot x + \det A = 0$ has a root in $N(R)$ and a root in $1 + N(R)$, where $N(R)$ denotes the set of all nilpotent elements in R. Therefore we completely decide which kind of 2×2 matrices over a commutative local ring is strongly nil clean. The strongly nil cleanness of a single triangular matrix over local rings is investigated as well.

2. Solvability of quadratic equations

Theorem 2.1. *Let R be a local ring. Then $A \in M_2(R)$ is strongly nil clean if and only if A is nilpotent or $I_2 - A$ is nilpotent or A is similar to a matrix $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, where $\lambda \in N(R), \mu \in 1 + N(R)$.*

Proof. If either A or $I_2 - A$ is nilpotent, then A is strongly nil clean. For any nilpotent elements $w_1, w_2 \in R$, $\begin{pmatrix} w_1 & 0 \\ 0 & 1+w_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} w_1 & 0 \\ 0 & w_2 \end{pmatrix}$ is a strongly nil clean decomposition. Thus, one direction is clear.

Conversely, assume that $A \in M_2(R)$ is strongly nil clean. Let $\pi : 2R \to 2R$ be the corresponding R-morphism of A. In view of [6, Definition 1.2.8], there exists a corresponding decomposition $2R = C \oplus D$ into π-invariant direct summands such that π is nilpotent on C and $1 - \pi$ is nilpotent on D. Since a local ring has invariant basis number and is projective-free, we have three cases: $C = 2R, D = 0$ or $C = 0, D = 2R$ or C and D are both rank 1 free modules. In this last case, π is diagonalizable; necessarily with one diagonal entry that is nilpotent and one that is 1 minus nilpotent. □

Corollary 2.2. *Let R be a strongly nil clean local ring. Then the following are equivalent*:

(1) *$A \in M_2(R)$ is strongly nil clean.*
(2) *A is nilpotent or $I_2 - A$ is nilpotent or A is similar to a diagonal matrix.*

Proof. (1) $\Rightarrow$ (2) is trivial from Theorem 2.1.

(2) $\Rightarrow$ (1) If A is nilpotent or $I_2 - A$ is nilpotent, then $A \in M_2(R)$ is strongly nil clean. Otherwise, A is similar to a diagonal matrix, and so there exists some

$P \in GL_2(R)$ such that $P^{-1}AP = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}$ for some $r_1, r_2 \in R$. Since R is a strongly nil clean local ring, it follows by [6, Proposition 3.2.2] that $J(R)$ is nil and $R/J(R) \cong \mathbb{Z}_2$. If $r_1, r_2 \in J(R)$, then they are both nilpotent; hence, $A \in M_2(R)$ is nilpotent. If $r_1, r_2 \notin J(R)$, it follows from $R/J(R) \cong \mathbb{Z}_2$ that $1 - r_1, 1 - r_2 \in J(R)$; hence, $1 - r_1, 1 - r_2 \in R$ are nilpotent. This implies that $I_2 - A$ are nilpotent. If $r_1 \in J(R)$ and $r_2 \notin J(R)$, then $r_1, r_2 - 1 \in R$ are nilpotent. Hence A is similar to $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} r_1 & 0 \\ 0 & r_2 - 1 \end{pmatrix}$. If $r_1 \notin J(R)$ and $r_2 \in J(R)$, A is similar to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} r_1 - 1 & 0 \\ 0 & r_2 \end{pmatrix}$, where $r_1 - 1, r_2 \in R$ are both nilpotent. Therefore we complete the proof by Theorem 2.1. □

Corollary 2.3. *Let R be a commutative local ring. Then $M_2(R)$ is not strongly nil clean.*

Proof. Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \in M_2(R)$. Then $A \in GL_2(R)$ and $I_2 - A \in GL_2(R)$. Thus, $A, I_2 - A \in M_2(R)$ are not nilpotent. Suppose that $A \in M_2(R)$ is strongly nil clean. In view of Theorem 2.1, A is similar to a diagonal matrix $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, where $\lambda \in N(R), \mu \in 1 + N(R)$. Hence, $-1 = \det A = \lambda\mu \in N(R)$, a contradiction. Therefore $A \in M_2(R)$ is not strongly nil clean. □

