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DYNAMICS OF TRANSCENDENTAL ENTIRE FUNCTIONS WITH SIEGEL DISKS AND ITS APPLICATIONS

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Abstract. We study the dynamics of transcendental entire functions with Siegel disks whose singular values are just two points. One of the two singular values is not only a superattracting fixed point with multiplicity more than two but also an asymptotic value. Another one is a critical value with free dynamics under iterations. We prove that if the multiplicity of the superattracting fixed point is large enough, then the restriction of the transcendental entire function near the Siegel point is a quadratic-like map. Therefore the Siegel disk and its boundary correspond to those of some quadratic polynomial at the level of quasiconformality. As its applications, the logarithmic lift of the above transcendental entire function has a wandering domain whose shape looks like a Siegel disk of a quadratic polynomial.

1. Introduction

Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. The *Fatou set* $F(f)$ is the set of normality in the sense of Montel for the family $\{f^n\}_{n=1}^{\infty}$, where $f^n = f \circ \cdots \circ f$ is n iterates of f. The *Julia set* $J(f)$ is the complement $\mathbb{C} \setminus F(f)$. Let z_0 be a fixed point of f with $f'(z_0) = e^{2\pi i\alpha}$, where $0 < \alpha < 1$ is irrational. The entire function f is *locally linearizable* at the fixed point z_0 if there exists a conformal map φ near z_0 with $\varphi(z_0) = 0$ such that $\varphi \circ f \circ \varphi^{-1}(z) = R_\alpha(z) = e^{2\pi i\alpha}z$. In this case, we call the irrational number α the *rotation number*. The entire function f is locally linearizable at z_0 if and only if z_0 belongs to the Fatou set. The Fatou component Δ containing z_0 is called the *Siegel disk* centered at z_0. The Siegel disk Δ is the largest domain on which f is conformally conjugate to the rotation R_α. For the irrational number α, we consider the continued fraction expansion $\alpha = [a_1, a_2, \ldots, a_n, \ldots]$, where each a_n is a positive integer. Then the sequence of rational numbers $p_n/q_n = [a_1, a_2, \ldots, a_n]$ converges to α. The irrational number α is a *Diophantine number of order* $\kappa \geq 2$ if there exists

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$\varepsilon > 0$ such that

$$\left|\alpha - \frac{p}{q}\right| > \frac{\varepsilon}{q^\kappa}$$

for all rational numbers p/q. The irrational number α belongs to the class of Diophantine numbers of order κ if and only if the sequence

$$\left\{\frac{q_{n+1}}{q_n^{\kappa-1}}\right\}_{n=1}^{\infty}$$

is bounded. In the case that $\kappa = 2$, the sequence $\{a_n\}_{n=1}^\infty$ is bounded if and only if $\{q_{n+1}/q_n\}_{n=1}^\infty$ is. Therefore Diophantine numbers of order 2 are said to be of *bounded type*. The irrational number α is a *Bryuno number* if the sum

$$\sum_{n=1}^{\infty} \frac{\log q_{n+1}}{q_n}$$

converges. Bryuno proved that if α is a Bryuno number, then f is locally linearizable at z_0. Yoccoz proved that if a quadratic polynomial $Q(z) = e^{2\pi i\alpha}z+z^2$ is locally linearizable at the origin, then α is a Bryuno number.

In this paper, we consider transcendental entire functions

$$f_c(z) = c\left(\frac{z}{c}\right)^n \exp\left\{\frac{1}{c}(\lambda - n)(z-c)\right\},$$

where $n \geq 2$, $c \in \mathbb{C} \setminus \{0\}$, $\lambda = e^{2\pi i\alpha}$ and $0 < \alpha < 1$ is a Bryuno number. Let $f = f_1$. The origin is a superattracting fixed point and c is the center of a Siegel disk Δ with rotation number α. The function f_c has the following properties:

- The origin is a superattracting fixed point with multiplicity n.
- c is a fixed point with multiplier $f_c'(c) = \lambda$.
- $cn/(n-\lambda)$ is a critical point and there is no other critical point.
- The origin is an asymptotic value.
- There is no singular value other than the origin and $f_c(cn/(n-\lambda))$.
- f_c is of finite order.

