Bull. Korean Math. Soc. **48** (2011), No. 4, pp. 697–704
DOI 10.4134/BKMS.2011.48.4.697

ON SUFFICIENT CONDITIONS FOR STRONGLY STARLIKE FUNCTIONS ASSOCIATED WITH A LINEAR OPERATOR

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Abstract. By using the method of differential subordinations, we derive some sufficient conditions for strongly starlike functions associated with a linear operator. All these results presented here are sharp.

1. Introduction and preliminaries

Let A_p denote the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_n z^{n+p} \quad (p \in \mathbb{N} = \{1, 2, 3, \dots\}), \tag{1.1}$$

which are analytic in the open unit disk $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. A function $f(z) \in A_p$ is called p-valent starlike in U if it satisfies

$$\mathrm{Re}\frac{zf'(z)}{f(z)} > 0 \quad (z \in U). \tag{1.2}$$

Also, a function $f(z) \in A_p$ is called p-valent strongly starlike of order α $(0 < \alpha \le 1)$ if it satisfies

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\pi}{2}\alpha \quad (z \in U). \tag{1.3}$$

For functions $f_j(z) \in A_p$ $(j = 1, 2)$ given by

$$f_j(z) = z^p + \sum_{n=1}^{\infty} a_{n,j} z^{n+p} \ (j = 1, 2),$$

we define the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

$$(f_1 * f_2)(z) = z^p + \sum_{n=1}^{\infty} a_{n,1} a_{n,2} z^{n+p} = (f_2 * f_1)(z).$$

Received August 14, 2009; Revised March 3, 2010.
2010 *Mathematics Subject Classification.* Primary 30C45; Secondary 30A10.
Key words and phrases. analytic function, starlike function, strongly starlike function, subordination, Hadamard product (or convolution).

Suppose that $f(z)$ and $g(z)$ are analytic in U. We say that the function $f(z)$ is subordinate to $g(z)$ in U, and we write $f(z) \prec g(z)$, if there exists an analytic function $w(z)$ in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$, such that $f(z) = g(w(z))$ $(z \in U)$. If $g(z)$ is univalent in U, then the following equivalence relationship holds true.

$$f(z) \prec g(z)\ (z \in U) \Longleftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

For a function $f(z) \in A_p$, we consider a linear operator $Q^{\lambda_1}_{p,\lambda_2} : A_p \to A_p$ as following:

$$Q^{\lambda_1}_{p,\lambda_2} f(z) = \begin{pmatrix} p + \lambda_1 + \lambda_2 - 1 \\ p + \lambda_2 - 1 \end{pmatrix} \frac{\lambda_1}{z^{\lambda_2}} \int_0^z \left(1 - \frac{t}{z}\right)^{\lambda_1 - 1} t^{\lambda_2 - 1} f(t)dt \tag{1.4}$$
$$(\lambda_1 > 0, \lambda_2 > -1 \quad \text{and} \quad f(z) \in A_p).$$

We note that

$$Q^{\lambda_1}_{p,\lambda_2} f(z) = z^p + \sum_{n=1}^{\infty} \frac{\Gamma(p + n + \lambda_2)\Gamma(p + \lambda_1 + \lambda_2)}{\Gamma(p + n + \lambda_1 + \lambda_2)\Gamma(p + \lambda_2)} a_{p+n} z^{p+n}$$
$$(\lambda_1 > 0, \lambda_2 > -1 \quad \text{and} \quad f(z) \in A_p).$$

In particular, we have

$$Q^0_{p,\lambda_2} f(z) = f(z) \quad \text{for } \lambda_1 = 0 \text{ and } f(z) \in A_p.$$

The operator $Q^{\lambda_1}_{p,\lambda_2}$ was introduced by Liu and Owa [4]. When $p = 1$, the operator $Q^{\lambda_1}_{p,\lambda_2}$ was first introduced by Jung et al. [2]. Many interesting subclasses of analytic functions, associated with the operator $Q^{\lambda_1}_{p,\lambda_2}$, have been considered by Jung et al. [2], Aouf et al. [1], Liu [3], Liu and Owa [4] and others.

In order to prove our main results, we need the following lemma.

Lemma. *Let the function $g(z)$ be analytic and univalent in U and let the functions $\theta(w)$ and $\varphi(w)$ be analytic in a domain D containing $g(U)$, with $\varphi(w) \neq 0$ $(w \in g(U))$. Set*

$$Q(z) = zg'(z)\varphi(g(z)) \quad \textit{and} \quad h(z) = \theta(g(z)) + Q(z)$$

and suppose that

(i) $Q(z)$ *is univalently starlike in* U

and

(ii) $\operatorname{Re} \frac{zh'(z)}{Q(z)} = \operatorname{Re}\left(\frac{\theta'(g(z))}{\varphi(g(z))} + \frac{zQ'(z)}{Q(z)}\right) > 0 \quad (z \in U)$.

