Bull. Korean Math. Soc. **48** (2011), No. 4, pp. 725–736
DOI 10.4134/BKMS.2011.48.4.725

NEW RESULT CONCERNING MEAN SQUARE EXPONENTIAL STABILITY OF UNCERTAIN STOCHASTIC DELAYED HOPFIELD NEURAL NETWORKS

Chuanzhi Bai

Abstract. By using the Lyapunov functional method, stochastic analysis, and LMI (linear matrix inequality) approach, the mean square exponential stability of an equilibrium solution of uncertain stochastic Hopfield neural networks with delayed is presented. The proposed result generalizes and improves previous work. An illustrative example is also given to demonstrate the effectiveness of the proposed result.

1. Introduction

In practical implementation of neural networks, the weight coefficients of the neurons depend on certain resistance and capacitance values, which are subject to uncertainties. It is important to ensure that the designed network is stable in the presence of these uncertainties. For the parameter uncertainties, there have been a great deal of robust stability criteria proposed by many researchers, see for example [1, 3, 8, 12, 15, 16, 17] and the references therein.

In the past few years, neural networks with stochastic perturbations have attracted increasing research attention in the neural network community since, in real nervous systems, the synaptic transmission is a noisy process brought on by random fluctuations from the release of neurotransmitters and other probabilistic causes. It has been revealed in [2] that a neural network could be stabilized or destabilized by certain stochastic inputs.

Accordingly, the stability analysis problem for stochastic neural networks and the synchronization problem for delayed neural networks with stochastic perturbation have been an important research issue, and some preliminary results have been published, see, for example, [6, 7, 9, 10, 13, 14, 20, 22, 23].

Recently, Wan and Sun [19] studied the mean square exponential stability of stochastically perturbed Hopfield-type neural networks with constant fixed

Received November 18, 2009; Revised January 5, 2010.
2010 *Mathematics Subject Classification.* Primary 34F05, 60H10, 93D09.
Key words and phrases. stochastic Hopfield neural networks, mean square exponential stability, linear matrix inequality.
This work was financially supported by NNSC 10771212 and NSJEO 06KJB110010.

time delays

$$
(1)\qquad
\begin{aligned}
dx_i(t) &= [-c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t-\tau_j))]dt \\
&\quad + \sum_{j=1}^{n} \sigma_{ij}(x_j(t)) dw_j(t), \\
x_i(t) &= \xi_i(t), \quad -\tau \le t \le 0, \quad i = 1, 2, \ldots, n,
\end{aligned}
$$

by means of variation parameter, inequality technique and stochastic analysis.

In this paper, motivated by [12], [17], [19], we consider the problem of mean square exponential stability for a class of uncertain stochastic Hopfield neural networks with discrete delays. By using the Lyapunov functional technique and stochastic analysis, an unified LMI approach is developed to establish sufficient conditions for the neural networks to be robustly, exponentially stable in the mean square. This condition is in terms of LMI, which can be readily verified by using standard numerical software [4], [5]. In the special case with certain case, the LMI condition given in this paper generalizes and improves those given in [19], [23].

2. Model description and preliminaries

Consider the following uncertain stochastic Hopfiled-type neural networks with fixed time delays

$$
(2)\qquad
\begin{aligned}
dx_i(t) &= [-c_i x_i(t) + \sum_{j=1}^{n} (a_{ij} + \Delta a_{ij}) f_j(x_j(t)) \\
&\quad + \sum_{j=1}^{n} (b_{ij} + \Delta b_{ij}) f_j(x_j(t-\tau_j))]dt + \sum_{j=1}^{n} \sigma_{ij}(x_j(t)) dw_j(t), \\
x_i(t) &= \xi_i(t), \quad -\tau \le t \le 0, \quad i = 1, 2, \ldots, n,
\end{aligned}
$$

or equivalently,

$$
(3)\qquad
\begin{aligned}
dx(t) &= [-Cx(t) + (A + \Delta A) f(x(t)) + (B + \Delta B) f(x(t-\tau))]dt \\
&\quad + \sigma(x(t)) dw(t), \\
x(t) &= \xi(t), \quad -\tau \le t \le 0,
\end{aligned}
$$

where

$$x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \in \mathbb{R}^n,$$

is the state vector, n denotes the number of neurons, and the superscript T to any vector (or matrix) denotes the transpose of that vector (or matrix),

$$f(x(t)) = (f_1(x_1(t)), f_2(x_2(t)), \ldots, f_n(x_n(t)))^T,$$

is the neuron activation function, while $f(x(t-\tau))$ denotes

$$f(x(t-\tau)) = (f_1(x_1(t-\tau_1)), f_2(x_2(t-\tau_2)), \ldots, f_n(x_n(t-\tau_n)))^T,$$

