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Asymptotic Normality for Threshold-Asymmetric GARCH Processes of Non-Stationary Cases

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Abstract

This article is concerned with a class of threshold-asymmetric GARCH models both for stationary case and for non-stationary case. We investigate large sample properties of estimators from QML(quasi-maximum likelihood) and QL(quasilikelihood) methods. Asymptotic distributions are derived and it is interesting to note for non-stationary case that both QML and QL give asymptotic normal distributions.

Keywords: Quasilikelihood, quasi-maximum likelihood, non-stationary, threshold.

1. Introduction

Since the seminal paper of Engle (1982) and Bollerslev (1986), GARCH-type models have been successfully used in econometrics and finance field. Asymptotic theory of the quasi-maximum likelihood estimator(QMLE) in GARCH context were first established by Weiss (1986) for ARCH model, and followed by Lumsdaine (1996) for GARCH(1, 1) processes. Further studies for general GARCH(p, q) models can be found in Berkes *et al.* (2003), Straumann and Mikosch (2006) and Lee and Lee (2009). We refer to Straumann (2005) for a recent comprehensive treatment on the estimation of GARCH models in a broader context. An estimator obtained from solving the quasi-likelihood score equation is referred to as the quasi-likelihood estimator(QLE). See, for instance, Hwang and Basawa (2011a). In the context of stationary GARCH processes, it is well documented that both QMLE and QLE are asymptotically normally distributed.

This short article is concerned with the non-stationary case. It is usual in stochastic processes that limiting distributions from non-stationary case are no longer normal distributions. As an illustration, in the simple AR(1) model, non-stationary distribution for the unit root case is given by a functional of Brownian motion and non-stationary explosive AR(1) model produces a non-normal limiting distribution which is a product of two independent random variables (*cf.* Fuller, 1996, Ch. 10). Interestingly enough, however, in the GARCH context, Jensen and Rahbek (2004a, 2004b) obtained normal limits of the QMLE for the non-stationary GARCH processes. Also, Hwang and Basawa (2005) established asymptotic normal distribution of the least squares estimation for the explosive non-stationary random coefficient AR(1) process which is closely related to ARCH models. The main contribution of this paper is to establish normal limits of the QMLE and QLE from a general class of threshold-asymmetric GARCH models for the non-stationary case. Section 2 describes the class of models and related QMLE and QLE are introduced. Non-stationary asymptotic distributions are obtained in Section 3 in terms of normal limits.

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2. Threshold-Asymmetric GARCH Processes: Stationary Case

Following Pan *et al.* (2008), consider a time series $\{\epsilon_t\}$ generated by a general class of threshold asymmetric GARCH processes(TAGARCH, for short) defined by

$$\epsilon_t = \sqrt{h_t} e_t,$$
$$h_t^\delta - \sum_{j=1}^{q} \beta_j h_{t-j}^\delta = \alpha_0 + \sum_{i=1}^{p} \left[\alpha_{i1} (\epsilon_{t-i}^+)^{2\delta} + \alpha_{i2} (\epsilon_{t-i}^-)^{2\delta} \right], \tag{2.1}$$

where

$$\delta > 0, \quad \alpha_0 > 0, \quad \alpha_{i1} \geq 0, \quad \alpha_{i2} \geq 0, \quad (i = 1, 2, \ldots, p), \qquad \beta_j \geq 0, \quad (j = 1, 2, \ldots, q).$$

Let $\phi = (\delta, \alpha_0, \alpha_{11}, \alpha_{12}, \ldots, \alpha_{p1}, \alpha_{p2}, \beta_1, \ldots, \beta_q)'$; $(2p + q + 2) \times 1$ denote a parameter vector and the true value of ϕ is given by $\phi^0 = (\delta^0, \alpha_0^0, \alpha_{11}^0, \alpha_{12}^0, \ldots, \alpha_{p1}^0, \alpha_{p2}^0, \beta_1^0, \ldots, \beta_q^0)'$. Define

$$\phi_t(\delta) = \left[\alpha_0 + \sum_{i=1}^{p} \alpha_{i1} (\epsilon_{t-i}^+)^{2\delta} + \sum_{i=1}^{p} \alpha_{i2} (\epsilon_{t-i}^-)^{2\delta} + \sum_{j=1}^{q} \beta_j h_{t-j}^\delta(\phi) \right]^{\frac{1}{\delta}}. \tag{2.2}$$

It will be assumed that

(A1) The innovation $\{e_t\}$ is non-degenerate and symmetrically distributed.

