Bull. Korean Math. Soc. **48** (2011), No. 4, pp. 705–712
DOI 10.4134/BKMS.2011.48.4.705

BOUNDARY CONTROLLABILITY OF SEMILINEAR NEUTRAL EVOLUTION SYSTEMS

Jong Yeoul Park and Jae Ug Jeong

Abstract. In this paper, we investigate the boundary control of semilinear neutral evolution systems with a nonlocal condition by using the Banach fixed point theorem.

1. Introduction

Let E and U be Banach spaces with norms $\|\cdot\|$ and $|\cdot|$, respectively. Let σ be a linear closed and densely defined operator in E and let τ be a linear operator with domain in E and range in some Banach space X.

We consider the following boundary control of semilinear neutral evolution systems with nonlocal condition:

$$\begin{aligned} &\frac{d}{dt}[y(t)+g(t,y(t))] = \sigma y(t) + f(t,y(t)), \quad t\in[0,T], \\ &\tau y(t) = B_1 u(t), \\ &y(0)+h(t_1,t_2,\ldots,t_p,y(\cdot)) = y_0, \end{aligned} \tag{1.1}$$

where $0 < t_1 < t_2 < \cdots < t_p \le T$. The operator $B_1 : U \to X$ is a linear continuous operator, the control function $u \in L^1(0,T;U)$, a Banach space of admissible control functions with U, $f : [0,T]\times E \to E$ and $h : [0,T]^p \times E \to E$ are given functions. The symbol $h(t_1,t_2,\ldots,t_p,y(\cdot))$ is used in the sense that in the place of $\cdot$ we can substitute only elements of the set $\{t_1,t_2,\ldots,t_p\}$. For example, $h(t_1,t_2,\ldots,t_p,y(\cdot))$ can be defined by the formula

$$h(t_1,t_2,\ldots,t_p,y(\cdot)) = c_1y(t_1)+c_2y(t_2)+\cdots+c_py(t_p),$$

where c_i $(i=1,2,\ldots,p)$ are given constants.

Dauer and Mahmudov [4] proved the existence of mild solutions to semilinear neutral evolution equations with nonlocal conditions. Benchohra and Ntouyes [3] consider the nonlocal Cauchy problems for neutral functional differential and integrodifferential inclusions in Banach spaces. Han and Park [5]

Received October 12, 2009.
2010 *Mathematics Subject Classification.* 93B05, 34A60.
Key words and phrases. controllability, Banach fixed point, nonlocal condition, semilinear neutral evolution equation.

studied the boundary controllability of semilinear systems with nonlocal condition. Balachandran and Anandhi [1] investigated the boundary controllability of delay integrodifferential systems in Banach space. Recently, Balachandran, Anandhi and Dauer [2] established the sufficient conditions for the boundary controllability of various types of nonlinear Sobolev-type systems including integrodifferential systems in Banach spaces.

The purpose of this paper is to study the boundary control of semilinear neutral evolution systems with nonlocal condition (1.1). This paper is organized as follows: In Section 2 we give some notations, hypotheses and definitions. In Section 3 we state the main result. In Section 4 we give some applications to illustrate our result.

2. Preliminaries

In this section, we describe necessary notations, hypotheses and definitions for the proof of the main theorem.

Let $A : E \to E$ be the linear operator defined by

$$D(A) = \{y \in D(\sigma) : \tau y = 0\}, \quad Ay = \sigma y \quad \text{for} \quad y \in D(A).$$

For the existence of a solution of (1.1), we need the following hypotheses:

($\mathbf{H_1}$) $D(\sigma) \subset D(\tau)$ and the restriction of τ to $D(\sigma)$ is continuous relative to graph norm of $D(\sigma)$.

($\mathbf{H_2}$) The operator A is the infinitesimal generator of a C_0-semigroup $S(t)$, $t \geq 0$, satisfying $\|S(t)\|_{L(E)} \leq L_1$ and $\|AS(t)\|_{L(E)} \leq L_2$.