Main Theorem. *If n is large enough, then there exist bounded simply connected domains U and V satisfying $1 \in U \Subset V$ such that $f : U \to V$ is a quadratic-like map.*

A map $g : U_1 \to U_2$ is a *polynomial-like map* of degree d if g is a holomorphic proper map of degree d, where U_1 and U_2 are topological disks with $U_1 \Subset U_2$. In the case that $d = 2$, we call it a *quadratic-like map*. The *filled-in Julia set* $K(g)$ of the polynomial-like map $g : U_1 \to U_2$ is defined as

$$K(g) = \{\, z \in U_1 : g^n(z) \in U_1 \text{ for all } n \geq 0 \,\}$$

and the *Julia set* $J(g)$ of the polynomial-like map is the boundary $J(g) = \partial K(g)$ of the filled-in Julia set. The dynamics and the structure of the Julia set of a polynomial-like map correspond to those of some genuine polynomial at the

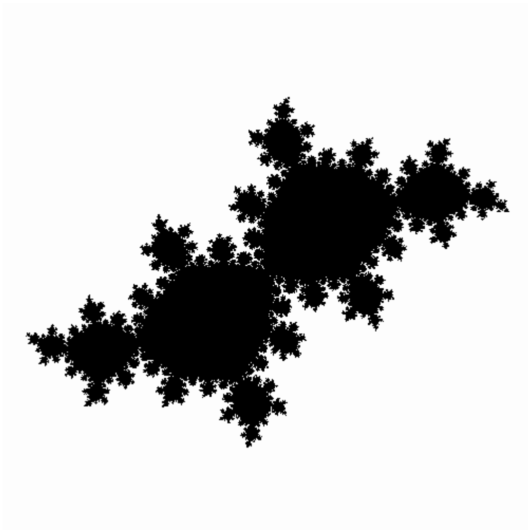

FIGURE 1. The Siegel disk of the quadratic polynomial $Q(z) = e^{2\pi i\alpha}z + z^2$ centered at the origin, where $\alpha = (\sqrt{5}-1)/2 = [1, 1, 1, \ldots]$.

level of quasiconformality (The Straightening Theorem). Therefore properties of the Siegel disk Δ of f centered at 1 and its boundary $\partial\Delta$ correspond to those of the Siegel disk $\mathcal{D}$ of Q centered at the origin and its boundary $\partial\mathcal{D}$. For examples:

1. *The function f is locally linearizable at* 1 *if and only if α is a Bryuno number.*
2. *If α is of bounded type, then the boundary $\partial\Delta$ is a quasicircle containing the critical point $n/(n-\lambda)$.*
3. *The boundary $\partial\Delta$ is a quasicircle but the critical point $n/(n-\lambda)$ is not on $\partial\Delta$ for some α.*

Topological and quasiconformal statements which hold for the Siegel disk of the quadratic polynomial Q centered at the origin hold for the Siegel disk of the transcendental entire function f centered at 1.

2. Siegel disks of quadratic polynomials and quadratic-like maps

Let $Q(z) = e^{2\pi i\alpha}z + z^2$, where $0 < \alpha < 1$ is irrational. The quadratic polynomial Q has the following properties:

- The point at infinity is a superattracting fixed point.
- The origin is a fixed point with multiplier $Q'(0) = e^{2\pi i\alpha}$.
- $-e^{2\pi i\alpha}/2$ is a critical point.

If the irrational number α is a Bryuno number, then there exists the Siegel disk $\mathcal{D}$ of Q centered at the origin. Many mathematicians have been interested in the boundary $\partial\mathcal{D}$ of the Siegel disk $\mathcal{D}$ for years and have obtained many results about it. A basic result about it is that *the boundary $\partial\mathcal{D}$ is contained in the postcritical set of* Q, namely

$$\partial\mathcal{D} \subset \overline{\{Q^n(-e^{2\pi i\alpha}/2)\}_{n=1}^{\infty}}.$$