(A2) For some $\Delta > 0$, $E|e_t|^\Delta < \infty$ and $P\{e_t^2 \geq t\} t^{-\mu}$ goes to zero as t tends to infinity for some $\mu > 0$.

(A3) The parameter space Θ is compact subset of R^d, $d = 2p + q + 2$ and ϕ^0 is an interior point of Θ, and the Lyapunov exponent $\gamma(\phi) < 0$ for all $\phi \in \Theta$ (*cf.* Pan *et al.*, 2008).

Rewrite (2.1) in a compact form of the vector equation with random coefficient as

$$X_t = A_t X_{t-1} + B, \tag{2.3}$$

where with $\kappa = 2p + q - 2$,

$$X_t = \left(h_{t+1}^\delta, \ldots, h_{t-q+2}^\delta, (\epsilon_t^+)^{2\delta}, (\epsilon_t^-)^{2\delta}, \ldots, \left(\epsilon_{t-p+2}^+\right)^{2\delta}, \left(\epsilon_{t-p+2}^-\right)^{2\delta} \right)' \in R^\kappa, \tag{2.4}$$

$B \equiv B(\phi) = (\alpha_0, 0, \ldots, 0)' \in R^\kappa$ and $\kappa \times \kappa$ matrix A_t is defined by

$$A_t \equiv A_t(\phi) = \begin{pmatrix} \tau_t' & \beta_q & \alpha' & \alpha_{p1} & \alpha_{p2} \\ I_{q-1} & 0 & 0 & 0 & 0 \\ \xi_{1t}' & 0 & 0 & 0 & 0 \\ \xi_{2t}' & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{2(p-2)} & 0 & 0 \end{pmatrix}, \tag{2.5}$$

with $\tau_t = (\beta_1 + \alpha_{11}(e_t^+)^{2\delta} + \alpha_{12}(e_t^-)^{2\delta}, \beta_2, \ldots, \beta_{q-1})' \in R^{q-1}$, I denoting the identity matrix $\xi_t^+ = (e_t^{+2\delta}, 0, \ldots, 0)' \in R^{q-1}$ and $\xi_t^- = (e_t^{-2\delta}, 0, \ldots, 0)' \in R^{q-1}$.

Due to Theorem 5 of Pan *et al.* (2008), the model (2.3) and hence the TAGARCH model in (2.1) has a unique strictly stationary solution under assumptions (A1) and (A2) if and only if

$$\gamma(\phi) < 0, \tag{2.6}$$

where $\gamma(\phi)$ is the top Lyapunov exponent of the sequence A_t, *i.e.*,

$$\gamma(\phi) = \lim_{t\to+\infty} \frac{1}{t}\|A_0 A_{-1}\cdots A_{-t}\|, \tag{2.7}$$

where $\|\cdot\|$ denotes a norm on R^κ. Consequently, in what follows, TAGARCH model can be referred to as stationary and non-stationary process according respectively to $\gamma(\phi) < 0$ and $\gamma(\phi) \geq 0$.

Under conditions (A1)–(A3), $h_t = h_t(\phi)$ may be expressed in terms of observation process $\{\epsilon_t\}$ as

$$\begin{aligned} h_t(\phi) = {} & \frac{\alpha_0}{1-\sum_{j=1}^q \beta_j} + \sum_{i=1}^p \alpha_{i1}\,(\epsilon_{t-i}^+)^{2\delta} + \sum_{i=1}^p \alpha_{i2}\,(\epsilon_{t-i}^-)^{2\delta} \\ & + \sum_{i=1}^p \alpha_{i1} \sum_{k=1}^\infty \sum_{j_1=1}^q \cdots \sum_{j_k=1}^q \beta_{j_1}\cdots\beta_{j_k}\left(\epsilon_{t-i-j_1-\cdots-j_k}^+\right)^{2\delta} \\ & + \sum_{i=1}^p \alpha_{i2} \sum_{k=1}^\infty \sum_{j_1=1}^q \cdots \sum_{j_k=1}^q \beta_{j_1}\cdots\beta_{j_k}\left(\epsilon_{t-i-j_1-\cdots-j_k}^-\right)^{2\delta}. \end{aligned} \tag{2.8}$$

