KYUNGPOOK Math. J. 51(2011), 233-239
DOI 10.5666/KMJ.2011.51.2.233

I-Monotonic and I-Convergent Sequences

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Abstract. In this article we study the noton of I-monotonic sequences. We prove the decomposition theorem and generalize some of the results on monotonic sequences. We also introduce I-convergent series and studied some results.

1. Introduction

Throughout the article w, c, c_0, ℓ_∞ denote the spaces of *all*, *convergent*, *null* and *bounded* sequences of real numbers respectively.

The notion of statistical convergence of sequences was introduced by Fast [2] and Schoenberg [8] independently. Later on it was further investigated from sequence space point of view and linked with Summability Theory by Salat [5], Rath and Tripathy [4], Tripathy ([9], [10], [11], [12], [13], [14]), Tripathy and Baruah [15], Tripathy and Sarma [19], Tripaty and Sen ([21], [22]) and many others. The notion depends on the density of the subsets of N.

Definition 1.1. A subset E of N, the set of all natural numbers, is said to have *asymptotic density* or *density* $\delta(E)$ if

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k)$$

exists, where χ_E is the *characteristic function* of E.

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Received July 23, 2010; revised January 7, 2011; accepted January 20, 2011.
2000 Mathematics Subject Classification: 40A05, 40A35, 40D09, 40D20, 46A45.
Key words and phrases: I-conergence, I-bounded, monotonic sequence, I-convergent series.

Definition 1.2. A complex sequence (x_k) is said to be *statistically convergent* to L if for every $\varepsilon > 0$, $\delta(\{k \in N : |x_k - L| \geq \varepsilon\}) = 0$. We write $stat - lim x_k = L$.

2. Definitions and preliminaries

The notion of *I-convergence* was studied at initial stage by Kostyrko, Salat and Wilczynski [3]. Later on it was studied by Salat, Tripathy and Ziman ([8], [9]), Demirci [1], Tripathy and Hazarika ([16], [17], [18]), Tripathy and Mahanta [20] and others.

Definition 2.1. Let X be a non-empty set. Then a family of sets $I \subseteq 2^X$ (power sets of X) is said to be an *ideal* if I is additive i.e. $A, B \in I \Rightarrow A \cup B \in I$ and hereditary i.e. $A \in I$,$B \subseteq A \Rightarrow B \in I$.

Definition 2.2. A non-empty family of sets $\Im \subset 2^X$ is said to be a *filter* on X if and only if $\phi \notin \Im$, for each $A, B \in \Im$ we have $A \cap B \in \Im$ and for each $A \in \Im$ and $B \supset A$, implies $B \in \Im$.

Definition 2.3. An ideal $I \subseteq 2^X$ is called *non-trivial* if $I \neq 2^X$.

Definition 2.4. A non-trivial ideal $I \subset 2^X$ is called *admissible* if and only if $I \supset \{\{x\} : x \in X\}$.

Definition 2.5. A non-trivial ideal I is *maximal* if there cannot exists any non-trivial ideal $J \neq I$ containing I as a subset.

For each ideal I, there is a filter $\Im(I)$ corresponding to I, i.e. $\Im(I) = \{K \subseteq N : K^c \in I\}$, where $K^c = N - K$.

Definition 2.6. A sequence $(x_k) \in w$ is said to be *I- convergent* to the number L if for every $\varepsilon > 0$, $\{k \in N : |x_k - L| \geq \varepsilon\} \in I$. We write $I - \lim x_k = L$.

Definition 2.7. A sequence $(x_k) \in w$ is said to be *I-null* if $L = 0$. We write $I - \lim x_k = 0$.

Definition 2.8. A sequence $(x_k) \in w$ is said to be *I-Cauchy* if for every $\varepsilon > 0$ there exists a number $m = m(\varepsilon)$ such that $\{k \in N : |x_k - x_m| \geq \varepsilon\} \in I$.

Definition 2.9. A sequence $(x_k) \in w$ is said to be *I-bounded* if there exists $M > 0$ such that $\{k \in N : |x_k| > M\} \in I$.

Definition 2.10. Let (x_k) and (y_k) be two sequences. We ay that $x_k = y_k$ for *almost all k relative to I* $(a.a.k.r.I)$, if $\{k \in N : x_k \neq y_k\} \in I$.

Throughout the paper, ℓ_∞^I, c^I, c_0^I denote the classes of *I-bounded, I-convergent*, and *I-null* sequence spaces respectively.

The usual convergence is a particular case of *I-convergence*. In this case $I = I_f$ (the ideal of all finite subsets of N).

The statistical convergence is a particular case of *I-convergence*. In this case $I = I_\delta$ (the ideal of all subsets of N of zero *asymptotic density*).