If $q(z)$ is analytic in U with $q(0) = g(0)$, $q(U) \subset D$ and

$$\theta(q(z)) + zq'(z)\varphi(q(z)) \prec \theta(g(z)) + zg'(z)\varphi(g(z)) = h(z) \quad (z \in U), \tag{1.5}$$

then $q(z) \prec g(z)$ $(z \in U)$ and $g(z)$ is the best dominant of (1.5).

The lemma is due to Miller and Mocanu [5, p. 132].

2. Sufficient conditions for strongly starlike functions

In this section, we assume that $\alpha, \lambda_0, \lambda, a, b \in \mathbb{R}$ and $\mu \in \mathbb{C}$.

Theorem 1. *Let*

$$0 < \alpha \leq 1, \ \lambda_0 a \geq 0, \ |b+1| \leq \frac{1}{\alpha} \quad \text{and} \quad |a-b-1| \leq \frac{1}{\alpha} \ . \tag{2.1}$$

If $f(z) \in A_p$ satisfies $Q^{\lambda_1}_{p,\lambda_2} f(z)(Q^{\lambda_1}_{p,\lambda_2} f(z))' \neq 0$ $(z \in U \setminus \{0\})$ and

$$\lambda_0 \left(\frac{z(Q^{\lambda_1}_{p,\lambda_2} f(z))'}{pQ^{\lambda_1}_{p,\lambda_2} f(z)}\right)^a + z\left(\frac{z(Q^{\lambda_1}_{p,\lambda_2} f(z))'}{pQ^{\lambda_1}_{p,\lambda_2} f(z)}\right)' \left(\frac{z(Q^{\lambda_1}_{p,\lambda_2} f(z))'}{pQ^{\lambda_1}_{p,\lambda_2} f(z)}\right)^b \prec h(z) \ (z \in U), \tag{2.2}$$

where

$$h(z) = \lambda_0 \left(\frac{1+z}{1-z}\right)^{a\alpha} + \left(\frac{1+z}{1-z}\right)^{(b+1)\alpha} \cdot \frac{2\alpha z}{1-z^2}, \tag{2.3}$$

then the function $Q^{\lambda_1}_{p,\lambda_2} f(z)$ is p-valent strongly starlike of order α in U. The number α is sharp for the function $f(z)$ defined by

$$\frac{z(Q^{\lambda_1}_{p,\lambda_2} f(z))'}{pQ^{\lambda_1}_{p,\lambda_2} f(z)} = \left(\frac{1+z}{1-z}\right)^\alpha. \tag{2.4}$$

Proof. We choose

$$q(z) = \frac{z(Q^{\lambda_1}_{p,\lambda_2} f(z))'}{pQ^{\lambda_1}_{p,\lambda_2} f(z)}, \quad g(z) = \left(\frac{1+z}{1-z}\right)^\alpha, \quad \theta(w) = \lambda_0 w^a \quad \text{and} \quad \varphi(w) = w^b$$

in Lemma. Clearly, the function $g(z)$ is analytic and univalently convex in U and

$$|\arg g(z)| < \frac{\pi}{2}\alpha \leq \frac{\pi}{2} \quad (z \in U). \tag{2.5}$$

The function $q(z)$ is analytic in U with $q(0) = g(0) = 1$ and $q(z) \neq 0 \quad (z \in U)$. The functions $\theta(w)$ and $\varphi(w)$ are analytic in a domain D containing $g(U)$ and $q(U)$, with $\varphi(w) \neq 0$ when $w \in g(U)$. For

$$-\frac{1}{\alpha} \leq b+1 \leq \frac{1}{\alpha},$$

the function $Q(z)$ given by

$$Q(z) = zg'(z)\varphi(g(z)) = \frac{2\alpha z}{(1-z)^{1+(b+1)\alpha}(1+z)^{1-(b+1)\alpha}}$$

is univalently starlike in U because

$$\begin{aligned} \mathrm{Re}\frac{zQ'(z)}{Q(z)} &= 1 + (1+(b+1)\alpha)\mathrm{Re}\frac{z}{1-z} - (1-(b+1)\alpha)\mathrm{Re}\frac{z}{1+z} \\ &> 1 - \frac{1}{2}(1+(b+1)\alpha) - \frac{1}{2}(1-(b+1)\alpha) = 0 \quad (z \in U). \end{aligned} \tag{2.6}$$