and

$$\xi(t) = (\xi_1(t), \xi_2(t), \dots, \xi_n(t))^T,$$

is the initial condition; τ_i, $i = 1, 2, \dots, n$, are the transmission delays, $\tau = \max_{1\le i\le n} \tau_i$; $C = \text{diag}(c_1, c_2, \dots, c_n)$ is a positive definite diagonal matrix, c_i represents the rate with which the ith unit will reset its potential to the resting state in isolation when disconnected from the network and the external stochastic perturbation; $A = (a_{ij})_{n\times n} \in \mathbb{R}^{n\times n}$ is referred to as the feedback matrix, $B = (b_{ij})_{n\times n} \in \mathbb{R}^{n\times n}$ represents the delayed feedback matrix, a_{ij} and b_{ij}, $i, j = 1, 2, \dots, n$, represent the weight coefficients of the neurons; $\Delta A = (\Delta a_{ij})$ and $\Delta B = (\Delta b_{ij})$ denote respectively, the parametric uncertainties in A and B. Moreover, $\sigma(x) = (\sigma_{ij}(x_j))_{n\times n}$, and

$$w(t) = (w_1(t), w_2(t), \dots, w_n(t))^T$$

is n-dimensional Brownian motion defined on a complete probability space (Ω, F, P) with a natural filtration $\{F_t\}_{t\ge 0}$ generated by $\{w(s) : 0 \le s \le t\}$, where we associate Ω with the canonical space generated by $w(t)$, and denote by F the associated σ-algebra generated by $\{w(t)\}$ with the probability measure P. Note that, $\xi = \{\xi(s) : -\tau \le s \le 0\}$ is $C([-\tau, 0]; \mathbb{R}^n)$-valued function, which is F_0-measurable $\mathbb{R}^n$-valued random variables, where $C([-\tau, 0]; \mathbb{R}^n)$ is the space of all continuous $\mathbb{R}^n$-value functions defined on $[-\tau, 0]$ with a norm $\|\xi\|_\tau = \sup\{|\xi(t)| : -\tau \le t \le 0\}$ and $|x|$ is the Euclidean norm of a vector $x \in \mathbb{R}^n$. $\xi \in L^2_{F_0}([-\tau, 0], \mathbb{R}^n)$, where $L^2_{F_0}([-\tau, 0], \mathbb{R}^n)$ is a $\mathbb{R}^n$-valued stochastic process.

For the vector $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$ and the matrix A, we define the norms as follows:

$$|x(t)| = \left[\sum_{i=1}^n |x_i(t)|^2\right]^{1/2}, \quad \|A\| = \sup\{|Ax| : |x| = 1\} = \sqrt{\lambda_{\max}(A^T A)},$$

where $\lambda_{\max}(\cdot)$ (respectively, $\lambda_{\min}(\cdot)$) means the largest (respectively, smallest) eigenvalue of A.

Throughout this paper, let us list the following assumption:

(H_1) $f_i(0) = \sigma_{ij}(0) = 0$, f_i and σ_{ij} are Lipschitz-continuous with Lipschitz constant $l_i > 0$ and $\nu_{ij} > 0$, respectively, for $i, j = 1, 2, \dots, n$.

It follows from [18], [21] that under the Assumption (H_1), (3) has a unique global solution on $t \ge 0$, which is denoted by $x(t; \xi)$, or, $x(t)$ if no confusion occurs. Clearly, (3) admits an equilibrium solution $x(t) \equiv 0$.

We are now ready to introduce the notion of robust global stability for the stochastic neural network (3) with parameter uncertainties and time-delays.

Definition 2.1. For the neural network (3) and every $\xi \in L^2_{F_0}([-\tau, 0], \mathbb{R}^n)$, the trivial solution (equilibrium point) is robustly, globally stable in the mean square if there exists a pair of positive constants λ and c such that

$$\mathbf{E}|x(t, \xi)|^2 \le c\mathbf{E}|\xi|^2 e^{-\lambda t}, \quad t \ge 0.$$

Let $C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+;\ \mathbb{R}_+)$ denote the family of all nonnegative functions $V(x,t)$ on $\mathbb{R}^n \times \mathbb{R}_+$ which is continuously twice differentiable in x and one differentiable in t. For each $V \in C^{2,1}([-\tau,\infty) \times \mathbb{R}^n;\ \mathbb{R}_+)$, define an operator φV, associated with the stochastic delayed Hopfield neural networks (3), from $\mathbb{R}_+ \times C([-\tau,0];\ \mathbb{R}^n)$ to $\mathbb{R}$ by

$$\varphi V = V_t(t,x) + V_x(t,x)[-Cx(t) + \overline{A}f(x(t)) + \overline{B}f(x(t-\tau))] + \frac{1}{2}\mathrm{trace}[\sigma(x)^T V_{xx}(t,x)\sigma(x)],$$

where

$$\overline{A} = A + \Delta A, \qquad \overline{B} = B + \Delta B,$$
$$V_t(t,x) = \frac{\partial V(t,x)}{\partial t}, \qquad V_x(t,x) = \left(\frac{\partial V(t,x)}{\partial x_1}, \frac{\partial V(t,x)}{\partial x_2}, \ldots, \frac{\partial V(t,x)}{\partial x_n}\right),$$
$$V_{xx}(t,x) = \left(\frac{\partial^2 V(t,x)}{\partial x_i \partial x_j}\right).$$