Let R be a commutative ring, and let $A = (a_{ij}) \in M_2(R)$. Denote $\mathrm{tr}(A) = a_{11} + a_{12}$ and $\det(A) = a_{11}a_{22} - a_{12}a_{21}$. Many authors characterized strongly cleanness by means of quadratic equations, but their techniques could not directly be extended to strongly nil cleanness. We will investigate the strongly nil cleanness of a single 2×2 matrix over commutative local rings in a new route. It is shown that such property can be characterized by a class of quadratic equations completely.

Lemma 2.4. *Let R be a commutative ring, and let $A = (a_{ij}) \in M_2(R)$. If $a_{21} \in U(R)$ and the equation $x^2 - \mathrm{tr}A \cdot x + \det A = 0$ has two roots $x_1, x_2 \in R$ such that $x_1 - x_2 \in U(R)$, then A is similar to a diagonal reduction.*

Proof. See [4, Lemma 6.1]. □

Theorem 2.5. *Let R be a commutative local ring. Then the following are equivalent*:

(1) *$A \in M_2(R)$ is strongly nil clean.*
(2) *A is nilpotent or $I_2 - A$ is nilpotent or the equation $x^2 - \mathrm{tr}A \cdot x + \det A = 0$ has a root in $N(R)$ and a root in $1 + N(R)$.*

Proof. (1) ⇒ (2) Let $A \in M_2(R)$ be strongly nil clean. Suppose that $A, I_2 - A \in M_2(R)$ are not nilpotent. In view of Theorem 2.1, A is similar to the matrix $B = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \in M_2(R)$, where $\lambda \in N(R), \mu \in 1 + N(R)$. Thus, $x^2 - \mathrm{tr}A \cdot x + \det A = \det(xI_2 - A) = \det(xI_2 - B) = (x - \lambda)(x - \mu)$. Hence, $x^2 - \mathrm{tr}A \cdot x + \det A = 0$ has a root $\lambda \in N(R)$ and a root $\mu \in 1 + N(R)$.

(2) ⇒ (1) Let $A \in M_2(R)$. If either $A \in M_2(R)$ or $I_2 - A \in M_2(R)$ is nilpotent, then $A \in M_2(R)$ is strongly nil clean. Otherwise, it follows by hypothesis that the equation $x^2 - \mathrm{tr}A \cdot x + \det A = 0$ has a root $x_1 \in N(R)$

and a root $x_2 \in 1 + N(R)$. Clearly, $x_1 - x_2 \in U(R)$. Further, we deduce that $\mathrm{tr}A = x_1 + x_2$ and $\det A = x_1x_2$. As $\det A \in N(R)$, $A \notin GL_2(R)$. Obviously, $\det(I_2 - A) = 1 - \mathrm{tr}A + \det A \in N(R)$, and so $I_2 - A \notin GL_2(R)$. According to [10, Lemma 4], there are some $\lambda \in J(R), \mu \in 1 + J(R)$ such that A is similar to $B = \left(\begin{smallmatrix} 0 & \lambda \\ 1 & \mu \end{smallmatrix}\right)$. Obviously, $x^2 - \mathrm{tr}B \cdot x + \det B = \det(xI_2 - B) = \det(xI_2 - A) = x^2 - \mathrm{tr}A \cdot x + \det A$; and so $x^2 - \mathrm{tr}B \cdot x + \det B = 0$ has a root in $1 + N(R)$ and a root in $N(R)$. According to Lemma 2.4, there exists a $P \in GL_2(R)$ such that $P^{-1}BP = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right)$ for some $\alpha \in N(R), \beta \in 1 + N(R)$. Thus, $P^{-1}BP = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) + \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta-1 \end{smallmatrix}\right)$ is a strongly nil clean expression. Therefore $A \in M_2(R)$ is strongly nil clean, as desired. □

