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A Modified Definition on the Process Capability Index C_{pk} Based on Median

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Abstract

This study proposes a modified definition about C_{pk} based on median as the centering parameter in order to more easily control the process since the mean does not represent any quantile of the asymmetric process distribution. Then we consider an estimate and derive the asymptotic normality for the estimate of the modified C_{pk}. In addition, we provide an example with asymmetric distributions and discuss the estimation for the limiting variance that are followed by some concluding remarks.

Keywords: Asymmetric normality, median, process capability index.

1. Introduction

It is an important issue for manufacturers to assess product quality and whether the product satisfies the given specification limits that achieves the target value since consumers or end-users require stronger product quality guarantees. Manufacturers should increase efforts to keep the production process stable and reduce the variability within the specification limits to maintain the production quality. As a methodology for measuring and assessing the ability of the production process, process capability indices(PCIs) have been defined in theory and successfully applied on the floor to maintain and enhance the product quality. Since Juran (1974) introduced the PCI in the quality control, the definition of PCI has been expanded in time and in order to accommodate various situations and environments on the floor. Among them, C_p and C_{pk} are the most popular and widely used definitions of PCI, which will be presented in the following section. For this, let LSL and USL be the upper and lower specification limits, respectively. In addition let μ and σ^2 (or σ) be the mean and variance(or standard deviation) of the production process, respectively. Then C_p and C_{pk} are defined as

$$C_p = \frac{\mathrm{USL} - \mathrm{LSL}}{6\sigma} \quad \text{and} \quad C_{pk} = \frac{\min\{\mu - \mathrm{LSL}, \mathrm{USL} - \mu\}}{3\sigma}.$$

C_p is the simplest definition among those of PCI and can be used for the symmetric distributions when the process mean μ is the centering point between USL and LSL. C_{pk} can be used for the symmetric distributions when μ is not the centering point between USL and LSL or for the asymmetric or skewed distributions. However we note that the process mean μ is also used for the centering parameter in the definition of C_{pk}. Since the mean μ may not be used to represent any quantile point of the process distribution, the meaning of the two intervals, (LSL, μ) and (μ, USL) may become more or less ambiguous when the distribution is asymmetric. Instead if we use median θ in this case, then the meaning of (LSL, θ) and (θ, USL) would be more sensible since median θ stands for

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50% point of the process distribution. Then this may help the managers working for the quality control identify the state of process whether the process can be controlled easily or not since they can understand the probability concerning the interval (LSL, θ) or (θ, USL). For this reason, it would be convenient to assess and control the production process using a modified C_{pk} based on median θ. In addition, we note that many statisticians and engineers agree that it is common to encounter data with a heavy-tailed or skewed distribution on the floor (*cf.* Gunter, 1989). Even for the case of C_p, when we make inference, sometimes we may not assume normality even though the symmetry is assumed for the process distribution. In this case, it would be better to consider applying the nonparametric method for the inference based on median for the location parameter. In this vein, Park (2009) proposed the control charts based on median and obtained control limits using the bootstrap method (*cf.* Efron, 1979) that has been widely used in nonparametric statistics for the computational methodology; however, there have been no known reports about a PCI which is based on median.

This research proposes a modified definition $C_{pk}(\theta)$ of C_{pk} based on the median to consider its estimation and derive asymptotic normality in the next section. Then we provide an example to compare the behavior of C_{pk} and $C_{pk}(\theta)$ using the asymmetric distributions and discuss an estimation procedure for the limiting variance of the asymptotic normality. Finally we comment on applying the bootstrap method as a re-sampling method for the estimation of the limiting variance.