($\mathbf{H_3}$) There exists a linear continuous operator $B : U \to E$ such that

$$\begin{aligned}
&\sigma B \in L(U, E), \quad \tau(Bu) = B_1 u \quad \text{for all} \quad u \in U,\\
&Bu \quad \text{is continuously differentiable},\\
&\|Bu\| \leq C\|B_1 u\|_X \quad \text{for all} \quad u \in U,
\end{aligned}$$

where C is some positive constant.

($\mathbf{H_4}$) For all $t \in [0, T]$ and $u \in U$, $S(t)Bu \in D(A)$. Moreover, there exists a positive function $\gamma \in L^1(0, T)$ such that

$$\|AS(t)B\|_{L(U,E)} \leq \gamma(t) \quad \text{a.e.} \quad t \in (0, T).$$

($\mathbf{H_5}$) The linear operator $W : L^2(J, U)/\mathrm{Ker}(W) \to E$ defined by

$$Wu = \int_0^T S(T-s)[AS(T-s) - S(T-s)\sigma]Bu(s)ds$$

has an invertible operator $\widetilde{W}^{-1}$ defined on $L^2(J, U)/\mathrm{Ker}\ W$ and there exists $L_3 > 0$ such that $\|W^{-1}\| \leq L_3$.

($\mathbf{H_6}$) The function $f : [0, T] \times E \to E$ is continuous in t and there exists a constant $L_4 > 0$ such that

$$\|f(t, y) - f(t, z)\| \leq L_4\|y - z\|$$

for $t \in [0,T]$, $y, z \in B_r$, where $B_r = \{y : \|y\| \le r\} \subset E$.

($\mathbf{H_7}$) The function $g : [0,T] \times E \to E$ is continuous in t and there exist constants $L_5, L_6 > 0$ such that

$$\|g(t,y) - g(t,z)\| \le L_5 \|y - z\|,$$
$$\|g(t,y)\| \le L_6 \|y\|$$

for $t \in [0,T]$ and $y, z \in B_r$.

($\mathbf{H_8}$) The function $h : [0,T]^p \times E \to E$ and there exists a constant $L_7 > 0$ such that

$$\|h(t_1, \ldots, t_p, z_1(\cdot)) - h(t_1, \ldots, t_p, z_2(\cdot))\| \le L_7 \sup_{t \in [0,T]} \|z_1(t) - z_2(t)\|$$

for $z_1, z_2 \in C([0,T]; B_r)$.

In terms A and B the system (1.1) can be written as follows:

$$\begin{aligned} &\frac{d}{dt}[y(t) + g(t, y(t))] = Az(t) + \sigma Bu(t) + f(t, y(t)), \quad t \in [0,T], \\ &y(t) = z(t) + Bu(t), \\ &y(0) + h(t_1, \ldots, t_p, y(\cdot)) = y_0. \end{aligned} \tag{2.1}$$

If u is continuously differentiable on $[0,T]$, then z can be defined as a mild solution to the following problem:

$$\begin{aligned} \frac{d}{dt}[z(t) + g(t, y(t))] &= Az(t) + \sigma Bu(t) - Bu'(t) + f(t, y(t)), \\ z(0) &= y_0 - h(t_1, \ldots, t_p, y(0)) - Bu(0). \end{aligned}$$

Thus, in this way we may define the solution y to the system (1.1) by the variation of constant formula

$$\begin{aligned} y(t) = &\, S(t)[y_0 - h(t_1, \ldots, t_p, y(\cdot)) - Bu(0) + g(0, y_0)] \\ &+ Bu(t) - \int_0^t AS(t-s) g(s, y(s)) ds \\ &+ \int_0^t S(t-s)[\sigma Bu(s) - Bu'(s) + f(s, y(s))] ds. \end{aligned} \tag{2.2}$$