Corollary 2.6. *Let R be a commutative local ring. Then $A \in M_2(R)$ is strongly nil clean if and only if one of the following holds*:

(1) *A is nilpotent or $I_2 - A$ is nilpotent.*

(2) $\mathrm{tr}A \in 1+N(R)$ *and the equation* $x-x^2 = (\mathrm{tr}A)^{-2} \det A$ *has a nilpotent root.*

Proof. (1) ⇒ (2) Let $A \in M_2(R)$ be strongly nil clean. Assume that $A, I_2 - A \in M_2(R)$ are not nilpotent. In view of Theorem 2.1, A is similar to $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right)$, where $\lambda \in N(R), \mu \in 1 + N(R)$. Thus, $\mathrm{tr}A = \lambda + \mu, \det A = \lambda\mu$. In view of Theorem 2.5, $y^2 - (\lambda + \mu)y + \lambda\mu = 0$ has a nilpotent root. Thus, so is the equation $y - (\lambda + \mu)^{-1}y^2 = (\lambda + \mu)^{-1}\lambda\mu$. Set $x = (\lambda + \mu)^{-1}y$. Then $(\lambda + \mu)x - (\lambda + \mu)x^2 = (\lambda + \mu)^{-1}\lambda\mu$. We infer that $x - x^2 = (\lambda + \mu)^{-2}\lambda\mu$. Therefore $x - x^2 = (\mathrm{tr}A)^{-2} \det A$ has a nilpotent root.

(2) ⇒ (1) Assume that A and I_2-A are not nilpotent. Then $\mathrm{tr}A \in 1+N(R)$ and the equation $y - y^2 = (\mathrm{tr}A)^{-2} \det A$ has a nilpotent root $\alpha \in R$. Clearly, $1-\alpha \in R$ is a root of the equation. Choose $x = \mathrm{tr}A\cdot y$. Then $x^2-\mathrm{tr}A\cdot x+\det A = 0$ has a root in $1+N(R)$ and a root in $N(R)$. By using Theorem 2.5, we complete the proof. □

Corollary 2.7. *Let R be a commutative local ring, and let $A \in M_2(R)$. Then the following are equivalent*:

(1) *$A \in M_2(R)$ is strongly nil clean.*

(2) *$A \in N\big(M_2(R)\big)$ or $I_2 - A \in N\big(M_2(R)\big)$, or* $\mathrm{tr}A \in 1 + N(R), \det A \in N(R)$ *and the equation* $x^2 - x = \frac{\det A}{\mathrm{tr}^2 A-4\det A}$ *is solvable.*

Proof. (1) ⇒ (2) In view of Corollary 2.6, we may assume that $\mathrm{tr}A \in 1+N(R)$ and the equation $x^2 - x = -\frac{\det A}{\mathrm{tr}^2 A}$ has a root $a \in N(R)$. Then $\det A \in N(R)$. It is easy to verify that

$$\begin{aligned} \big(a(2a-1)^{-1}\big)^2 - \big(a(2a-1)^{-1}\big) &= \frac{\det A}{\mathrm{tr}^2 A\cdot\big(4(a^2-a)+1\big)} \\ &= \frac{\det A}{\mathrm{tr}^2 A\cdot\big(-4(\mathrm{tr}A)^{-2}\det A+1\big)} \\ &= \frac{\det A}{\mathrm{tr}^2 A-4\det A}. \end{aligned}$$

Therefore the equation $x^2 - x = \frac{\det A}{\mathrm{tr}^2 A-4\det A}$ is solvable.