In the following, $P > Q$ ($P \geq Q$) means that the matrix $P - Q$ is positive definite ($P-Q$ is semi-positive definite, respectively), 0 denotes the null matrix or null vector of appropriate dimension, and I denotes the identity matrix of appropriate dimension.

The uncertainty ΔA is assumed to satisfy

$$\Delta A = HFE, \tag{4}$$

where H and E are known constant matrices of appropriate dimensions, and F is an unknown matrix representing the parameter uncertainty, which satisfies

$$F^T F \leq I. \tag{5}$$

The uncertainty model of (4) and (5) is well known [21]. The matrices H and E characterize how the uncertain parameters in F enter A. The F can always be restricted as (5) by appropriate choosing H and E, i.e., there is no loss of generality in choosing F as in (5). Similarly, the uncertainty ΔB is assumed to be of the form

$$\Delta B = H_1 F_1 E_1, \quad (F_1)^T F_1 \leq I.$$

In order to obtain our main result, we need the following lemmas:

Lemma 2.2 ([4])**.** *For a given matrix* $S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix}$ *with* $S_{11} = S_{11}^T$, $S_{22} = S_{22}^T$, *then the following conditions are equivalent*:

(i) $S < 0$;

(ii) $S_{22} < 0,\ S_{11} - S_{12}S_{22}^{-1}S_{12}^T < 0.$

Lemma 2.3 ([18])**.** *Let* U, V, W, *and* M *be real matrices of appropriate dimensions with* M *satisfying* $M = M^T$. *Then*

$$M + UVW + W^T V^T U^T < 0$$

for all $V^T V \leq I$ *if and only if there exists a scalar* $\varepsilon > 0$ *such that*

$$M + \varepsilon^{-1} U U^T + \varepsilon W^T W < 0.$$

3. Main result

Theorem 3.1. *Suppose that* (H_1) *holds. If there exist positive definite matrix* $P > 0$*, positive definite diagonal matrix* $D = \mathrm{diag}\{d_i > 0\} \in \mathbb{R}^{n\times n}$*, positive semidefinite diagonal matrix* $K = \mathrm{diag}\{k_i \geq 0\} \in \mathbb{R}^{n\times n}$*, and positive scalars* $\delta > 0$*,* $\varepsilon > 0$ *and* $\varepsilon_1 > 0$ *such that the following LMI*:

(6)
$$\begin{bmatrix} -2PC + \delta I & PA + K & PB & PH_1 & PH \\ A^{\mathrm{T}}P + K & D + \varepsilon E^{\mathrm{T}}E - 2KL & 0 & 0 & \\ B^{\mathrm{T}}P & 0 & -D + \varepsilon_1 E_1^{\mathrm{T}} E_1 & 0 & 0 \\ H_1^{\mathrm{T}} P & 0 & 0 & -\varepsilon_1 I & 0 \\ H^{\mathrm{T}} P & 0 & 0 & 0 & -\varepsilon I \end{bmatrix} < 0,$$

where $L = \mathrm{diag}(1/l_1, 1/l_2, \ldots, 1/l_n)$ $(l_i$ *are as in* $(H_1))$*. Moreover, if*

$$\max_{1\leq j\leq n} \sum_{i=1}^{n} \nu_{ij}^2 \leq \frac{\delta}{\|P\|} \tag{7}$$

holds, where ν_{ij} *are as in* (H_1)*. Then the dynamics of the neural network* (3) *is robustly, globally stable in the mean square.*

Proof. Firstly, we will prove that (6) holds which implies

$$N = \begin{bmatrix} 2PC - \delta I & -P\overline{A} - K & -P\overline{B} \\ -(\overline{A})^T P - K & -D + 2KL & 0 \\ -(\overline{B})^T P & 0 & D \end{bmatrix} > 0. \tag{8}$$