Because the differentiability of the controller u represents an unrealistic and severe requirement, we are led to extend the concept of the solution to (1.1) for the general inputs $u \in L^2(0,T;U)$. Integrating by parts in (2.2) with hypothesis (H_4) we obtain

$$\begin{aligned} y(t) = &\, S(t)[y_0 - h(t_1, \ldots, t_p, y(\cdot)) + g(0, y_0)] \\ &- \int_0^t AS(t-s) g(s, y(s)) ds - \int_0^t AS(t-s) Bu(s) ds \\ &+ \int_0^t S(t-s)(\sigma Bu(s) + f(s, y(s))) ds, \end{aligned} \tag{2.3}$$

which is well defined. In addition, it is called a mild solution of the system (1.1).

Definition 2.1. The system (1.1) is said to be controllable on the interval $[0,T]$ if for every y_0, $a \in E$, there exists a control $u \in L^2([0,T];U)$ such that the solution $y(\cdot)$ of (1.1) satisfies $y(T) = a$.

Define the linear operator W from $L^2(J,U)$ into E by

$$Wu = \int_0^T [AS(T-s) - S(T-s)\sigma]Bu(s)ds,$$

then we can check that the operator W is well defined. We desire to transfer the nonlinear system (2.1) from $y(0) = y_0 - h(t_1,\dots,t_p,y(\cdot))$ to $y(T) = a$.

3. A main result

Theorem 3.1. *If the hypotheses* (H_1)-(H_8), *then the system* (1.1) *is controllable on* $[0,T]$ *provided*

$$L_1L_7 + L_1L_4T + L_2L_5T + (L_2 + L_1\|\sigma\|)\|B\|L_3 \\ \big\{L_7T + L_2L_5T + L_1L_4T\big\} = k, \quad 0 < k < 1.$$

Proof. Using the invertible operator $\widetilde{W}$, for arbitrary function $y(\cdot)$ we define the control

$$u(t) = \widetilde{W}^{-1}\Big\{a - S(T)[y_0 - h(t_1,\dots,t_p,y(\cdot)) + g(0,y_0)] \\ + \int_0^T AS(T-s)g(s,y(s))ds - \int_0^T S(T-s)f(s,y(s))ds\Big\}(t).$$

Now, using this control, we show that operator Ψ defined in the following has a fixed point. Define the operator Ψ on $C([0,T];B_r)$ by

$$(\Psi w)(t) \\ = S(t)[y_0 - h(t_1,\dots,t_p,w(\cdot)) + g(0,y_0)] - \int_0^t AS(t-s)g(s,w(s))ds \\ - \int_0^t [AS(t-s) - S(t-s)\sigma]B\widetilde{W}^{-1}\Big\{a - S(T)[y_0 - h(t_1,\dots,t_p,w(\cdot)) \\ + g(0,y_0)] + \int_0^T AS(T-\tau)g(\tau,w(\tau))d\tau \\ - \int_0^T S(T-\tau)f(\tau,w(\tau))d\tau\Big\}(s)ds + \int_0^t S(t-s)f(s,w(s))ds. \tag{2.4}$$

First of all, we show that Ψ maps $C([0,T];B_r)$ into itself. By hypotheses, there exists $M > 0$ such that $\int_0^T \gamma(t)dt \le M$ and we can choose $M_1, M_2, M_3 > 0$

such that

$$M_1 = \max_{s\in[0,T]} \|f(s,0)\|,\ M_2 = \max_{s\in[0,T]} \|g(s,0)\|,\ M_3 = \max_{w\in C([0,t];B_r)} g(0,w).$$