(2) $\Rightarrow$ (1) Assume that A and $I_2 - A$ are not in $N\big(M_2(R)\big)$. Then $\text{tr}A \in 1 + N(R), \det A \in N(R)$ and the equation $x^2 - x = \frac{\det A}{\text{tr}^2 A - 4\det A}$ has a root $a \in R$. Clearly, $b := 1 - a \in R$ is a root of this equation. One easily checks that $\big(a(2a-1)^{-1}\text{tr}A\big)^2 - \text{tr}A \cdot \big(a(2a-1)^{-1}\text{tr}A\big) + \det A = -\frac{\text{tr}^2 A \cdot (a^2 - a)}{4(a^2-a)+1} + \det A = 0$. Thus, the equation $x^2 - \text{tr}A \cdot x + \det A = 0$ has roots $a(2a-1)^{-1}\text{tr}A$ and $b(2b-1)^{-1}\text{tr}A$. Clearly, $ab = -\frac{\det A}{\text{tr}^2 A - 4\det A} \in N(R)$. In addition, $a + b = 1$. As R is local, either a or b is invertible in R. This implies that either $a \in N(R)$ or $b \in N(R)$. Thus, $x^2 - \text{tr}A \cdot x + \det A = 0$ has a root in $1 + N(R)$ and a root in $N(R)$. According to Theorem 2.5, we obtain the result. $\square$

Let $\mathbb{Z}_4 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}\}$. Then $\mathbb{Z}_4$ is a commutative local ring. Let $A = \begin{pmatrix} \overline{1} & \overline{1} \\ \overline{2} & \overline{2} \end{pmatrix} \in M_2\big(\mathbb{Z}_4\big)$. Then $\text{tr}A = \overline{3} \in \overline{1} + N\big(\mathbb{Z}_4\big)$ and $\det A = \overline{0} \in N\big(\mathbb{Z}_4\big)$. It is easy to see that $x^2 - x = \overline{0}$ has a root $\overline{0}$. Therefore $A \in M_2\big(\mathbb{Z}_4\big)$ is strongly nil clean by Corollary 2.7. In this case, neither A nor $I_2 - A$ is nilpotent in $M_2\big(\mathbb{Z}_4\big)$.

3. Triangular form

In [6], Diesl characterized triangular strongly nil clean rings. He showed that a local ring R is strongly nil clean if and only if $T_n(R)$ is strongly nil clean for every positive integer n ([6, Theorem 3.2.5]). The aim of this section is to give an explicit description of a single triangular matrix over local rings, though the ring of all triangular matrix maybe not strongly nil clean. Let $a \in R$. $l_a : R \to R$ and $r_a : R \to R$ denote, respectively, the abelian group endomorphisms given by $l_a(r) = ar$ and $r_a(r) = ra$ for all $r \in R$. Thus, $l_a - r_b$ is an abelian group endomorphism such that $(l_a - r_b)(r) = ar - rb$ for any $r \in R$. Following Diesl, a local ring R is bleached provided that for any $a \in U(R), b \in J(R)$, $l_a - r_b, l_b - r_a$ are both surjective. The class of beached local rings contains many familiar examples, e.g., commutative local rings, local rings with nil Jacobson radical, local rings for which some power of each element of their Jacobson radicals is central (cf. [6, Example 2.1.11]).

Proposition 3.1. *Let R be a local ring, and let $A = (A_{ij}) \in T_n(R)$. Then $A \in T_n(R)$ is strongly nil clean if and only if each $A_{ii} \in N(R)$ or $1 + N(R)$.*

Proof. Let $A \in T_n(R)$ be strongly nil clean. Then we can find an idempotent $E = (e_{ij}) \in T_n(R)$ such that $EA = AE$ and $A - E \in T_n(R)$ is nilpotent. Clearly, each e_{ii} is 0 or 1. This infers that each $A_{ii} \in N(R)$ or $1 + N(R)$.