2.1. Quasi-maximum likelihood estimation

The QMLE (which will be denoted by ϕ_{ML}) based on the sample $\epsilon_1, \ldots, \epsilon_n$ is defined by the value maximizing the presumed normal likelihood, viz.,

$$\phi_{ML} = \mathrm{argmax}_{\phi\in\Theta} L_n(\phi),$$

where

$$L_n(\phi) = \sum_{t=1}^n -\frac{1}{2}\left\{\log h_t(\phi) + \frac{\epsilon_t^2}{h_t(\phi)}\right\} = \sum_{t=1}^n l_t(\phi). \tag{2.9}$$

Define

$$\Lambda(\phi) = E\left\{\frac{\partial l_t(\phi)}{\partial\phi\partial\phi'}\right\} \quad \text{and} \quad \Omega(\phi) = E\left\{\frac{\partial l_t(\phi)}{\partial\phi}\frac{\partial l_t(\phi)}{\partial\phi'}\right\}.$$

For the stationary case, one can obtain the consistency and asymptotic normality of ϕ_{ML} after combining Lemmas 4 and 5 of Pan *et al.* (2008) and central limit theorem for martingales. Details are omitted.

Proposition 1. *Under (A1)–(A3) plus $Ee_t^4 < \infty$, we have*

(i) $\phi_{ML} \xrightarrow{a.s.} \phi^0$,

(ii) $\sqrt{n}\left(\phi_{ML} - \phi^0\right) \xrightarrow{d} N\left(0, \Lambda_0^{-1}\Omega_0\Lambda_0^{-1}\right)$, *where* $\Lambda_0 = \Lambda(\phi^0)$ *and* $\Omega_0 = \Omega(\phi^0)$.

2.2. Quasilikelihood estimation

Using the notation E_{t-1}, for the conditional expectation given F_{t-1}, σ-field generated $\epsilon_{t-1}, \epsilon_{t-2}, \ldots$, viz., $E_{t-1}(\cdot) = E(\cdot|F_{t-1})$, consider a sequence of martingale differences $\{g_t(\phi)\}$ with respect to the increasing σ-field $\{F_t\}$. Note that $E_{t-1}g_t(\phi) = 0$ and it is assumed that $E_{t-1}[\partial g_t(\phi)/\partial\phi] \neq 0$ and $E_{t-1}[g_t^2(\theta)] < \infty$. The quasilikelihood estimator (ϕ_{QL}) is obtained by solving $U_n(\phi) = 0$ where $U_n(\phi)$ denotes a quasilikelihood score function which is defined by

$$U_n(\phi) = \sum_{t=1}^{n} g_t(\theta) \frac{E_{t-1}\left[\partial g_t(\phi)/\partial\phi\right]}{E_{t-1}\left[g_t^2(\phi)\right]}. \tag{2.10}$$

See Hwang and Basawa (2011a). When we take $g_t(\phi) = \epsilon_t^2 - h_t(\phi)$, we readily have

$$E_{t-1}\left[g_t^2(\phi)\right] = E_{t-1}\epsilon_t^4 - h_t^2(\phi) \quad \text{and} \quad E_{t-1}\left[\frac{\partial g_t(\phi)}{\partial(\phi)}\right] = -\frac{\partial h_t(\phi)}{\partial\phi}$$

and therefore we have a quasilikelihood score function defined by

$$U_n(\phi) = -\sum_{t=1}^{n} \frac{\partial h_{t-1}(\phi)}{\partial\phi}\left[E_{t-1}\epsilon_t^4 - h_t^2(\phi)\right]^{-1}\left(\epsilon_t^2 - h_t(\phi)\right). \tag{2.11}$$

For the stationary case, ϕ_{QL} obtained from solving $U_n(\phi) = 0$ in (2.11) is seen to be asymptotically normal. Refer to Heyde (1997).