Also, because $w(\cdot)$ in h is continuous on $[0,T]$, we take

$$M_4 = \max_{w\in C([0,T];B_r)} \|h(t_1,\dots,t_p,w(\cdot))\|.$$

From (2.4) we have

$$\begin{aligned}
&\|(\Psi w)(t)\| \\
&\le \|S(t)y_0\| + \|S(t)h(t_1,\dots,t_p,w(\cdot))\| + \|S(t)g(0,y_0)\| \\
&\quad + \|\int_0^t AS(t-s)g(s,w(s))\| + \|\int_0^t [AS(t-s) - S(t-s)\sigma]B\widetilde{W}^{-1}\Big\{a \\
&\quad - S(T)[y_0 - h(t_1,\dots,t_p,w(\cdot)) + g(0,y_0)] \\
&\quad + \int_0^T AS(T-\tau)g(\tau,w(\tau))d\tau - \int_0^T S(T-\tau)f(\tau,w(\tau))d\tau\Big\}(s)ds\| \\
&\quad + \|\int_0^t S(t-\tau)f(s,w(s))ds\| \\
&\le L_1\|y_0\| + L_1M_4 + L_1M_3 + L_2\int_0^t \Big[\|g(s,w(s)) - g(s,0)\| + \|g(s,0)\|\Big]ds \\
&\quad + \int_0^t \Big[\|AS(t-s)B\|_{L(U,E)} + L_1\|\sigma B\|\Big]\|\widetilde{W}^{-1}\|\Big\{[\|a\| + L_1\|y_0\| + L_1M_4 \\
&\quad + L_1M_3 + L_2\int_0^T \Big[\|g(\tau,w(\tau)) - g(\tau,0)\| + \|g(\tau,0)\|\Big]d\tau \\
&\quad + L_1\int_0^T \Big[\|f(\tau,w(\tau)) - f(\tau,0)\| + \|f(\tau,0)\|\Big]d\tau\Big\}(s)ds \\
&\quad + L_1\int_0^t \Big[\|f(s,w(s)) - f(s,0)\| + \|f(s,0)\|\Big]ds \\
&\le L_1\|y_0\| + L_1M_4 + L_1M_3 + L_2TL_5r + L_2M_2T + \Big[MT + L_1T\|\sigma B\|\Big]L_3\Big\{\|a\| \\
&\quad + L_1\|y_0\| + L_1M_4 + L_1M_3 + L_1L_4Tr + L_1TM_1 + L_2L_5Tr + L_2TM_2\Big\} \\
&\quad + L_1L_4Tr + L_1TM_1.
\end{aligned}$$

Thus Ψ maps $C([0,T];B_r)$ into itself.

Now we show that Ψ is a contraction on $C([0,T];B_r)$.

Indeed,

$$\begin{aligned}
&\|(\Psi w)(t) - (\Psi\widetilde{w})(t)\| \\
&\le \|S(t)\|\|h(t_1,\dots,t_p,w(\cdot)) - h(t_1,\dots,t_p,\widetilde{w}(\cdot))\|
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t \|AS(t-s)\|\|g(s,w(s))-g(s,\widetilde{w}(s))\|ds \\
&+\int_0^t \|\big[AS(t-s)-S(t-s)\sigma\big]B\|\|\widetilde{W}^{-1}\|\Big\{\|h(t_1,\dots,t_p,w(\cdot)) \\
&-h(t_1,\dots,t_p,\widetilde{w}(\cdot))\| + \int_0^T \|AS(t-\tau)\|\|g(\tau,w(\tau))-g(\tau,\widetilde{w}(\tau))\|d\tau \\
&+\int_0^T \|S(T-\tau)\|\|f(\tau,w(\tau))-f(\tau,\widetilde{w}(\tau))\|d\tau\Big\}(s)ds \\
&+\int_0^t \|S(t-s)\|\|f(s,w(s))-f(s,\widetilde{w}(s))\|ds \\
\le\ & L_1L_7\sup_{t\in[0,T]}\|w(t)-\widetilde{w}(t)\| + L_2L_5\int_0^t \|w(s)-\widetilde{w}(s)\|ds \\
&+(L_2+L_1\|\sigma\|)\|B\|\|\widetilde{W}^{-1}\|\int_0^t\Big\{L_7\sup_{t\in[0,T]}\|w(t)-\widetilde{w}(t)\| \\
&+\int_0^t L_2L_5\|w(\tau)-\tilde{w}(\tau)\|d\tau + \int_0^T L_1L_4\|w(\tau)-\widetilde{w}(\tau)\|d\tau\Big\}(s)ds \\
&+L_1L_4\int_0^t \|w(s)-\widetilde{w}(s)\|ds \\
\le\ & \Big[L_1L_7+L_2L_5T+(L_2+L_1\|\sigma\|)\|B\|L_3\Big\{L_7T+L_2L_5T+L_1L_4T\Big\} \\
&+L_1L_4T\Big]\sup_{t\in[0,T]}\|w(t)-\widetilde{w}(t)\| = k\sup_{t\in[0,T]}\|w(t)-\widetilde{w}(t)\|,
\end{aligned}
$$