2. Definition, Estimate and Asymptotic Normality for $C_{pk}(\theta)$

We first propose a modified definition of $C_{pk}(\theta)$ based on a process median θ as follows.

$$C_{pk}(\theta) = \frac{\min\{\theta - \mathrm{LSL}, \mathrm{USL} - \theta\}}{3\sigma}.$$

In order to estimate $C_{pk}(\theta)$, suppose that we have a sample $X_1, \ldots, X_n$ from the production process having a continuous but unknown distribution function F with a median θ. We assume that F may have the finite fourth moment for the technical reason of our discussion for the asymptotic normality. From the sample $X_1, \ldots, X_n$, let $\hat{\theta}$ be the $[n/2]+1^{st}$ order statistic, where $[x]$ denotes the largest integer part of the real number x. We note that $\hat{\theta}$ is a sample median and a consistent estimate of θ. In addition, let s^2 be the unbiased estimate of σ^2 such that

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2.$$

Then one may propose an estimate $\hat{C}_{pk}(\theta)$ of $C_{pk}(\theta)$ simply as follows:

$$\hat{C}_{pk}(\theta) = \frac{\min\left\{\hat{\theta} - \mathrm{LSL}, \mathrm{USL} - \hat{\theta}\right\}}{3s}.$$

It is obvious that $\hat{C}_{pk}(\theta)$ is a consistent estimate of $C_{pk}(\theta)$ since $\hat{\theta}$ and s are consistent estimates of θ and σ, respectively. For the inference on $C_{pk}(\theta)$ such as confidence interval and/or hypothesis testing, one has to obtain the distribution of $\hat{C}_{pk}(\theta)$. Since the exact distribution of $\hat{C}_{pk}(\theta)$ is too complicated to be derived, we consider the asymptotic normality of $\sqrt{n}(\hat{C}_{pk}(\theta) - C_{pk}(\theta))$ instead. The asymptotic normalities for $\hat{C}_{pk}$ based on the sample mean, has been considered by Chan *et al.* (1990) and Chen and Pearn (1997) under some conditions. Now we state the results for the asymptotic normality in the following Theorem.

Theorem 1. *With the assumption that F has the finite fourth moment, we have that*

$$\sqrt{n}\left(\hat{C}_{pk}(\theta) - C_{pk}(\theta)\right)$$

converges in distribution to a normal random variable with mean 0 and variance $V(C_{pk}(\theta))$ such as

(i) if $\min\{\theta - LSL, USL - \theta\} = \theta - LSL$

$$V\left(C_{pk}(\theta)\right) = \frac{1}{9\sigma^4}\left\{\frac{\sigma^2}{4f^2(\theta)} + \frac{(\theta - LSL)^2(\mu_4 - \sigma^4)}{4\sigma^2} - 2\sigma(\theta - LSL)\rho_{\theta\sigma}\right\}.$$

(ii) if $\min\{\theta - LSL, USL - \theta\} = USL - \theta$

$$V\left(C_{pk}(\theta)\right) = \frac{1}{9\sigma^4}\left\{\frac{\sigma^2}{4f^2(\theta)} + \frac{(USL - \theta)^2(\mu_4 - \sigma^4)}{4\sigma^2} + 2\sigma(USL - \theta)\rho_{\theta\sigma}\right\},$$

where

$$\rho_{\theta\sigma} = -\frac{1}{4\sigma f(\theta)}\left\{2\int_{-\infty}^{\theta} x^2 dF(x) - \left(\sigma^2 - \mu^2\right) - 4\mu\int_{-\infty}^{\theta} x dF(x)\right\}.$$

Proof: Before we proceed to prove the asymptotic normality, we first note that one may prove that $\hat{\theta}$ converges with probability one(w.p.1) to θ using Lemma which follows in the sequel, with Slutsky's Theorem. This in turn implies that $\hat{\theta} - \text{LSL}$ converges w.p.1 to $\theta - \text{LSL}$ for the case (i) $\min\{\theta - \text{LSL}, \text{USL} - \theta\} = \theta - \text{LSL}$. Therefore even though $\min\{\theta - \text{LSL}, \text{USL} - \theta\} = \theta - \text{LSL}$ does not guarantee that

$$\min\left\{\hat{\theta} - \text{LSL}, \text{USL} - \hat{\theta}\right\} = \hat{\theta} - \text{LSL} \tag{2.1}$$

for some finite numbers of n, eventually (2.1) will hold for sufficiently large n and thereafter. Therefore it would be enough to consider the asymptotic normality for the case (i) that