Proposition 2. *Under the same conditions as in Proposition 1, we have*

$$\sqrt{n}\left(\phi_{QL} - \phi^0\right) \xrightarrow{d} N\left(0, J(\phi^0)^{-1}\right),$$

where $J(\phi^0) = plim\ n^{-1}Var(U_n(\phi))$ *with* $Var(\cdot)$ *being evaluated at* $\phi = \phi^0$.

3. Non-Stationary Case

It is often the case in practice to assume that the ARCH/GARCH process is ergodic stationary so appropriate laws of large numbers apply (Jensen and Rahbek, 2004a). Recently Jensen and Rahbek (2004a) have showed that QMLE of the ARCH parameter is asymptotically normal with the same rate of convergence $\sqrt{n}$ even for the non-stationary explosive case. In a subsequent paper of Jensen and Rahbek (2004b), they obtained a similar result for an estimator for GARCH(1, 1) processes. Although, in their papers, estimators are restricted QMLE for which the true intercept coefficient α_0 is assumed to be known, Jensen and Rahbek (2004a, 2004b) were the first to consider the asymptotic theory of the QMLE for non-stationary ARCH/GARCH models. See also Linton *et al.* (2010). In order to discuss non-stationary case, we shall confine ourselves to the first order TAGARCH(1, 1) process. A simple TAGARCH(1, 1) process is given by $\epsilon_t = \sqrt{h_t}e_t$ and

$$h_t(\theta) = \alpha_0 + \alpha_{11}\left(\epsilon_{t-1}^+\right)^2 + \alpha_{12}\left(\epsilon_{t-1}^-\right)^2 + \beta_1 h_{t-1}(\theta), \quad t = 1, \ldots, n, \tag{3.1}$$

where $\theta = (\alpha_{11}, \alpha_{12}, \beta_1, \alpha_0) :\ 4 \times 1$.

3.1. Non-stationary QMLE

For a non-stationarity, we assume $\gamma(\phi) \geq 0$ (see Equation (2.7)), equivalently, assume that true parameters satisfy (C1) given by

$$(\mathbf{C1}) \qquad E \log\left[\alpha_{11}^0 (e_t^+)^2 + \alpha_{12}^0 (e_t^-)^2 + \beta_1^0\right] \geq 0. \tag{3.2}$$

It is noted that the parameter vector θ of the TAGARCH(1, 1) model is denoted by $\theta = (\alpha_{11}, \alpha_{12}, \beta_1, \alpha_0)$ rather than ϕ, with α_{11}, α_{12}, β_1 and α_0 being all positive. Denote the true parameter values by $\theta^0 = (\alpha_{11}^0, \alpha_{12}^0, \beta_1^0, \alpha_0^0)$. Adapting Jensen and Rahbek (2004a, 2004b) techniques originally used for standard (and symmetric) GARCH(1, 1) to our (asymmetric) TAGARCH(1,1) model, one can obtain (*cf.* Lemma 1 and Theorem 1 of Jensen and Rahbek (2004b)) asymptotic normality of the non-stationary QMLE. Assume that α^0 is known, a priori, as in Jensen and Rahbek (2004b). The QMLE of $(\alpha_{11}, \alpha_{12}, \beta_1)$ is denoted loosely by θ_{ML}.

Theorem 1. *Assume that the true likelihood is indeed normal. Then, under (C1) of a non-stationarity, the QMLE (and hence the exact maximum likelihood estimator) θ_{ML} is consistent and asymptotically normal, viz.,*

$$\sqrt{n}\left[\theta_{ML} - \left(\alpha_{11}^0, \alpha_{12}^0, \beta_1^0\right)\right] \xrightarrow{d} N(0, \Omega).$$

Here Ω is given by

$$\Omega = plim\; n^{-1}\left(\frac{\partial^2 l_n(\theta^0)}{\partial(\alpha_{11}, \alpha_{12}, \beta_1)\partial(\alpha_{11}, \alpha_{12}, \beta_1)'}\right).$$

Remark 1. It is interesting to note that convergence rate of the QMLE is still given by the square root of the sample size even for the non-stationary case.