The critical point belongs to the Julia set $J(Q)$. However we don't know whether it is on the boundary $\partial\mathcal{D}$ or not in general. Herman proved that *if the irrational number α is of Herman-Yoccoz type* (*every analytic circle diffeomorphism with rotation number α is analytically linearizable*), *then the critical point is on the boundary $\partial\mathcal{D}$.* The Siegel disk $\mathcal{D}$ is a simply connected domain. Hence there exists a conformal map between the Siegel disk and the unit disk. However we don't know whether the boundary $\partial\mathcal{D}$ is a closed Jordan curve or not in general. Many mathematicians contributed to the result that *if the irrational number α is of bounded type*, *then the boundary $\partial\mathcal{D}$ is a quasicircle containing the critical point.* Petersen and Zakeri proved that *if the irrational number α is of David type* (*the irrational number* $\alpha = [a_1, a_2, \dots, a_n, \dots]$ *satisfies that* $\log a_n = O(\sqrt{n})$ *as n tends to* ∞), *then the boundary $\partial\mathcal{D}$ is a David circle containing the critical point.* David circles are closed Jordan curves. Moreover *there exists an irrational number α of David type for which the boundary $\partial\mathcal{D}$ is a David circle but not a quasicircle.* Therefore the above David-type result contains the bounded-type one. Herman proved that *there exists a Bryuno number α such that the boundary $\partial\mathcal{D}$ is quasicircle but the critical point is not on $\partial\mathcal{D}$.* Figure 1 is the Siegel disk $\mathcal{D}$, where the rotation number $\alpha = (\sqrt{5}-1)/2 = [1, 1, 1, \dots]$ is of bounded type. Hence the boundary $\partial\mathcal{D}$ is a quasicircle containing the critical point.

Polynomial-like maps

A map $g : U_1 \to U_2$ is a *polynomial-like map* of degree d if g is a holomorphic proper map of degree d, where U_1 and U_2 are topological disks with $U_1 \Subset U_2$. In the case that $d = 2$, we call it a *quadratic-like map.* The *filled-in Julia set* $K(g)$ of the polynomial-like map $g : U_1 \to U_2$ is defined as

$$K(g) = \{\, z \in U_1 : g^n(z) \in U_1 \text{ for all } n \geq 0 \,\}$$

and the *Julia set* $J(g)$ of the polynomial-like map is the boundary $J(g) = \partial K(g)$ of the filled-in Julia set. Two polynomial-like maps g and h are *hybrid equivalent* if there exists a quasiconformal map φ defined on a neighborhood of $K(g)$ such that $\bar{\partial}\varphi = 0$ on $K(g)$ and $\varphi \circ g = h \circ \varphi$. The following theorem is called The Straightening Theorem. It means that the dynamics and the structure of the Julia set of a polynomial-like map correspond to those of some genuine polynomial at the level of quasiconformality.

Theorem 2.1 (The Straightening Theorem)**.** *Every polynomial-like map g is hybrid equivalent to some genuine polynomial p of the same degree. Moreover if $K(g)$ is connected, then the polynomial p is unique up to affine conjugation.*

The Straightening Theorem indicates that topological and quasiconformal statements which hold for the Siegel disk $\mathcal{D}$ of the quadratic polynomial Q hold for Siegel disks of quadratic-like maps.

3. Siegel disks of transcendental entire functions

3.1. Preliminaries

Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. The function f is a *transcendental entire function* if f does not extend to the point at infinity. In this case, the point at infinity is an essential singularity. A point $v \in \mathbb{C}$ is a *critical value* of f if $v = f(c)$, where c is a critical point of f. A point $v \in \mathbb{C}$ is an *asymptotic value* of f if there exists a path $\gamma : [0, 1) \to \mathbb{C}$ such that $\gamma(t) \to \infty$ and $f(\gamma(t)) \to v$ as $t \to 1$. The function f is said to be of *finite order* if there exist positive constants a and r such that

$$|f(z)| \leq \exp |z|^a$$

for $|z| > r$. The infimum of all positive numbers a for which the inequality holds is the *order* of f. Let $\{a_j\}$ be the zeros of f listed with multiplicities. The *rank* p of f is the least positive integer such that

$$\sum_{a_j \neq 0} |a_j|^{-(p+1)} < \infty.$$

If f has a finite number of zeros, then it has rank zero. Suppose that f is of finite order λ. Then the canonical Weierastrass product

$$f(z) = e^{g(z)} P(z)$$

has the property that g is a polynomial of degree $q \leq \lambda$. The *genus* μ of f is the maximum of p and q.