Further, we have

$$\theta(g(z)) + Q(z) = \lambda_0 \left(\frac{1+z}{1-z}\right)^{a\alpha} + \frac{2\alpha z}{(1-z)^{1+(b+1)\alpha}(1+z)^{1-(b+1)\alpha}}$$
$$= h(z),$$

where $h(z)$ is given by (2.3), and so

$$\frac{zh'(z)}{Q(z)} = \frac{\theta'(g(z))}{\varphi(g(z))} + \frac{zQ'(z)}{Q(z)}$$
$$= \lambda_0 a(g(z))^{a-b-1} + \frac{zQ'(z)}{Q(z)}. \tag{2.7}$$

Also, for

$$|a-b-1| \leq \frac{1}{\alpha},$$

we find that

$$\left|\arg(g(z))^{a-b-1}\right| \leq |a-b-1| \cdot \frac{\alpha\pi}{2} \leq \frac{\pi}{2} \quad (z \in U). \tag{2.8}$$

Therefore, it follows from (2.1) and (2.5) to (2.8) that

$$\mathrm{Re}\frac{zh'(z)}{Q(z)} > 0 \quad (z \in U).$$

The other conditions of Lemma are also satisfied. Hence we conclude that

$$q(z) = \frac{z(Q^{\lambda_1}_{p,\lambda_2} f(z))'}{pQ^{\lambda_1}_{p,\lambda_2} f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha} = g(z) \quad (z \in U)$$

and $g(z)$ is the best dominant of (2.2). By (2.5) we see that the function $Q^{\lambda_1}_{p,\lambda_2} f(z)$ is p-valent strongly starlike of order α in U.

Furthermore, for the function $f(z)$ defined by (2.4), we have

$$\lambda_0(q(z))^a + zq'(z)(q(z))^b = h(z),$$

which shows that the number α is sharp. The proof of Theorem 1 is now completed. □

Theorem 2. *Let*

$$0 < \alpha \leq 1, \;\; \lambda(b+2) \geq 0, \;\; (b+1)\mathrm{Re}\mu \geq 0 \quad \textit{and} \quad |b+1| \leq \frac{1}{\alpha}. \tag{2.9}$$

If $f(z) \in A_p$ *satisfies* $Q^{\lambda_1}_{p,\lambda_2} f(z)(Q^{\lambda_1}_{p,\lambda_2} f(z))' \neq 0$ $(z \in U \setminus \{0\})$ *and*

$$\lambda \left(\frac{z(Q^{\lambda_1}_{p,\lambda_2} f(z))'}{pQ^{\lambda_1}_{p,\lambda_2} f(z)}\right)^{b+2} + \mu \left(\frac{z(Q^{\lambda_1}_{p,\lambda_2} f(z))'}{pQ^{\lambda_1}_{p,\lambda_2} f(z)}\right)^{b+1}$$
$$+ z \left(\frac{z(Q^{\lambda_1}_{p,\lambda_2} f(z))'}{pQ^{\lambda_1}_{p,\lambda_2} f(z)}\right)' \left(\frac{z(Q^{\lambda_1}_{p,\lambda_2} f(z))'}{pQ^{\lambda_1}_{p,\lambda_2} f(z)}\right)^{b} \prec h(z) \quad (z \in U), \tag{2.10}$$

where

$$h(z) = \left(\frac{1+z}{1-z}\right)^{(b+1)\alpha} \left(\mu + \lambda\left(\frac{1+z}{1-z}\right)^{\alpha} + \frac{2\alpha z}{1-z^2}\right), \tag{2.11}$$

then the function $Q^{\lambda_1}_{p,\lambda_2} f(z)$ *is p-valent strongly starlike of order* α *in* U. *The number* α *is sharp for the function* $f(z)$ *defined by* (2.4).