$$\begin{aligned}
\sqrt{n}\left(\hat{C}_{pk}(\theta) - C_{pk}(\theta)\right) &= \sqrt{n}\left(\frac{\min\left\{\hat{\theta} - \text{LSL}, \text{USL} - \hat{\theta}\right\}}{3s} - \frac{\min\left\{\theta - \text{LSL}, \text{USL} - \theta\right\}}{3\sigma}\right) \\
&= \sqrt{n}\left(\frac{\hat{\theta} - \text{LSL}}{3s} - \frac{\theta - \text{LSL}}{3\sigma}\right) \\
&= \frac{\sqrt{n}}{3}\left\{\frac{\sigma\left(\hat{\theta} - \theta\right) - (s - \sigma)(\theta - \text{LSL})}{\sigma s}\right\}.
\end{aligned}$$

Then we need the results of the asymptotic normalities for

$$\sqrt{n}\left(\hat{\theta} - \theta\right) \quad \text{and} \quad \sqrt{n}(s - \sigma)$$

and the limiting form $\rho_{\theta\sigma}$ of $n\,\text{Cov}(\hat{\theta} - \theta, s - \sigma)$, the covariance between $\sqrt{n}(\hat{\theta} - \theta)$ and $\sqrt{n}(s - \sigma)$. In order to derive the asymptotic normality for $\sqrt{n}(\hat{\theta} - \theta)$, we will apply the Bahadur Representation

Theorem (*cf.* Serfling, 1980), which will be stated in the following lemma. For this, let F_n be the empirical distribution function and defined using the indicator function $I(\cdot)$ as

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I(X_i \le x).$$

We note that F_n is a strongly consistent estimate of F. In the following lemma, we denote f as the probability density function(pdf) of F. In addition, we assume that $f(x) > 0$ for some neighborhood of θ. □

Lemma 1. *(Bahadur Representation Theorem)*

For $0 < p < 1$, let ξ_p be the p^{th} quantile of F. Then $F(\xi_p) = p$. Assume that F is twice differentiable at ξ_p and $F'(\xi_p) = f(\xi_p) > 0$. Sequence $\{k_n\}$, $1 \le k_n \le n$, with positive integers satisfies the following: as $n \to \infty$, for $\tau \ge 1/2$,

$$\frac{k_n}{n} = p + o\left(\frac{(\log n)^{\tau}}{n^{\frac{1}{2}}}\right). \tag{2.2}$$

Then we have w.p.1 that

$$\hat{\xi}_p = \xi_p + \frac{k_n/n - F_n(\xi_p)}{f(\xi_p)} + O\left(n^{-\frac{3}{4}}(\log n)^{\left(\frac{1}{2}\right)(\tau+1)}\right).$$

For the proof of Lemma, you may refer to Serfling (1980). For the proof of Theorem using Lemma, we take $\xi_p = \theta$ and $p = 1/2$. Also by choosing $k_n = [2/n] + 1$, we see that $k_n/n = 1/2 + O(n^{-1})$ that satisfies the condition (2.2) in Lemma. Then using the Central Limit Theorem with Slutsky's Theorem, we have that

$$\sqrt{n}\left(\hat{\theta} - \theta\right) \xrightarrow{d} N\left(0, \frac{1}{(2f(\theta))^2}\right), \tag{2.3}$$

where $\xrightarrow{d}$ means the convergence in distribution.

For the asymptotic normality of $\sqrt{n}(s - \sigma)$, we may use the results of the asymptotic normality of $\sqrt{n}(s^2 - \sigma^2)$ and Δ-method (*cf.* Bickel and Doksum, 1977). From Serfling (1980, p.114, Problem 8), we see that

$$\sqrt{n}\left(s^2 - \sigma^2\right) \xrightarrow{d} N\left(0, \mu_4 - \sigma^4\right),$$

where $\mu_4 = E\{(X - \mu)^4\}$. Then taking $g(x) = \sqrt{x}$, since $g'(x) = 1/(2\sqrt{x})$, from the Δ-method, we obtain that

$$\sqrt{n}(s - \sigma) = \sqrt{n}\left(g\left(s^2\right) - g\left(\sigma^2\right)\right) \xrightarrow{d} N\left(0, \frac{\mu_4 - \sigma^4}{4\sigma^2}\right). \tag{2.4}$$