Indeed, according to Lemma 2.2 (Schur complement), (6) is equivalent to

$$-\begin{bmatrix} 2PC - \delta I & -PA - K & -PB & -PH_1 \\ -A^T P - K & -D + 2KL - \varepsilon E^{\mathrm{T}} E & 0 & 0 \\ -B^T P & 0 & D - \varepsilon_1 E_1^{\mathrm{T}} E_1 & 0 \\ -H_1^T P & 0 & 0 & \varepsilon_1 I \end{bmatrix}$$
$$+ \begin{bmatrix} -PH \\ 0 \\ 0 \\ 0 \end{bmatrix} \varepsilon^{-1} I [-H^T P \ 0 \ 0 \ 0] < 0,$$

which can be written as

$$
(9)\quad \begin{aligned} &-\begin{bmatrix} 2PC-\delta I & -PA-K & -PB & -PH_1 \\ -A^TP-K & -D+2KL & 0 & 0 \\ -B^TP & 0 & D-\varepsilon_1 E_1^{\mathrm{T}}E_1 & 0 \\ -H_1^TP & 0 & 0 & \varepsilon_1 I \end{bmatrix} \\ &+\varepsilon^{-1}\begin{bmatrix} PH \\ 0 \\ 0 \\ 0 \end{bmatrix}[H^TP\ 0\ 0\ 0]+\varepsilon\begin{bmatrix} 0 \\ E^T \\ 0 \\ 0 \end{bmatrix}[0\ E\ 0\ 0]<0. \end{aligned}
$$

From (9) and Lemma 2.3, noting that $F^TF \leq I$, we get

$$
(10)\quad \begin{aligned} &-\begin{bmatrix} 2PC-\delta I & -PA-K & -PB & -PH_1 \\ -A^TP-K & -D+2KL & 0 & 0 \\ -B^TP & 0 & D-\varepsilon_1 E_1^{\mathrm{T}}E_1 & 0 \\ -H_1^TP & 0 & 0 & \varepsilon_1 I \end{bmatrix} \\ &+\begin{bmatrix} PH \\ 0 \\ 0 \\ 0 \end{bmatrix}F[0\ E\ 0\ 0]+\begin{bmatrix} 0 \\ E^T \\ 0 \\ 0 \end{bmatrix}F^T[H^TP\ 0\ 0\ 0]<0, \end{aligned}
$$

which implies

$$
(11)\quad -\begin{bmatrix} 2PC-\delta I & -P\overline{A}-K & -PB & -PH_1 \\ -(\overline{A})^TP-K & -D+2KL & 0 & 0 \\ -B^TP & 0 & D-\varepsilon_1 E_1^{\mathrm{T}}E_1 & 0 \\ -H_1^TP & 0 & 0 & \varepsilon_1 I \end{bmatrix}<0.
$$

Using Lemma 2.2 again, (11) is equivalent to

$$
\begin{aligned} &-\begin{bmatrix} 2PC-\delta I & -P\overline{A}-K & -PB \\ -(\overline{A})^TP-K & -D+2KL & 0 \\ -B^TP & 0 & D-\varepsilon_1 E_1^{\mathrm{T}}E_1 \end{bmatrix} \\ &+\begin{bmatrix} -PH_1 \\ 0 \\ 0 \end{bmatrix}\varepsilon_1^{-1}I[-H_1^TP\ 0\ 0]<0, \end{aligned}
$$

which can be written as

$$
(12)\quad \begin{aligned} &-\begin{bmatrix} 2PC-\delta I & -P\overline{A}-K & -PB \\ -(\overline{A})^TP-K & -D+2KL & 0 \\ -B^TP & 0 & D \end{bmatrix} \\ &+\varepsilon_1^{-1}\begin{bmatrix} PH_1 \\ 0 \\ 0 \end{bmatrix}[H_1^TP\ 0\ 0]+\varepsilon_1\begin{bmatrix} 0 \\ 0 \\ E_1^T \end{bmatrix}[0\ 0\ E_1]<0. \end{aligned}
$$

By (12) and Lemma 2.3, similar to that of (10) and (11), we obtain that

$$-\begin{bmatrix} 2PC-\delta I & -P\overline{A}-K & -P\overline{B} \\ -(\overline{A})^T P-K & -D+2KL & 0 \\ -(\overline{B})^T P & 0 & D \end{bmatrix} < 0,$$

that is $N > 0$, a.e., (8) holds.

Secondly, we construct the following positive definite Lyapunov functional:

$$V(t, x(t)) = x^T(t)Px(t) + \sum_{i=1}^{n} d_i \int_{t-\tau_i}^{t} f_i^2(x_i(s))ds, \tag{13}$$

where $P = P^T > 0$ and $d_i > 0$, $i = 1, 2, \ldots, n$. By (H_1) and condition (7), we get that

$$\begin{aligned} &\frac{1}{2}\text{trace}(\sigma^T(x(t))V_{xx}\sigma(x(t))) = \text{trace}(\sigma^T(x(t))P\sigma(x(t))) \\ &\le \|P\|\text{trace}(\sigma^T(x(t))\sigma(x(t)) \le \|P\| \sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij}^2(x_j(t)) \\ &\le \|P\| \max_{1\le j\le n} \sum_{i=1}^{n} \nu_{ij}^2 \sum_{j=1}^{2} x_j^2(t) = \|P\| \max_{1\le j\le n} \sum_{i=1}^{n} \nu_{ij}^2 x^T(t)x(t) \\ &\le \delta x^T(t)Ix(t). \end{aligned} \tag{14}$$