where $k = L_1L_7 + L_2L_5T + (L_2 + L_1\|\sigma\|)\|B\|L_3\big(L_7T + L_2L_5T + L_1L_4T\big) + L_1L_4T$. Because $0 < k < 1$, the operator Ψ is a contraction on $C([0,T];B_r)$. Applying the Banach fixed point theorem we get a unique fixed point for ψ in $C([0,T],B_r)$ and this point is the mild solution of the system (1.1). Consequently, the system (1.1) is controllable on $[0,T]$. □

4. An application

Let Ω be a bounded and open subset of $\mathbb{R}^n$ with a sufficiently smooth boundary Γ of class C^∞. We consider the boundary control neutral evolution system:

$$
\begin{aligned}
\frac{\partial}{\partial t}[y(t,x)+g(t,y(t))] &= \triangle y(t) + f(t,y(t)) \quad \text{in} \quad Q=(0,T)\times\Omega, \\
y(t,0) &= u(t,0) \quad \text{on} \quad \Sigma=(0,T)\times\Gamma, \\
y(0,x)+h(y(T^*,x)) &= y_0(x) \quad \text{for} \quad x\in\Omega, \quad T^*\in[0,T],
\end{aligned}
\tag{4.1}
$$

where $u \in L^2(\Sigma)$, $y_0 \in L^2(\Omega)$, $f \in L^2(Q)$ and $g \in L^2(Q)$. Moreover, we assume that the functions f, g, h are satisfied the following conditions:

$$\|f(t,y) - f(t,z)\| \le c_1\|y - z\|, \quad t \in [0,T],$$
$$\|g(t,y) - g(t,z)\| \le c_2\|y - z\|, \quad t \in [0,T],$$
$$\|g(t,y)| \le c_3\|y\|, \quad t \in [0,T]$$

and

$$\|h(w_1(T^*,x)) - h(w_2(T^*,x))\| \le c_4 \sup_{t\in[0,T]} \|w_1(t) - w_2(t)\|,$$

where c_1, c_2, c_3, c_4 are positive constants, $y, z \in B_r$ and $w_1, w_2 \in C([0,T], B_r)$.

To formulate this as a boundary control system of the form (1.1) we define $E = L^2(\Omega)$, $X = H^{-\frac{1}{2}}(\Gamma)$, $U = L^2(\Gamma)$ $B_1 = I$, $D(\sigma) = \{y \in L^2(\Omega) : \triangle y \in L^2(\Omega)\}$ and $\sigma = \triangle$. The operator τ is the trace operator $\tau y = y|_\Gamma$ which is well defined and belong to $H^{-\frac{1}{2}}(\Gamma)$ for each $y \in D(\sigma)$. The operator A is given by

$$A = \triangle, \quad D(A) = H^1(\Omega) \cap H^2(\Omega).$$

To verify (H_3) and (H_4) we define the linear operator $B : U(L^2(\Gamma)) \to L^2(\Omega)$ by $Bu = w_u$, where $w_u \in L^2(\Omega)$ is the unique solution to the Dirichlet boundary value problem:

$$\triangle w_u = 0 \quad \text{in} \quad \Omega,$$

$$w_u = u \quad \text{in} \quad \Gamma.$$

In other words,

$$\int_\Omega w_u \triangle\psi dx = \int_\Gamma u\frac{\partial\psi}{\partial\nu}dx \quad \text{for all} \quad \psi \in H_0^1(\Omega) \cap H^2(\Omega). \tag{4.2}$$