Let $A \in T_n(R)$. Conversely, assume that each $A_{ii} \in N(R)$ or $1 + N(R)$. Define the collection $\{e_{ii}\}_{i=1}^n$ of idempotents of R be setting $e_{ii} = 0$ if $A_{ii} \in N(R)$ and $e_{ii} = 1$ if $A_{ii} \in 1 + N(R)$. If $e_{ii} \neq e_{jj}$, then $A_{ii} \in 1 + N(R), A_{jj} \in N(R)$ or $A_{ii} \in N(R), A_{jj} \in 1 + N(R)$. Assume that $A_{ii} \in N(R), A_{jj} \in 1 + N(R)$. Write $A_{ii}^k = 0$. As in the proof of [6, Example 2.1.11], it is easy to verify that $\big(l_{A_{ii}} - r_{A_{jj}}\big)^{-1} = l_{A_{ii}^{-1}} + l_{A_{ii}^{-2}} r_{A_{jj}} + \cdots + l_{A_{ii}^{-k}} r_{A_{jj}^{k-1}}$. Thus, the equation $l_{A_{ii}} - r_{A_{jj}}$ is surjective. Assume that $A_{ii} \in N(R), A_{jj} \in 1 + N(R)$.

Similarly, the equation $l_{A_{ii}} - r_{A_{jj}}$ is surjective. According to [6, Lemma 2.1.6], we can find an idempotent $E \in T_n(R)$ such that $AE = EA$ and $E_{ij} = e_{ij}$. This implies that $A - E \in T_n(R)$ is nilpotent. Therefore $A \in T_n(R)$ is strongly nil clean, as asserted. $\square$

Let R be a local ring. Immediately, we deduce that a lower matrix $\begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix}$ is strongly nil clean in the ring of all lower triangular matrices if and only if an upper matrix $\begin{pmatrix} a_{11} & a_{21} \\ 0 & a_{22} \end{pmatrix} \in T_2(R)$ is strongly nil clean in the ring of all upper triangular matrices if and only if each $a_{ii} \in N(R)$ or $1 + N(R)$. Let R be a local ring, and let $\mathcal{T}(R) = \{\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \mid a_{11}, a_{22}, a_{33}, a_{21}, a_{23} \in R\}$. Then $\mathcal{T}(R)$ is a 3×3 subring of $M_3(R)$ under the usual addition and multiplication. In fact, $\mathcal{T}(R)$ possesses the similar form of both the ring of all lower triangular matrices and the ring of all upper triangular matrices. The strong cleanness of $\mathcal{T}(R)$ for some special local rings is well known. A natural problem asks if the strongly nil cleanness of such subrings of $M_n(R)$ $\big(T_n(R)\big)$ coincides with that of R. This inspires us to consider the strongly nil cleanness of $\mathcal{T}(R)$.

Theorem 3.2. *Let R be a local ring. Then $A \in \mathcal{T}(R)$ is strongly nil clean if and only if each $A_{ii} \in N(R)$ or $1 + N(R)$.*

Proof. (1) $\Rightarrow$ (2) Let $A = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \in \mathcal{T}(R)$. Then there exists an idempotent $E = \begin{pmatrix} e_{11} & 0 & 0 \\ e_{21} & e_{22} & e_{23} \\ 0 & 0 & e_{33} \end{pmatrix} \in \mathcal{T}(R)$ such that $EA = AE$ and $A - E \in \mathcal{T}(R)$ is nilpotent. As R is local, each e_{ii} is 0 or 1. Clearly, $A - E \in \mathcal{T}(R)$ is nilpotent if and only if each $a_{ii} - e_{ii} \in R$ is nilpotent. This implies that each $a_{ii} \in N(R)$ or $1 + N(R)$, as required.

(2) $\Rightarrow$ (1) Let $A = (a_{ij}) \in \mathcal{T}(R)$.

Case 1. $a_{11}, a_{22}, a_{33} \in 1 + N(R)$. Then $A = I_2 + (A - I_2)$ is a strongly nil clean decomposition.

Case 2. $a_{11}, a_{22}, a_{33} \in N(R)$. Choose $E = 0$. Then $A = E + (A - E)$ is a strongly nil clean decomposition.