By Ito's formula and (14), we can calculate φV along system (3):

$$\begin{aligned} \varphi V &= 2x^T(t)P[-Cx(t) + \overline{A}f(x(t)) + \overline{B}f(x(t-\tau))] + f^T(x(t))Df(x(t)) \\ &\quad - f^T(x(t-\tau))Df(x(t-\tau)) + \frac{1}{2}\text{trace}\sigma^T(x(t))V_{xx}(t,x)\sigma(x(t)) \\ &\le -2x^T(t)PCx(t) + 2x^T(t)P\overline{A}f(x(t)) + 2x^T(t)P\overline{B}f(x(t-\tau)) \\ &\quad + f^T(x(t))Df(x(t)) - f^T(x(t-\tau))Df(x(t-\tau)) + \delta x^T(t)Ix(t). \end{aligned} \tag{15}$$

After some rearrangement, (15) can be expressed as

$$\begin{aligned} \varphi V \le& -[x^T(t)\ f^T(x(t))\ f^T(x(t-\tau))]N \begin{bmatrix} x(t) \\ f(x(t)) \\ f(x(t-\tau)) \end{bmatrix} \\ &- 2f^T(x(t))K[x(t) - Lf(x(t))], \end{aligned} \tag{16}$$

where $K = \text{diag}\{k_i\} \in \mathbb{R}^{n\times n}$ is a positive semi-definite diagonal matrix ($k_i \ge 0, i = 1, 2, \ldots, n$), and N as in (8). Since $N > 0$, there exists a constant $\delta_* > 0$ such that

$$N - \begin{bmatrix} \delta_* I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} > 0. \tag{17}$$

Owing to (H_1), the last term of (16) is non-positive. So, from (16) and (17), we get

$$\varphi V \leq -\delta_* x^T(t)x(t) = -\delta_* |x(t)|^2. \tag{18}$$

By (H_1), one can compute

$$\begin{aligned} &\sum_{i=1}^{n} d_i \int_{t-\tau_i}^{t} f_i^2(x_i(s))ds = \sum_{i=1}^{n} d_i \int_{-\tau_i}^{0} f_i^2(x_i(t+s))ds \\ &\leq \sum_{i=1}^{n} d_i l_i^2 \int_{-\tau_i}^{0} x_i^2(t+s)ds \leq \int_{-\tau}^{0} \sum_{i=1}^{n} d_i l_i^2 x_i^2(t+s)ds \\ &= \int_{-\tau}^{0} x(t+s)^{\mathrm{T}} G x(t+s)ds, \end{aligned} \tag{19}$$

where $G = \mathrm{diag}(d_1 l_1^2, d_2 l_2^2, \ldots, d_n l_n^2) > 0$. It is easy to check that there exists a unique positive number $\gamma > 0$ such that

$$\gamma \|P\| + \|G\| e^{\gamma\tau} = \delta_* + \|G\|, \tag{20}$$

where $\delta_* > 0$ as in (17).

The Ito's formula shows that for any $t \geq 0$

$$\begin{aligned} e^{\gamma t} V(t, x(t)) = V(0, x(0)) &+ \int_0^t e^{\gamma s} [\gamma V(s, x(s)) + \varphi V(s, x(s))]ds \\ &+ \int_0^t e^{\gamma s} V_x(x(s), s)\sigma(x(s))dw(s). \end{aligned} \tag{21}$$

Taking expectations in (21), we get

$$\begin{aligned} \mathbf{E}(e^{\gamma t} V(t, x(t))) \leq \mathbf{E}(\xi^{\mathrm{T}}(0)P\xi(0)) &+ \|G\| \int_{-\tau}^{0} \mathbf{E}|\xi(s)|^2 ds \\ &- [\delta_* - \gamma\|P\|] \int_0^t e^{\gamma s} \mathbf{E}|x(s)|^2 ds \\ &+ \gamma \int_0^t e^{\gamma s} ds \int_{-\tau}^{0} \mathbf{E}x^{\mathrm{T}}(s+\theta)Gx(s+\theta)d\theta. \end{aligned} \tag{22}$$

For $t \geq 0$, we have

$$\begin{aligned} &\int_0^t e^{\gamma s} ds \int_{-\tau}^{0} \mathbf{E}x^{\mathrm{T}}(s+\theta)Gx(s+\theta)d\theta \\ &= \int_0^t \mathbf{E}x^{\mathrm{T}}(u)Gx(u)du \int_u^{u+\tau} e^{\gamma s} ds + \int_{-\tau}^{0} \mathbf{E}x^{\mathrm{T}}(u)Gx(u)du \int_0^{u+\tau} e^{\gamma s} ds \\ &\leq \frac{e^{\gamma\tau} - 1}{\gamma} \|G\| \int_0^t e^{\gamma u} \mathbf{E}|x(u)|^2 du + \frac{e^{\gamma\tau} - 1}{\gamma} \|G\| \int_{-\tau}^{0} \mathbf{E}|\xi(u)|^2 du. \end{aligned} \tag{23}$$