Finally we may obtain $\rho_{\theta\sigma}$ using Slutsky's Theorem and Lemma as follows:

$$\rho_{\theta\sigma} = -\frac{1}{4\sigma f(\theta)}\left\{2\int_{-\infty}^{\theta} x^2 dF(x) - \left(\sigma^2 - \mu^2\right) - 4\mu\int_{-\infty}^{\theta} x dF(x)\right\}. \tag{2.5}$$

Table 1: Comparison of PCI values for C_{pk} and $C_{pk}(\theta)$

α	Mean	Median	Variance	LSL	USL	C_{pk}	$C_{pk}(\theta)$
1/2	2	$(\log 2)^2$	20	0.0000	28.0722	0.1491	0.0358
1	1	$\log 2$	1	0.0050	5.2983	0.3317	0.2294
2	$\sqrt{\pi}/2$	$\sqrt{\log 2}$	$1 - \pi/4$	0.7808	2.3018	0.5867	0.5481

The derivation will appear in Appendix. Then using the results, (2.3), (2.4) and (2.5), with the fact that s is a consistent estimate of σ, we may obtain the asymptotic normality for the case (i).

For the case (ii), using the following equation for sufficiently large enough n

$$\sqrt{n}\left(\hat{C}_{pk}(\theta) - C_{pk}(\theta)\right) = \frac{\sqrt{n}}{3}\left\{\frac{-\sigma\left(\hat{\theta} - \theta\right) - (s - \sigma)(\mathrm{USL} - \theta)}{\sigma s}\right\},$$

we may obtain the result with the same arguments used for (i).

3. Example and Some Concluding Remarks

In order to provide some comparison between C_{pk} and $C_{pk}(\theta)$, we consider the following Weibull distribution family such that for any $\alpha > 0$

$$f(x) = \alpha x^{\alpha-1}\exp[-x^{\alpha}], \quad 0 < x < \infty.$$

We consider three cases by varying the value of α such as 1/2, 1 and 2. We choose $w_{0.005}$ and $w_{0.995}$ for LSL and USL for each case, where w_p is the p^{th} quantile point. We summarized all the relevant quantities in Table 1 with the values of C_{pk} and $C_{pk}(\theta)$ for each case. You may refer to Randles and Wolfe (1979) for the means and variances of Weibull distributions. The table shows that the difference between C_{pk} and $C_{pk}(\theta)$ becomes smaller as α increases. This may happen since the differences between the means and medians become smaller as α increases. We note that the shape of the pdfs moves toward symmetry as α increases. In addition, we note that all the values of $C_{pk}(\theta)$ are smaller than those of C_{pk}. Therefore, when the underlying distribution is not symmetric, we have to pay more attention to the production process to maintain the quality level and $C_{pk}(\theta)$ can take the role better than C_{pk} for this end since $C_{pk}(\theta)$ is more sensitive for the change of asymmetry of the process distribution.

The applications of median as a location parameter to the industrial field have rarely been studied and reported even though there may exist some merits such that the normality for the process distribution is not required. This may come from the facts that the researches for the mean have been widely studied and the results are relatively simple and easy to apply whereas those for median are rare and even complicated with especially the derivation of the asymptotic normality. However the mean may become less meaningful and can provide distorted information as a location parameter when the asymmetry gets severe. In addition, we note that the product quality may be controlled by some criteria with probability such as the control limits for the control charts. Therefore the use of median should be considered seriously for the asymmetric process distribution.

For the statistical inferences such as confidence interval and testing hypotheses for $C_{pk}(\theta)$ using the asymptotic normality, we have to obtain a consistent estimate of the limiting variance $V(C_{pk}(\theta)$. For this, first of all, we have to estimate $f(\theta)$. Then from Serfling (1980), for any suitable positive real number b_n, we may have

$$f_n\left(\hat{\theta}\right) = \frac{F_n\left(\hat{\theta} + b_n\right) - F_n\left(\hat{\theta} - b_n\right)}{2b_n},$$

as an estimate of $f(\theta)$. Serfling (1980) summarized extensively the asymptotic properties of the density estimation that can be used as criteria for the choice of value of b_n. In addition, we may obtain consistent estimates of

$$\int_{-\infty}^{\theta} x dF(x) \quad \text{and} \quad \int_{-\infty}^{\theta} x^2 dF(x)$$

by

$$\int_{-\infty}^{\hat{\theta}} x dF_n(x) = \frac{1}{n}\sum_{i=1}^{n} X_i I\left(X_i \le \hat{\theta}\right) \quad \text{and} \quad \int_{-\infty}^{\hat{\theta}} x^2 dF_n(x) = \frac{1}{n}\sum_{i=1}^{n} X_i I\left(X_i^2 \le \hat{\theta}\right).$$