Here $\frac{\partial\psi}{\partial\nu}$ denotes the outward derivative of ψ which is well defined as an element of $H^{\frac{1}{2}}(\Gamma)$. By (4.2), it follows by a standard argument that

$$\|w_u\|_{L^2(\Omega)} \le C\|u\|_{H^{-\frac{1}{2}}(\Gamma)} \quad \text{for all} \quad u \in H^{-\frac{1}{2}}(\Gamma), \tag{4.3}$$

$$\|w_u\|_{H^1(\Omega)} \le C\|u\|_{H^{\frac{1}{2}}(\Gamma)} \quad \text{for all} \quad u \in H^{\frac{1}{2}}(\Gamma). \tag{4.4}$$

The inequality (4.3) implies the hypothesis (H_3).

Next it follows by an interpolation argument involving estimates (4.3) and (4.4) that (see [6])

$$\|AS(t)B\|_{L(L^2(\Gamma),L^2(\Gamma))} \le Ct^{-\frac{3}{4}} \quad \text{for all} \quad t > 0.$$

And the hypothesis (H_4) holds with $\gamma(t) = Ct^{-\frac{3}{4}}$. Thus all the conditions stated in Theorem 3.1 are satisfied if we take $k(0 < k < 1)$. Therefore the system (4.1) is controllable on $[0,T]$.

5. Conclusions

We have studied the boundary control of semilinear neutral evolution systems with a nonlocal condition, which is new and allow us to develop the boundary controllability of various impulsive neutral evolution systems. An application is provided to illustrate the applicability of the new result. The result presented in this paper extend and improve the corresponding ones announced by Han and Park [5], Balachandran and Anandhi [1], Balachandran, Anandhi and Dauer [2], and others.

References

[1] K. Balachandran and E. R. Anandhi, *Boundary controllability of integrodifferential systems in Banach spaces*, Proc. Indian Acad. Sci. Math. Sci. **111** (2001), no. 1, 127–135.

[2] K. Balachandran, E. R. Anandhi, and J. P. Dauer, *Boundary controllability of Sobolev-type abstract nonlinear integrodifferential systems*, J. Math. Anal. Appl. **277** (2003), no. 2, 446–464.

[3] M. Benchohra and S. K. Ntouyas, *Nonlocal Cauchy problems for neutral functional differential and integrodifferential inclusions in Banach spaces*, J. Math. Anal. Appl. **258** (2001), no. 2, 573–590.

[4] J. P. Dauer and N. I. Mahmudov, *Integral inequalities and mild solutions of semilinear neutral evolution equations*, J. Math. Anal. Appl. **300** (2004), no. 1, 189–202.

[5] H.-K. Han and J. Y. Park, *Boundary controllability of differential equations with nonlocal condition*, J. Math. Anal. Appl. **230** (1999), no. 1, 242–250.

[6] D. Washburn, *A bound on the boundary input map for parabolic equations with application to time optimal control*, SIAM J. Control Optim. **17** (1979), no. 5, 652–671.

Jong Yeoul Park
Department of Mathematics
Pusan National University
Pusan 609-735, Korea
E-mail address: `jyepark@pusan.ac.kr`

Jae Ug Jeong
Department of Mathematics
Dongeui University
Pusan 614-714, Korea
E-mail address: `jujeong@deu.ac.kr`