Substituting (20) and (23) into (22), we obtain

$$\mathbf{E}(e^{\gamma t} V(t, x(t)) \leq c,$$

where

$$c = \mathbf{E}(\xi^{\mathrm{T}}(0)P\xi(0)) + e^{\gamma\tau}\|G\| \int_{-\tau}^{0} \mathbf{E}|\xi(s)|^2 ds < \infty,$$

since $\xi \in L^2_{F_0}([-\tau, 0], \mathbb{R}^n)$, a.e., $\int_{-\tau}^{0} \mathbf{E}|\xi(s)|^2 ds < \infty$. Hence, we have from the definition of V that

$$\mathbf{E}|x(t)|^2 \leq c(\lambda_{\min}(P))^{-1} e^{-\gamma t}, \qquad t \geq 0,$$

that is, Eq. (3) is exponentially stable in mean square. □

4. An example and remarks

In this section, we will make some comments and give an example to illustrate that the conditions given in this paper are more useful as compared with those in [19], [23].

Example 4.1. Consider a two-neural delayed stochastic neural network with parameter uncertainties

$$\begin{aligned} & d\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \\ (24) \quad = & -C\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} dt + (A + \triangle A)\begin{pmatrix} f_1(x_1(t)) \\ f_2(x_2(t)) \end{pmatrix} dt \\ & + (B + \triangle B)\begin{pmatrix} f_1(x_1(t-\tau_1)) \\ f_2(x_2(t-\tau_2)) \end{pmatrix} dt + \begin{pmatrix} \nu_{11}x_1(t) & \nu_{12}x_2(t) \\ \nu_{21}x_1(t) & \nu_{22}x_2(t) \end{pmatrix} dw(t), \end{aligned}$$

where

$$C = \begin{pmatrix} 0.9 & 0 \\ 0 & 0.8 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -1 \\ -1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0.1 \\ -0.1 & 0 \end{pmatrix},$$

$$\triangle A = HFE, \ \triangle B = H_1 F_1 E_1,$$

$$E = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.1 \end{pmatrix}, \quad H = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.2 \end{pmatrix},$$

$$E_1 = \begin{pmatrix} 0.1 & 0.1 \\ 0 & 0 \end{pmatrix}, \quad H_1 = \begin{pmatrix} 0 & 0 \\ -0.1 & -0.1 \end{pmatrix}.$$

F, F_1 are two unknown matrix representing the parameter uncertainty, which satisfy

$$F^T F \leq I, \quad F_1^T F_1 \leq I,$$

$$f_1(x) = 0.909 \arctan x, \ f_2(x) = \sin x, \tag{25}$$

$$\nu_{11} = 0.2, \quad \nu_{12} = 0.2, \quad \nu_{21} = 0.25, \quad \nu_{22} = 0.15. \tag{26}$$

The activation functions f_i in this example are satisfy Assumption (H_1) with

$$l_1 = 0.9093, \quad l_2 = 1. \tag{27}$$

Thus, we have $L = \begin{bmatrix} 1.1 & 0 \\ 0 & 1 \end{bmatrix}$.

By using the Matlab toolbox, we find a solution to the LMI (6) are as follows:

$$P = \begin{bmatrix} 3.4241 & -0.9874 \\ -0.9874 & 2.5831 \end{bmatrix}, \quad D = \begin{bmatrix} 1.0791 & 0 \\ 0 & 1.0791 \end{bmatrix},$$

$$K = \begin{bmatrix} 2.8799 & 0 \\ 0 & 2.8799 \end{bmatrix}, \quad \delta = 0.4104,\ \varepsilon = 3.5085,\ \varepsilon_1 = 3.1974.$$

Since the Lipschitz constants of functions σ_{ij} $(i, j = 1, 2)$ in this example with ν_{ij}, we have

$$\max_{1\le j\le 2} \sum_{i=1}^{2} \nu_{ij}^2 = 0.085 < \frac{\delta}{\|P\|} = 0.093.$$

Therefore, by Theorem 3.1 in this paper, we have that the delayed stochastic neural network (24) with parameter uncertainties is robustly global stable in square mean.