Then using the plug-in method, we may obtain a consistent estimate of the limiting variance $V(C_{pk}(\theta)$ together with s^2 and $\hat{\theta}$.

As a matter of fact, the expressions of the limiting variance $V(C_{pk}(\theta)$ in Theorem are very messy to be estimated. Therefore if one considers to make inferences with the estimated variance proposed as in the previous paragraph, then one may confront the problems of the efficiency and accuracy for the result of the analysis. One statistical method to avoid this situation may be to use the bootstrap method. Franklin and Wasserman (1992) considered to apply the bootstrap method for obtaining the confidence intervals with several versions of bootstrap methods (*cf.* Efron and Tibshirani, 1993; Shao and Tu, 1995) for C_p and C_{pk} in this regard. Also Cho *et al.* (1999) considered to use the bootstrap method in this direction. We may compare the efficiency between the plug-in and bootstrap methods by obtaining the coverage probabilities of the confidence intervals through simulation study. This will be one of our research topics in the future and appear in a suitable journal.

Appendix:

In this appendix, we derive the limiting form $\rho_{\theta\sigma}$ of $n\text{Cov}(\hat{\theta} - \theta, s - \sigma)$. Then it is enough to derive the limiting form of $nE[(\hat{\theta} - \theta)(s - \sigma)]$. For this, first of all, we note that

$$n\left(\hat{\theta} - \theta\right)(s - \sigma) = n\frac{\left(\hat{\theta} - \theta\right)\left(s^2 - \sigma^2\right)}{s + \sigma}.$$

If we apply Lemma again, then we have w.p.1 that

$$\begin{aligned} n\frac{\left(\hat{\theta} - \theta\right)\left(s^2 - \sigma^2\right)}{s + \sigma} &= n\frac{\{1/2 - F_n(\theta)\} + O\left(n^{-3/4}(\log n)^{(1/2)(\tau+1)}\right)}{(s + \sigma)f(\theta)}\left(s^2 - \sigma^2\right) \\ &= n\frac{1/n\sum_{i=1}^{n}\{1/2 - I(X_i \le \theta)\} + O\left(n^{-3/4}(\log n)^{(1/2)(\tau+1)}\right)}{(s + \sigma)f(\theta)}\left(s^2 - \sigma^2\right). \end{aligned} \tag{A.1}$$

Since s is an unbiased estimate of σ, it is enough to consider the numerator of $(A.1)$. Then the right-hand side of the numerator of $(A.1)$ can be re-arranged as

$$\begin{aligned} &\sum_{i=1}^{n}\left\{\frac{1}{2} - I(X_i \le \theta)\right\}\left(s^2 - \sigma^2\right) + nK_n\left(s^2 - \sigma^2\right) \\ &= \sum_{i=1}^{n}\left\{\frac{1}{2} - I(X_i \le \theta)\right\}s^2 - \sum_{i=1}^{n}\left\{\frac{1}{2} - I(X_i \le \theta)\right\}\sigma^2 + nK_n\left(s^2 - \sigma^2\right), \end{aligned}$$

where $K_n = O(n^{-3/4}(\log n)^{(1/2)(\tau+1)})$. Since s^2 is an unbiased estimate of σ^2 and K_n is a quantity which is not related with s^2,

$$nE\left\{K_n\left(s^2-\sigma^2\right)\right\} = nK_nE\left(s^2-\sigma^2\right) = 0.$$