Proof. Let

$$q(z) = \frac{z(Q^{\lambda_1}_{p,\lambda_2} f(z))'}{pQ^{\lambda_1}_{p,\lambda_2} f(z)},\ g(z) = \left(\frac{1+z}{1-z}\right)^{\alpha},\ \theta(w) = \lambda w^{b+2} + \mu w^{b+1},\ \varphi(w) = w^b$$

in Lemma. Clearly, the functions $q(z), g(z), \theta(w), \varphi(w)$ and $Q(z){=}zg'(z)\varphi(g(z))$ satisfy the conditions of Lemma respectively. Further, we have

$$\begin{aligned}\theta(g(z)) + Q(z) &= \lambda\left(\frac{1+z}{1-z}\right)^{(b+2)\alpha} + \mu\left(\frac{1+z}{1-z}\right)^{(b+1)\alpha} \\ &\quad + \frac{2\alpha z}{(1-z)^{1+(b+1)\alpha}(1+z)^{1-(b+1)\alpha}} \\ &= h(z),\end{aligned}$$

where $h(z)$ is given by (2.11), and so

$$\begin{aligned}\frac{zh'(z)}{Q(z)} &= \frac{\theta'(g(z))}{\varphi(g(z))} + \frac{zQ'(z)}{Q(z)} \\ &= \lambda(b+2)g(z) + \mu(b+1) + \frac{zQ'(z)}{Q(z)}.\end{aligned}$$

Now, for

$$\lambda(b+2) \geq 0 \quad \text{and} \quad (b+1)\mathrm{Re}\mu \geq 0,$$

we have

$$\mathrm{Re}\frac{zh'(z)}{Q(z)} > 0 \quad (z \in U).$$

The other conditions of Lemma are also satisfied. Hence we obtain the desired result of the theorem.

Furthermore, for the function $f(z)$ defined by (2.4), we have

$$\lambda(q(z))^{b+2} + \mu(q(z))^{b+1} + zq'(z)(q(z))^b = h(z),$$

which shows that the number α is sharp. The proof of Theorem 2 is completed. □

Theorem 3. *Let*

$$0 < \alpha \leq 1, \quad \mu > 0 \quad \textit{and} \quad 0 \leq (b+1)\alpha \leq 1. \tag{2.12}$$

If $f(z) \in A_p$ *satisfies* $Q^{\lambda_1}_{p,\lambda_2} f(z)(Q^{\lambda_1}_{p,\lambda_2} f(z))' \neq 0$ $(z \in U \setminus \{0\})$ *and for* $z \in U$

(2.13)
$$\left|\arg\left\{\mu\left(\frac{z(Q^{\lambda_1}_{p,\lambda_2} f(z))'}{pQ^{\lambda_1}_{p,\lambda_2} f(z)}\right)^{b+1} + z\left(\frac{z(Q^{\lambda_1}_{p,\lambda_2} f(z))'}{pQ^{\lambda_1}_{p,\lambda_2} f(z)}\right)'\left(\frac{z(Q^{\lambda_1}_{p,\lambda_2} f(z))'}{pQ^{\lambda_1}_{p,\lambda_2} f(z)}\right)^{b}\right\}\right| < \frac{\pi}{2}\beta,$$

where

$$\beta = (b+1)\alpha + \frac{2}{\pi}\tan^{-1}\left(\frac{\alpha}{\mu}\right), \tag{2.14}$$

then

$$\left|\arg\left(\frac{z(Q^{\lambda_1}_{p,\lambda_2} f(z))'}{Q^{\lambda_1}_{p,\lambda_2} f(z)}\right)\right| < \frac{\pi}{2}\alpha \quad (z \in U). \tag{2.15}$$

This shows that the function $Q^{\lambda_1}_{p,\lambda_2} f(z)$ *is p-valent strongly starlike of order* α *in* U. *The bound* β *in* (2.13) *is the largest number such that* (2.15) *holds true.*

Proof. By taking

$$\lambda = 0, \quad \mu > 0 \quad \text{and} \quad 0 \le b+1 \le \frac{1}{\alpha}$$

in Theorem 2, we see that if

(2.16)
$$\mu\left(\frac{z(Q^{\lambda_1}_{p,\lambda_2} f(z))'}{pQ^{\lambda_1}_{p,\lambda_2} f(z)}\right)^{b+1} + z\left(\frac{z(Q^{\lambda_1}_{p,\lambda_2} f(z))'}{pQ^{\lambda_1}_{p,\lambda_2} f(z)}\right)'\left(\frac{z(Q^{\lambda_1}_{p,\lambda_2} f(z))'}{pQ^{\lambda_1}_{p,\lambda_2} f(z)}\right)^{b} \prec h(z),\ z \in U,$$

where

$$h(z) = \left(\frac{1+z}{1-z}\right)^{(b+1)\alpha}\left(\mu + \frac{2\alpha z}{1-z^2}\right), \tag{2.17}$$

then (2.15) is true.