Case 3. $a_{11}, a_{22} \in 1 + N(R), a_{33} \in N(R)$. As in the proof of Proposition 3.1, we can find some $e_{23} \in R$ such that $a_{22}e_{23} - e_{23}a_{33} = a_{23}$. Choose $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e_{23} \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{T}(R)$. Then $E = E^2$ and $A = E+(A-E)$, where $A-E \in N(\mathcal{T}(R))$. In addition,

$$EA = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} + e_{23}a_{33} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{22}e_{23} \\ 0 & 0 & 0 \end{pmatrix} = AE.$$

Hence $A \in \mathcal{T}(R)$ is strongly nil clean.

Case 4. $a_{11}, a_{22} \in N(R), a_{33} \in 1 + N(R)$. As in the proceeding discussion, we can find some $e_{23} \in R$ such that $a_{22}e_{23} - e_{23}a_{33} = -a_{23}$. Choose $E = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e_{23} \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{T}(R)$. Then $E = E^2$ and $A = E+(A-E)$, where $A-E \in N(\mathcal{T}(R))$.

In addition,

$$EA = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e_{23}a_{33} \\ 0 & 0 & a_{33} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a_{22}e_{23} + a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} = AE.$$

Hence $A \in \mathcal{T}(R)$ is strongly nil clean.

Case 5. $a_{11}, a_{33} \in 1 + N(R), a_{22} \in N(R)$. Thus, we can find some $e_{21}, e_{23} \in R$ such that $a_{22}e_{21} - e_{21}a_{11} = -a_{21}$ and $a_{22}e_{23} - e_{23}a_{33} = -a_{23}$. Choose $E = \left(\begin{smallmatrix} 1 & 0 & 0 \\ e_{21} & 0 & e_{23} \\ 0 & 0 & 1 \end{smallmatrix}\right) \in \mathcal{T}(R)$. Then $E = E^2$ and $A = E + (A - E)$, where $A - E \in N(\mathcal{T}(R))$. In addition,

$$EA = \begin{pmatrix} a_{11} & 0 & 0 \\ e_{21}a_{11} & 0 & e_{23}a_{33} \\ 0 & 0 & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} + a_{22}e_{21} & 0 & a_{22}e_{23} + a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} = AE.$$

Hence $A \in \mathcal{T}(R)$ is strongly nil clean.

Case 6. $a_{11}, a_{33} \in N(R), a_{22} \in 1+N(R)$. We can find some $e_{21}, e_{23} \in R$ such that $a_{22}e_{21} - e_{21}a_{11} = a_{21}$ and $a_{22}e_{23} - e_{23}a_{33} = a_{23}$. Choose $E = \left(\begin{smallmatrix} 0 & 0 & 0 \\ e_{21} & 1 & e_{23} \\ 0 & 0 & 0 \end{smallmatrix}\right) \in \mathcal{T}(R)$. Then $E = E^2$ and $A = E + (A - E)$, where $A - E \in N(\mathcal{T}(R))$. In addition,

$$\begin{aligned} EA &= \begin{pmatrix} 0 & 0 & 0 \\ e_{21}a_{11} + a_{21} & a_{22} & a_{23} + e_{23}a_{33} \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ a_{22}e_{21} & a_{22} & a_{22}e_{23} \\ 0 & 0 & 0 \end{pmatrix} = AE. \end{aligned}$$

Hence $A \in \mathcal{T}(R)$ is strongly nil-clean.

Case 7. $a_{11} \in 1 + N(R), a_{22}, a_{33} \in N(R)$. Clearly, we can find some $e_{21} \in R$ such that $a_{22}e_{21} - e_{21}a_{11} = -a_{21}$. Choose $E = \left(\begin{smallmatrix} 1 & 0 & 0 \\ e_{21} & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right) \in \mathcal{T}(R)$. Then $E = E^2$ and $A = E + (A - E)$, where $A - E \in N(\mathcal{T}(R))$. In addition,

$$EA = \begin{pmatrix} a_{11} & 0 & 0 \\ e_{21}a_{11} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} + a_{22}e_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = AE.$$

Hence $A \in \mathcal{T}(R)$ is strongly nil clean.