Theorem 3.1. *An entire function of finite genus μ is also of finite order μ and $\lambda < \mu + 1$.*

Theorem 3.2 (The Hadamard Factorization Theorem)**.** *An entire function of finite order λ is also of finite genus μ and $\mu \leq \lambda$.*

3.2. Siegel disks and dynamics

Let $f(z) = p(z) \exp(q(z))$, where p and q are polynomials. Zakeri proved the following theorem about Siegel disks of transcendental entire functions.

Theorem 3.3 (Zakeri, [30])**.** *Suppose that the origin is a fixed point of f with multiplier $f'(0) = e^{2\pi i\alpha}$. If α is of bounded type, then the boundary of the Siegel disk centered at the origin is a quasicircle containing at least one critical point of f.*

Zakeri also proved useful theorems about the function f in [30]. Suppose that the degree of the polynomial q is at least one.

Theorem 3.4 (Zakeri, [30])**.** *The function f has a unique asymptotic value at the origin.*

Theorem 3.5 (Zakeri, [30])**.** *Suppose that V is a simply connected domain in $\mathbb{C} \setminus \{0\}$ and U is a connected component of $f^{-1}(V)$. Then $f : U \to V$ is a proper map.*

Hereafter we consider transcendental entire functions

$$f_c(z) = c\left(\frac{z}{c}\right)^n \exp\left\{\frac{1}{c}(\lambda - n)(z - c)\right\},$$

where $n \geq 2$, $c \in \mathbb{C} \setminus \{0\}$, $\lambda = e^{2\pi i\alpha}$ and $0 < \alpha < 1$ is a Bryuno number. Let $f = f_1$. The origin is a superattracting fixed point and c is the center of a Siegel disk Δ with rotation number α. The function f_c has the following properties:

- The origin is a superattracting fixed point with multiplicity n.
- c is a fixed point with multiplier $f_c'(c) = \lambda$.
- $cn/(n - \lambda)$ is a critical point and there is no other critical point.
- The origin is an asymptotic value (Theorem 3.4).
- There is no singular value other than the origin and $f_c(cn/(n - \lambda))$.
- f_c is of finite order.

Lemma 3.6. *Two functions f_{c_1} and f_{c_2} are conjugate via a conformal map for all non-zero complex numbers c_1 and c_2.*

Proof. It is clear that $\varphi \circ f_{c_1} \circ \varphi^{-1} = f_{c_2}$, where $\varphi(z) = (c_2/c_1)z$. □

Let $D(\varepsilon, r)$ be the disk with center $(1 + \varepsilon)c$ and radius r,

$$D(\varepsilon, r) = \{ z \in \mathbb{C} : |z - (1 + \varepsilon)c| < r \}.$$

Lemma 3.7. *If $|c_0|$ is large enough, then the following three statements hold for a small positive number ε_0 and a large integer n_0 :*

(1) *The inequality $|f_{c_0}(z) - c_0| > 3r$ holds on the circle $\partial D(\varepsilon_0, r)$.*
(2) *The inequality $r + \varepsilon_0|c_0| < 2r$ holds.*
(3) *The critical value $f_{c_0}(c_0 n_0/(n_0 - \lambda))$ is in the disk $D(0, 2r)$.*

Proof. Let $\varepsilon > 0$ be small enough and $n \geq 2$ large enough such that $\Phi(\varepsilon, n)$ is near 1, where

$$\Phi(\varepsilon, n) = \left| e^{\lambda\varepsilon}\left(\frac{1 + \varepsilon}{e^{\varepsilon}}\right)^n - 1 \right|.$$