Remark 4.1. In Example 4.1, if we choose $F = F_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, then it is easy to check that $F^T F \le I$ and $F_1^T F_1 \le I$. Moreover, we obtain

$$\triangle A = HFE = \begin{pmatrix} 0.06 & 0 \\ 0 & 0.02 \end{pmatrix}, \quad \triangle B = H_1 F_1 E_1 = \begin{pmatrix} 0 & 0 \\ -0.01 & -0.01 \end{pmatrix},$$

and

$$\overline{A} = A + \triangle A = (\bar{a}_{ij}) = \begin{pmatrix} 0.06 & -1 \\ -1 & -0.98 \end{pmatrix},$$

$$\overline{B} = B + \triangle B = (\bar{b}_{ij}) = \begin{pmatrix} 0 & 0.1 \\ -0.11 & -0.01 \end{pmatrix}.$$

Thus, network (24) reduces to

(28)

$$\begin{aligned} d\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} &= -\begin{pmatrix} 0.9 & 0 \\ 0 & 0.8 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} dt + \begin{pmatrix} 0.06 & -1 \\ -1 & -0.98 \end{pmatrix}\begin{pmatrix} f_1(x_1(t)) \\ f_2(x_2(t)) \end{pmatrix} dt \\ &\quad + \begin{pmatrix} 0 & 0.1 \\ -0.11 & -0.01 \end{pmatrix}\begin{pmatrix} f_1(x_1(t-\tau_1)) \\ f_2(x_2(t-\tau_2)) \end{pmatrix} dt + \begin{pmatrix} \nu_{11}x_1(t) & \nu_{12}x_2(t) \\ \nu_{21}x_1(t) & \nu_{22}x_2(t) \end{pmatrix} dw(t) \end{aligned}$$

f_i and ν_{ij} are as in (25) and (26), respectively. By Example 4.1, we know that the delayed stochastic neural network (28) is exponentially stable in mean square.

But, the criteria of exponentially stable in mean square in [18] fail in neural network (28). In fact, we have

$$\bar{L} = (4\nu_{ij}^2) = \begin{pmatrix} 0.16 & 0.25 \\ 0.16 & 0.09 \end{pmatrix},$$

$$D_1 = \mathrm{diag}(4a_1c_1^{-1}, 4a_2c_2^{-1}) = \mathrm{diag}(4.4578, 9.1315),$$

$$D_2 = \mathrm{diag}(4b_1c_1^{-1}, 4b_2c_2^{-1}) = \mathrm{diag}(0.0444, 0.05),$$

where

$$a_i = \sum_{j=1}^{2} \bar{a}_{ij}^2 l_j^2, \quad b_i = \sum_{j=1}^{2} \bar{b}_{ij}^2 l_j^2, \quad i = 1, 2,$$

l_j are as in (27), $c_1 = 0.9$ and $c_2 = 0.8$. Thus,

$$\rho(C^{-1}(D_1\bar{K} + D_2\bar{K} + \bar{L})) = 16.8657 > 1,$$

where $\bar{K} = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, $\rho(M)$ denotes the spectral radius of a square matrix M. Hence the condition given in Theorem 3.1 in [19] is not satisfied. This implies that the obtained result generalizes and improves those given in [19].

Remark 4.2. For a two-neural delayed stochastic neural network (28), according to the notation of the paper [23], we have

$$c_1 = 0.9,\ c_2 = 0.8,\ a_{11} = 0.06,\ a_{12} = -1,\ a_{21} = -1,\ a_{22} = -0.98,$$
$$b_{11} = 0,\ b_{12} = 0.1,\ b_{21} = -0.11, b_{22} = -0.01,\ \alpha_1 = \alpha_2 = 0.9093,$$
$$\beta_1 = \beta_2 = 1,\ L_{11} = 0.2,\ L_{12} = 0.2,\ L_{21} = 0.25,\ L_{22} = 0.15.$$

Thus, we obtain that

$$\begin{aligned} &-2c_1 + \sum_{j=1}^{2} |a_{1j}|\alpha_j + \sum_{j=1}^{2} |a_{j1}|\alpha_1 + \sum_{j=1}^{2} |b_{1j}|\beta_j + \sum_{j=1}^{2} |b_{j1}|\beta_1 + \sum_{j=1}^{2} L_{j1}^2 \\ &= 0.4402 > 0, \end{aligned}$$

$$\begin{aligned} &-2c_2 + \sum_{j=1}^{2} |a_{2j}|\alpha_j + \sum_{j=1}^{2} |a_{j2}|\alpha_2 + \sum_{j=1}^{2} |b_{2j}|\beta_j + \sum_{j=1}^{2} |b_{j2}|\beta_2 + \sum_{j=1}^{2} L_{j2}^2 \\ &= 2.2933 > 0, \end{aligned}$$

which implies that the condition (A2) of Theorem 3.1 in [23] is not satisfied. Hence the criteria of exponentially stable in mean square in [23] fail in neural network (28). But, by Example 4.1, we know that the delayed stochastic neural network (28) is exponentially stable in mean square. So, our result obtained here generalizes and improves those given in [23].