Case 8. $a_{11} \in N(R), a_{22}, a_{33} \in 1 + N(R)$. Then we can find some $e_{21} \in R$ such that $a_{22}e_{21} - e_{21}a_{11} = a_{21}$. Choose $E = \left(\begin{smallmatrix} 0 & 0 & 0 \\ e_{21} & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right) \in \mathcal{T}(R)$. Then $E = E^2$ and $A = E + (A - E)$, where $A - E \in N(\mathcal{T}(R))$. In addition,

$$EA = \begin{pmatrix} 0 & 0 & 0 \\ e_{21}a_{11} + a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ a_{22}e_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} = AE.$$

Hence $A \in \mathcal{T}(R)$ is strongly nil clean.

Therefore $\mathcal{T}(R)$ is strongly nil clean. □

Corollary 3.3. *Let R be a local ring. Then the following are equivalent*:

(1) *R is strongly nil clean.*
(2) *$T_3(R)$ is strongly nil clean.*
(3) *$\mathcal{T}(R)$ is strongly nil clean.*

Proof. (1) ⇔ (2) is obvious by [6, Theorem 3.2.5].

(2) ⇒ (3) According to [6, Proposition 3.2.2], $R/J(R) \cong \mathbb{Z}_2$ and $J(R)$ is nil. Thus, each $a_{ii} \in N(R)$ or $1 + N(R)$. Therefore $\mathcal{T}(R)$ is strongly nil clean from Theorem 3.2.

(3) ⇒ (1) Suppose that $\mathcal{T}(R)$ is strongly nil clean. Let $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then $R \cong E\mathcal{T}(R)E$. Thus, R is strongly nil clean from [6, Corollary 3.2.4]. □

Corollary 3.4. *Let R be a local ring. Then the following are equivalent*:

(1) *R is strongly nil clean.*
(2) *The subring ring $\{\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \mid$ each $a_{ij} \in R\}$ is strongly nil clean.*
(3) *The subring ring $\{\begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \mid$ each $a_{ij} \in R\}$ is strongly nil clean.*

Proof. (1) ⇔ (3) Construct a map $\varphi : \mathcal{T}(R) \to \{\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \mid \text{each } a_{ij} \in R\}$ given by $\varphi(A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} A \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ for any $A \in \mathcal{T}(R)$. One easily checks that φ is a ring isomorphism. Therefore we complete the proof by Corollary 3.3.

(1) ⇔ (2) is symmetric. □

Analogously, we can derive the following.

Proposition 3.5. *Let R be a local ring. Then the following are equivalent*:

(1) *R is strongly nil clean.*
(2) *The subring ring $\{\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & 0 & a_{33} \end{pmatrix} \mid$ each $a_{ij} \in R\}$ is strongly nil clean.*
(3) *The subring ring $\{\begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} \mid$ each $a_{ij} \in R\}$ is strongly nil clean.*

Let $\mathbb{Z}_4 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}\}$. One directly verifies that $\mathbb{Z}_4$ is a commutative local, strongly nil clean ring. According to Corollary 3.3 and Proposition 3.5, the rings

$$\begin{pmatrix} \mathbb{Z}_4 & 0 & 0 \\ \mathbb{Z}_4 & \mathbb{Z}_4 & 0 \\ \mathbb{Z}_4 & \mathbb{Z}_4 & \mathbb{Z}_4 \end{pmatrix}, \begin{pmatrix} \mathbb{Z}_4 & 0 & 0 \\ 0 & \mathbb{Z}_4 & 0 \\ \mathbb{Z}_4 & 0 & \mathbb{Z}_4 \end{pmatrix}, \begin{pmatrix} \mathbb{Z}_4 & 0 & 0 \\ \mathbb{Z}_4 & \mathbb{Z}_4 & \mathbb{Z}_4 \\ 0 & 0 & \mathbb{Z}_4 \end{pmatrix}$$

are all strongly nil clean, but the full matrix ring $M_3(\mathbb{Z}_4)$ is not strongly nil clean.

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