For $z = (1+\varepsilon)c + re^{i\theta}$, we obtain that

$$\begin{aligned}|f_c(z) - c| &= |c| \cdot \left| \left(\frac{z}{c}\right)^n \exp\left\{\frac{1}{c}(\lambda - n)(z - c)\right\} - 1 \right| \\ &= |c| \cdot \left| \left(1 + \varepsilon + \frac{re^{i\theta}}{c}\right)^n \exp\left\{(\lambda - n)\left(\varepsilon + \frac{re^{i\theta}}{c}\right)\right\} - 1 \right| \\ &= |c| \cdot \left| \exp\left\{\lambda\left(\varepsilon + \frac{re^{i\theta}}{c}\right)\right\} \cdot \left\{\frac{1 + \varepsilon + \frac{re^{i\theta}}{c}}{\exp\left(\varepsilon + \frac{re^{i\theta}}{c}\right)}\right\}^n - 1 \right|.\end{aligned}$$

If $|c_0|$ is large enough, then

$$|f_{c_0}(z) - c_0| > \frac{1}{2}\Phi(n, \varepsilon) \cdot |c_0| > \frac{1}{3}\Phi(n, \varepsilon) \cdot |c_0| > 3r.$$

If ε is small enough, then the inequality $r + \varepsilon|c_0| < 2r$ holds. Since

$$\lim_{\varepsilon \to 0} \Phi(\varepsilon, n) = 0, \quad \lim_{n \to \infty} \Phi(\varepsilon, n) = 1$$

and

$$\begin{aligned}&\lim_{n\to\infty} f_{c_0}\left(\frac{c_0 n}{n - \lambda}\right) \\ &= \lim_{n\to\infty} c_0 \left(\frac{n}{n-\lambda}\right)^n e^{-\lambda} \\ &= \lim_{n\to\infty} c_0 \left(\frac{n}{n-\lambda}\right)^\lambda \left(\frac{n}{n-\lambda}\right)^{n-\lambda} e^{-\lambda} \\ &= \lim_{n\to\infty} c_0 \left(\frac{1}{1-\lambda/n}\right)^\lambda \left\{\left(1 + \frac{\lambda}{n-\lambda}\right)^{(n-\lambda)/\lambda}\right\}^\lambda e^{-\lambda} \\ &= c_0 \cdot 1 \cdot e^\lambda \cdot e^{-\lambda} = c_0,\end{aligned}$$

there exist a small positive number ε_0 and a large integer n_0 such that the inequality

$$|f_{c_0}(z) - c_0| > \frac{1}{2}\Phi(\varepsilon, n) \cdot |c_0| > \frac{1}{3}\Phi(\varepsilon_0, n_0) \cdot |c_0| > 3r$$

holds on the circle $\partial D(\varepsilon_0, r)$, the inequality $r + \varepsilon_0|c_0| < 2r$ holds and the critical value $f_{c_0}(c_0 n_0/(n_0 - \lambda))$ is in the disk $D(0, 2r)$. □

Remark 3.8. (i) The above statement (2) implies that the fixed point c_0 is in the disk $D(\varepsilon_0, r)$ and $D(\varepsilon_0, r) \Subset D(0, 2r)$.

(ii) The above three statements (1), (2) and (3) hold for all $n \geq n_0$.