For $z = e^{i\theta}$ $(\theta \in \mathbb{R})$, $z \neq 1$ and $z \neq -1$, we get

$$\frac{z}{1-z} = -\frac{1}{2} + \frac{i}{2}\cot\frac{\theta}{2}, \quad \frac{z}{1+z} = \frac{1}{2} + \frac{i}{2}\tan\frac{\theta}{2}, \tag{2.18}$$

$$\frac{1+z}{1-z} = \frac{1+e^{i\theta}}{1-e^{i\theta}} = \cot\frac{\theta}{2}e^{\frac{\pi}{2}i} \neq 0. \tag{2.19}$$

The following two cases arise.

(i) If

$$k(\theta) = \cos\frac{\theta}{2}\sin\frac{\theta}{2} = \frac{1}{2}\sin\theta > 0,$$

then we deduce from (2.17) to (2.19) that

$$h(e^{i\theta}) = \left(\cot\frac{\theta}{2}\right)^{(b+1)\alpha} e^{\frac{1}{2}(b+1)\alpha\pi i}\left(\mu + i\frac{\alpha}{2}\left(\cot\frac{\theta}{2} + \tan\frac{\theta}{2}\right)\right),$$

which yields

$$\arg h(e^{i\theta}) = \frac{1}{2}(b+1)\alpha\pi + \tan^{-1}\left(\frac{\alpha}{2\mu k(\theta)}\right) \tag{2.20}$$

for $\mu > 0, e^{i\theta} \neq 1$ and $e^{i\theta} \neq -1$. Let $\theta_1 = \frac{\pi}{2}$. Then

$$0 < k(\theta) \leq k(\theta_1) = \frac{1}{2} \tag{2.21}$$

and it follows from (2.12), (2.20) and (2.21) that

$$\begin{aligned}\pi > \arg h(e^{i\theta}) \geq \arg h(e^{i\theta_1}) &= \frac{1}{2}(b+1)\alpha\pi + \tan^{-1}\left(\frac{\alpha}{\mu}\right)\\ &= \frac{\pi}{2}\beta > 0. \qquad (2.22)\end{aligned}$$

(ii) If $k(\theta) < 0$, then it follows from (2.17) to (2.19) that

$$h(e^{i\theta}) = \left(-\cot\frac{\theta}{2}\right)^{(b+1)\alpha} e^{-\frac{1}{2}(b+1)\alpha\pi i}\left(\mu + i\frac{\alpha}{2}\left(\cot\frac{\theta}{2} + \tan\frac{\theta}{2}\right)\right),$$

and so

$$\arg h(e^{i\theta}) = -\frac{1}{2}(b+1)\alpha\pi + \tan^{-1}\left(\frac{\alpha}{2\mu k(\theta)}\right) \tag{2.23}$$

for $\mu > 0,\ e^{i\theta} \neq 1$ and $e^{i\theta} \neq -1$. Let $\theta_2 = -\frac{\pi}{2}$. Then

$$0 > k(\theta) \geq k(\theta_2) = -\frac{1}{2} \tag{2.24}$$

and from (2.12), (2.23) and (2.24) we have

$$\begin{aligned}-\pi < \arg h(e^{i\theta}) \leq \arg h(e^{i\theta_2}) &= -\frac{1}{2}(b+1)\alpha\pi - \tan^{-1}\left(\frac{\alpha}{\mu}\right)\\ &= -\frac{\pi}{2}\beta < 0. \qquad (2.25)\end{aligned}$$

Noting that $h(0) = \mu > 0$, we find from (2.22) and (2.25) that $h(U)$ properly contains the angular region $-\frac{\pi}{2}\beta < \arg w < \frac{\pi}{2}\beta$ in the complex w-plane. Consequently, if $f(z) \in A_p$ satisfies (2.13), then the subordination relation (2.16) holds true, and so we have the assertion (2.15) of Theorem 3.

Furthermore, for the function $f(z) \in A_p$ defined by (2.4), we have (2.15) and

$$\mu\left(\frac{z(Q^{\lambda_1}_{p,\lambda_2}f(z))'}{pQ^{\lambda_1}_{p,\lambda_2}f(z)}\right)^{b+1} + z\left(\frac{z(Q^{\lambda_1}_{p,\lambda_2}f(z))'}{pQ^{\lambda_1}_{p,\lambda_2}f(z)}\right)'\left(\frac{z(Q^{\lambda_1}_{p,\lambda_2}f(z))'}{pQ^{\lambda_1}_{p,\lambda_2}f(z)}\right)^{b} = h(z).$$

Hence, by using (2.22) and (2.25), we conclude that the bound β in (2.13) is the best possible. This completes our proof. □

Acknowledgement. I would like to express sincere thanks to the referee for careful reading and suggestions which helped me to improve the paper.

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