Now for each i, since $E\{I(X_i \le \theta)\} = F(\theta) = 1/2$, also we have that

$$E\left[\sum_{i=1}^{n}\left\{\frac{1}{2} - I(X_i \le \theta)\right\}\sigma^2\right] = \sigma^2\sum_{i=1}^{n}E\left\{\frac{1}{2} - I(X_i \le \theta)\right\} = 0.$$

Therefore it is enough to consider that

$$\begin{aligned} E\left[\sum_{i=1}^{n}\left\{\frac{1}{2} - I(X_i \le \theta)\right\}s^2\right] &= \frac{n}{2}E\left(s^2\right) - E\left\{\sum_{i=1}^{n}I(X_i \le \theta)s^2\right\} \\ &= \frac{n}{2}\sigma^2 - E\left\{\sum_{i=1}^{n}I(X_i \le \theta)s^2\right\}. \end{aligned}$$

In addition, we note that

$$\begin{aligned} \sum_{i=1}^{n}I(X_i \le \theta)s^2 &= \frac{1}{n-1}\left\{\sum_{i=1}^{n}I(X_i \le \theta)\right\}\left\{\sum_{i=1}^{n}X_i^2 - n\bar{X}^2\right\} \\ &= \frac{1}{n-1}\left\{\sum_{i=1}^{n}I(X_i \le \theta)\right\}\left\{\frac{n-1}{n}\sum_{i=1}^{n}X_i^2 - \frac{1}{n}\sum\sum_{i\neq j}X_iX_j\right\} \\ &= \frac{1}{n}\left\{\sum_{i=1}^{n}I(X_i \le \theta)\right\}\left\{\sum_{i=1}^{n}X_i^2\right\} - \frac{1}{n(n-1)}\left\{\sum_{i=1}^{n}I(X_i \le \theta)\right\}\left\{\sum\sum_{i\neq j}X_iX_j\right\} \\ &= P_1 - P_2, \qquad \text{say.} \end{aligned}$$

Then for P_1, we have that

$$P_1 = \frac{1}{n}\sum_{i=1}^{n}I(X_i \le \theta)X_i^2 + \frac{1}{n}\sum\sum_{i\neq j}I(X_i \le \theta)X_j^2.$$

For each i, we have

$$E\left\{\sum_{i=1}^{n}I(X_i \le \theta)X_i^2\right\} = \int_{-\infty}^{\theta}x^2dF(x)$$

In addition, we have that since X_i and X_j are independent for $i \neq j$

$$E\left\{I(X_i \le \theta)X_j^2\right\} = E\{I(X_i \le \theta)\}E\left(X_j^2\right) = \frac{\sigma^2+\mu^2}{2}.$$

Thus, we have that

$$E(P_1) = \int_{-\infty}^{\theta}x^2dF(x) + (n-1)\frac{\sigma^2+\mu^2}{2}.$$

In addition, $n(n-1)P_2$ can be rewritten as

$$n(n-1)P_2 = 2\sum\sum_{i\neq j} I(X_i \leq \theta)X_iX_j + \sum\sum\sum_{i\neq j\neq k} I(X_k \leq \theta)X_iX_j.$$

Since

$$E\left\{I(X_i \leq \theta)X_iX_j\right\} = E\left\{I(X_i \leq \theta)X_i\right\}E(X_j) = \mu\int_{-\infty}^{\theta} xdF(x)$$

and

$$E\left\{I(X_k \leq \theta)X_iX_j\right\} = E\left\{I(X_k \leq \theta)\right\}E(X_i)E(X_j) = \frac{\mu^2}{2},$$

we have that

$$E(P_2) = 2\mu\int_{-\infty}^{\theta} xdF(x) + (n-2)\frac{\mu^2}{2}.$$

Thus, we have that

$$\begin{aligned} E\left[\sum_{i=1}^{n}\left\{\frac{1}{2} - I(X_i \leq \theta)\right\}s^2\right] &= \frac{n}{2}\sigma^2 - E(P_1) + E(P_2) \\ &= -\int_{-\infty}^{\theta} x^2dF(x) + \frac{\sigma^2-\mu^2}{2} + 2\mu\int_{-\infty}^{\theta} xdF(x). \end{aligned}$$

Finally we obtain the expression of $\rho_{\theta\sigma}$ in (2.1) with the fact that s is a consistent estimate of σ.

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