3.2. Non-stationary QLE

To discuss the quasilikelhood estimator θ_{QL}, consider the following quasilikelihood estimating function $U_n(\theta)$ defined in (2.11) with θ replacing ϕ

$$U_n(\theta) = \sum_{t=1}^{n} u_t(\theta) : \; 4 \times 1 \text{ vector}, \tag{3.3}$$

where $\{u_t(\theta)\}$ is a sequence of martingale differences defined by

$$u_t(\theta) = -\frac{\partial h_{t-1}(\theta)}{\partial \theta}\left[E_{t-1}\epsilon_t^4 - h_t^2(\theta)\right]^{-1}\left(\epsilon_t^2 - h_t(\theta)\right).$$

Denote by $\xi_n(\theta)$ the sum of conditional covariance matrices corresponding to $u_t(\theta)$, viz.,

$$\xi_n(\theta) = \sum_{t=1}^{n} \mathrm{Var}(u_t(\theta)|F_{t-1}) = \sum_{t=1}^{n} E\left(u_t(\theta)u_t^T(\theta)\middle|F_{t-1}\right) : \; 4 \times 4 \text{ matrix}. \tag{3.4}$$

Notice that the vector $U_n(\theta)$ constitutes a (4×1)-dimensional martingale. It will be assumed throughout that a law of large numbers is valid for the martingale $U_n(\theta)$;

(C2) There exists a nonsingular (non-random) matrix Δ such that

$$\Delta = \text{plim}\left[-\xi_n^{-\frac{1}{2}}\left(\frac{\partial U_n(\theta)}{\partial\theta}\right)\xi_n^{-\frac{1}{2}}\right]. \tag{3.5}$$

(C2) can be verified using an appropriate law of large numbers. We refer to Hall and Heyde (1980) for central limit theorems and laws of large numbers for martingales.

Using random norm $\xi_n^{1/2}(\theta)$ instead of $\sqrt{n}$, we obtain asymptotic normality of θ_{QL} as a solution of the equation $U_n(\theta) = 0$. Here, the half matrix $\xi_n^{1/2}$ is defined by the symmetric matrix obtained from the spectral decomposition of $\xi_n(\theta)$.

Theorem 2. *Assume that $Ee_t^4 < \infty$. Then under (C1) and (C2), we have*

$$\xi_n^{\frac{1}{2}}\left(\theta^0\right)\left(\theta_{QL} - \theta^0\right) \xrightarrow{d} N\left(0, \Delta^{-1}\Delta^{-T}\right),$$

where Δ is defined in (C2) and Δ^{-T} denotes the inverse of Δ^{-1}.

Proof: Recently, Hwang and Basawa (2011b) discussed asymptotic distributions of martingale estimating equations and thus we adapt the main arguments as in Hwang and Basawa (2011b). It follows from (C2) and a Taylor's expansion of $U_n(\theta)$ at θ_{QL} that

$$\xi_n^{\frac{1}{2}}\left(\theta_{QL} - \theta^0\right) = \Delta^{-1}\xi_n^{-\frac{1}{2}}U_n\left(\theta^0\right) + o_p(1). \tag{3.6}$$

Using the Cramer-Wold device, for given non-zero constant vector a of size (4×1), consider

$$a^T\xi_n^{-\frac{1}{2}}U_n\left(\theta^0\right) = \sum_{t=1}^{n}\left[a^T\xi_n^{-\frac{1}{2}}u_t\left(\theta^0\right)\right]$$

for which the term inside square bracket is seen to be a sequence of martingale array differences for each fixed n. Thus by using the central limit theorem for martingale arrays with random norms (*cf.*, Theorem 3.2 of Hall and Heyde (1980)), we conclude

$$\xi_n^{-\frac{1}{2}}U_n\left(\theta^0\right) \xrightarrow{d} N(0, I_k)$$

which in turn implies the theorem via (3.6). □

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