Acknowledgement. The author would like to thank the referees for their valuable suggestions and comments.

References

[1] M. Basin, J. Perez, and R. Martinez-Zuniga, *Optimal filtering for nonlinear polynomial systems over linear observations with delay*, Int. J. Innovative Comput. Inform. Control **2** (2006), no. 4, 863–874.

[2] S. Blythe, X. Mao, and X. X. Liao, *Stability of stochastic delay neural networks*, J. Franklin Inst. **338** (2001), no. 4, 481–495.

[3] E. K. Boukas and N. F. Al-Muthairi, *Delay-dependent stabilization of singular linear systems with delays*, Int. J. Innovative Comput. Inform. Control **2** (2006), no. 2, 283–291.

[4] S. Boyd, L. EI Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, SIAM Philadelphia, PA, 1994.

[5] P. Gahinet, A. Nemirovski, A. J. Laub, and M. Chilali, *LMI Control Toolbox: for Use with Matlab*, The Math Works, Natick, 1995.

[6] C. Huang, P. Chen, Y. He, L. Huang, and W. Tan, *Almost sure exponential stability of delayed Hopfield neural networks*, Appl. Math. Lett. **21** (2008), no. 7, 701–705.

[7] H. Huang, D. W. C. Ho, and J. Lam, *Stochastic stability analysis of fuzzy Hopfield neural networks with time-varying delays*, IEEE Trans. Circuits Syst. II **52** (2005), no. 5, 251–255.

[8] O. M. Kwon, J. H. Park, and S. M. Lee, *On delay-dependent robust stability of uncertain neutral systems with interval time-varying delays*, Appl. Math. Comput. **203** (2008), no. 2, 843–853.

[9] T. Li, A. Song, and S. Fei, *Robust stability of stochastic Cohen-Grossberg neural networks with mixed time-varying delays*, Neurocomputing **73** (2009), no. 1-3, 542–551.

[10] X. Liu and T. Chen, *Robust μ-stability for uncertain stochastic neural networks with unbounded time-varying delays*, Phys. A. **387** (2008), no. 12, 2952–2962.

[11] X. Mao, *Exponential Stability of Stochastic Differential Equations*, Marcel Dekker, New York, 1994.

[12] J. H. Park, *Further note on global exponential stability of uncertain cellular neural networks with variable delays*, Appl. Math. Comput. **188** (2007), no. 1, 850–854.

[13] J. H. Park and O. M. Kwon, *Synchronization of neural networks of neutral type with stochastic perturbation*, Modern Phys. Lett. B **23** (2009), no. 14, 1743–1751.

[14] J. H. Park, S. M. Lee, and H. Y. Jung, *LMI optimization approach to synchronization of stochastic delayed discrete-time complex networks*, J. Optim. Theory Appl. **143** (2009), no. 2, 357–367.

[15] J. Qiu, J. Zhang, and P. Shi, *Robust stability of uncertain linear systems with time-varying delay and nonlinear perturbations*, Proc. IMechE, Part I, J. Syst. Control Eng. **220** (2006), no. 5, 411–416.

[16] V. Singh, *Novel LMI condition for global robust stability of delayed neural networks*, Chaos Solitons Fractals **34** (2007), no. 2, 503–508.

[17] ______, *Global robust stability of delayed neural networks: An LMI approach*, IEEE Trans. Circuits Syst. II **52** (2005), no. 1, 33–36.

[18] ______, *Robust stability of cellular neural networks with delay: linear matrix inequality approach*, IEE Proc. Contr. Theory Appl. **151** (2004), no. 1, 125–129.

[19] L. Wan and J. Sun, *Mean square exponential stability of stochastic delayed Hopfield neural networks*, Phys. Lett. A **343** (2005), 306–318.

[20] Z. Wang, S. Lauria, J. Fang, and X. Liu, *Exponential stability of uncertain stochastic neural networks with mixed time-delays*, Chaos Solitons Fractals **32** (2007), no. 1, 62–72.

[21] Z. Wang, Y. Liu, K. Fraser, and X. Liu, *Stochastic stability of unicertain Hopfield neural networks with discrete and distributed delays*, Phys. Lett. A **354** (2006), no. 4, 288–297.

[22] J. Zhang, P. Shi, and J. Qiu, *Novel robust stability criteria for uncertain stochastic Hopfield neural networks with time-varying delays*, Nonlinear Anal. Real World Appl. **8** (2007), no. 4, 1349–1357.

[23] Q. Zhou and L. Wan, *Exponential stability of stochastic delayed Hopfield neural networks*, Appl. Math. Comput. **199** (2008), no. 1, 84–89.

School of Mathematical Science
Huaiyin Normal University
Huaian, Jiangsu 223300, P. R. China
E-mail address: czbai8@sohu.com