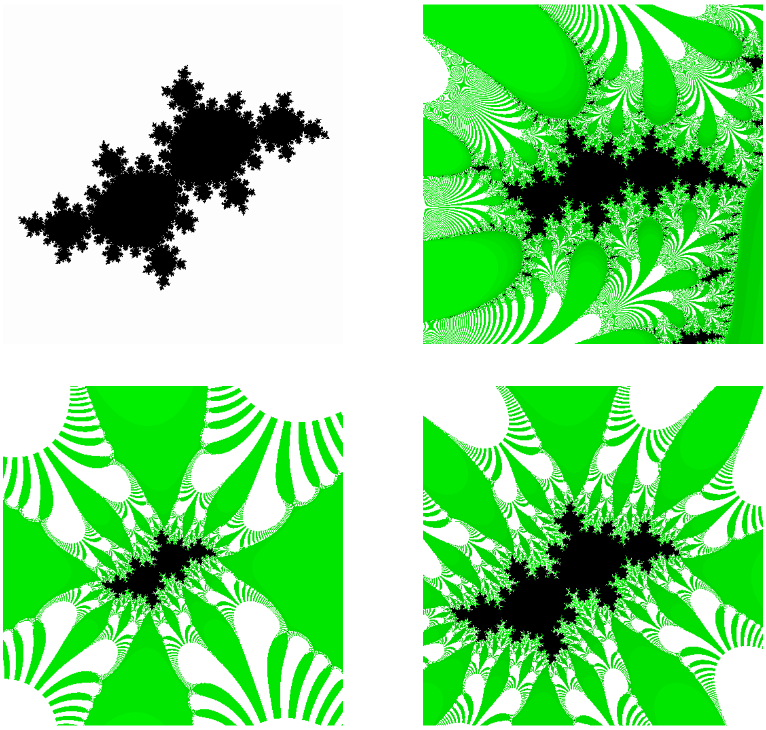

FIGURE 2. The upper left-hand figure is the Siegel disk $\mathcal{D}$ of the quadratic polynomial $Q(z) = e^{2\pi i\alpha}z + z^2$ centered at the origin, where the rotation number $\alpha = (\sqrt{5}-1)/2 = [1, 1, 1, \ldots]$. Another all figures are the Siegel disk Δ of the transcendental entire function f centered at 1. Black regions are the Siegel disk Δ and its preimages. Green ones are the attracting basin at the origin. The upper right-hand one is in the case that $n = 2$. The lower left-hand one is in the case that $n = 32$. It would be a "quadratic-polynomial-type" Siegel disk. The lower right-hand one is the enlargement of the lower left-hand one.

Theorem 3.9. *Let c_0, ε_0 and n_0 be as in Lemma 3.7. Suppose that $n \geq n_0$ is large enough, $V = D(0, 2r)$ and U is the connected component of $f_{c_0}^{-1}(V)$ containing the fixed point c_0. Then $f_{c_0} : U \to V$ is a quadratic-like map.*

Proof. By Lemma 3.7, the inclusion $U \Subset V$ holds and the critical point $c_0 n/(n-\lambda)$ is in U. By Theorem 3.5, the restriction $f_{c_0} : U \to V$ is a proper map. Therefore $f_{c_0} : U \to V$ is a quadratic-like map. $\square$

By Lemma 3.6, we obtain the Main Theorem:

Main Theorem. *If n is large enough, then there exist bounded simply connected domains U and V satisfying $1 \in U \Subset V$ such that $f : U \to V$ is a quadratic-like map.*

3.3. Corollaries

Suppose that n is large enough. Let U and V be above. By the Main Theorem, the restriction $f : U \to V$ is a quadratic-like map. Hence the Siegel disk Δ of f centered at 1 is in U and the shape looks like the Siegel disk $\mathcal{D}$ of the quadratic polynomial Q. Therefore we call Siegel disks of quadratic-like maps "quadratic-polynomial-type" Siegel disks. The Straightening Theorem indicates that topological and quasiconformal statements which hold for the Siegel disk $\mathcal{D}$ hold for the Siegel disk Δ. For examples:

Corollary 3.10. *The function f is locally linearizable at 1 if and only if the rotation number α is a Bryuno number.*

Corollary 3.11. *If the rotation number α is of bounded type, then the boundary $\partial\Delta$ is a quasicircle containing the critical point $n/(n-\lambda)$.*

Corollary 3.12. *The boundary $\partial\Delta$ is a quasicircle but the critical point $n/(n-\lambda)$ is not on $\partial\Delta$ for some α.*

4. Applications

Definition 4.1. Let g be a transcendental entire function and U a Fatou component. The Fatou component U is a *wandering domain* if all $g^m(U)$ with $m \geq 1$ are contained different Fatou components. The Fatou component U is a *Baker domain* of period p if U is a periodic Fatou component of period p such that the point at infinity is on the boundary ∂U and $g^{pm}(z) \to \infty$ for all $z \in U$ as $m \to \infty$.

Let $0 < \alpha < 1$ be a Bryuno number. We consider the logarithmic lift of f,

$$\tilde{f}(z) = nz + (\lambda - n)(e^z - 1).$$

Then the functional equation $\exp \circ \tilde{f} = f \circ \exp$ holds. The Fatou component $B = \tilde{f}(B)$ such that $\exp B$ is the immediate basin of f at the origin is an invariant Baker domain of $\tilde{f}$. The origin is a fixed point of $\tilde{f}$ with multiplier $\lambda = e^{2\pi i\alpha}$. Let $\tilde{\Delta}$ be the Siegel disk of $\tilde{f}$ centered at the origin and $\tilde{\Delta}_k$ the Fatou component containing $2\pi ki$, where k is a non-zero integer. Since the functional equation $\exp \circ \tilde{f} = f \circ \exp$ holds, the exponential map projects $\tilde{\Delta}$ and $\tilde{\Delta}_k$ down to the Siegel disk Δ of f centered at 1. The behavior of $2\pi ki$ is

$$2\pi ki \overset{\tilde{f}}{\mapsto} 2\pi kni \overset{\tilde{f}}{\mapsto} 2\pi kn^2 i \overset{\tilde{f}}{\mapsto} \cdots \overset{\tilde{f}}{\mapsto} 2\pi kn^m i \overset{\tilde{f}}{\mapsto} \cdots$$

or $\tilde{f}^m(2\pi ki) = 2\pi kn^m i$. Therefore $\{\tilde{\Delta}_{\pm k}\}_{k:\,\text{a prime number}}$ is a family of infinitely many wandering domains having distinct orbits.

Summary

The logarithmic lift $\tilde{f}$ has the following properties:

- The Fatou component $B = \tilde{f}(B)$ such that $\exp B$ is the immediate basin of f at the origin is an invariant Baker domain.
- The Siegel disk $\tilde{\Delta}$ of $\tilde{f}$ centered at the origin projects down to the Siegel disk Δ of f centered at 1 via the exponential map.
- Any Fatou component $\tilde{\Delta}_k$ containing $2\pi ki$ is a wandering domain and projects down to the Siegel disk Δ of f centered at 1 via the exponential map.
- $\{\tilde{\Delta}_{\pm k}\}_{k:\text{ a prime number}}$ is a family of infinitely many wandering domains having distinct orbits.
- $\omega_k = \log|n/(n-\lambda)| + i\cdot[\arg\{n/(n-\lambda)\} + 2\pi k]$ are critical points.

The following is an application of the Main Theorem:

Corollary 4.2. *If n is large enough, then the following statements hold*:

(1) *The Siegel disk $\tilde{\Delta}$ of $\tilde{f}$ centered at the origin is a "quadratic-polynomial-type" Siegel disk.*

(2) *The shape of Wandering domains $\tilde{\Delta}_k$ containing $2\pi ki$ looks like the Siegel disk Δ of f centered at 1.*

Corollary 4.2 indicates that if the irrational number α is of bounded type, then

- the boundary $\partial\tilde{\Delta}$ is a quasicircle containing the critical point ω_0,
- the boundary $\partial\tilde{\Delta}_k$ is also a quasicircle containing the critical point ω_k.

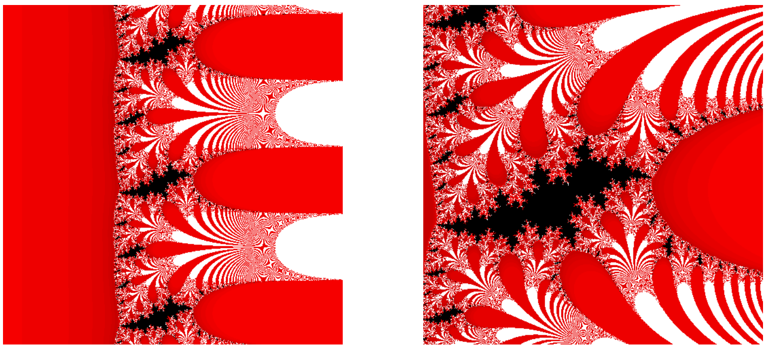

Figure 3. The two figures are in the case that $n = 2$ and the rotation number $\alpha = (\sqrt{5}-1)/2 = [1, 1, 1, \dots]$. The left-hand one is the Siegel disk $\tilde{\Delta}$ centered at the origin and wandering domains $\tilde{\Delta}_1$ and $\tilde{\Delta}_{-1}$. The right-hand one is the enlargement of the left-hand one. Black regions are the Siegel disk Δ or wandering domains $\tilde{\Delta}_k$ and their preimages. Red ones are the Baker domain B and its preimages.

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