Definition 2.11. Let $E \subset N$ and $d_n(E) = \frac{1}{s_n}\sum_{k=1}^{n} \frac{\chi_E(k)}{k}$, for $n \in N$, where $s_n = \sum_{k=1}^{n} \frac{1}{k}$. If $\lim_{n\to\infty} d_n(E)$ exists, then it is called as the *logarithmic density* of E. Clearly $I_d = \{E \subset N : d(E) = 0\}$ is an ideal.

Definition 2.12. Let $T = (t_{nk})$ be a *regular non − negative matrix*. For $E \subset N$, define $d_T^{(n)}(E) = \sum_{k=1}^{\infty} t_{nk}\chi_E(k)$, for all $n \in N$. If $\lim_{n\to\infty} d_T^{(n)}(E) = d_T(E)$ exists, then $d_T(E)$ is called as *T-density* of E. Clearly $I_{d_T} = \{E \subset N : d_T(E) = 0\}$ is an ideal.

Note 2.1. I_δ and I_d are particular cases of I_{d_T}.

(i) *Asymptotic density*, for

$$t_{nk} = \begin{cases} \frac{1}{n}, & \text{if } n \le k; \\ 0, & \text{otherwise.} \end{cases}$$

(ii) *Logarithmic density*, for

$$t_{nk} = \begin{cases} \frac{k^{-1}}{s_n}, & \text{if } n \le k; \\ 0, & \text{otherwise.} \end{cases}$$

The notion of I-monotonic sequence was studied by Salat, Tripathy and Ziman [6].

Definition 2.13. A real sequence (x_k) is said to be *I-monotonic incerasing* (*I-monotonic decreasing*), if there is a set $\{k_1 < k_2 < ...\} \in \Im(I)$ such that $x_{k_i} \le x_{k_{i+1}}$ $(x_{k_i} \ge x_{k_{i+1}})$.

Definition 2.14. A sequence (x_k) is said to be *I-monotonic*, if there exists a set $\{k_1 < k_2 < ...\} \in \Im(I)$ such that $x_{k_i} \le x_{k_{i+1}}$ or $x_{k_i} \ge x_{k_{i+1}}$.

Definition 2.15. A sequence $(x_k) \in w$ is said to be I^*-*convergent* to L, if and only if there exists a subset $M = \{m_1 < m_2 < ...\} \in \Im(I)$ such that $\lim_{i\to\infty} |x_{m_i} - L| = 0$.

Definition 2.16. A sequence space E is said to be *symmetric* if $(x_k) \in E$ implies $(x_{\pi(k)}) \in E$, where π is a permutation of N.

The following results will be used for establishing some results of this article.

Lemma 1(Kostyrko, Salat and Wilczynski [3], Proposition 3.2)**.** *Let I be an admissible ideal. If $I^* - \lim x_n = L$, then $I - \lim x_n = L$.*

Lemma 2 (Salat, Tripathy and Ziman [7], Lemma 2.5)**.** *Let $K \in \mathfrak{F}(I)$ and $M \subseteq N$.*

If $M \notin I$, then $M \cap K \notin I$.

3. Main results on I- monotonic sequences

Theorem 3.1. *A real sequence (x_n) is I-monotonic if and only if there exists a monotonic sequence (y_n) such that $x_n = y_n$ for a.a.n.r.I.*

Proof. Let (x_n) be a monotonic increasing sequence , Then there exists a subset $K = \{k_i : i \in N\}$ of N such that $K \in I$ and (x_{k_i}) is monotonic increasing. Let us construct (y_n) as $x_n = y_n$ if $n = k_i$ and $y_n = x_{k_i}$ if $k_i < n < k_{i+1}, i \in N$.

Then clearly (y_n) is monotonic increasing and $x_n = y_n a.a.n.r.I$.

Conversely suppose that (x_n) is such that $x_n = y_n a.a.n.r.I$ and (y_n) is monotonic increasing.

Let $H = \{k \in N : x_k \neq y_k\} \subset N$. Clearly $H \in I$ and $x_{k_i} \leq x_{k_{i+1}}$, for all $k_i \in N - H$, for $i \in N$.

This completes the proof. □

The proof of the following results are easy, so omitted.

Result 3.1. *Let $x_n \leq y_n$, for $n \in K \subset N$ with $K \in I$, $I - lim x_n = \infty$, then $I - lim y_n = \infty$ and if $I - lim y_n = -\infty$, then $I - lim x_n = -\infty$.*

Result 3.2. *Let $x_n \geq \alpha$, for all $n \in K \subset N$ with $K \in I$.* If *$I - lim x_n = L$, then $L \geq \alpha$.*

Result 3.3. *If a sequence is I-convergent, then it is I-bounded.*

Remark 3.1. If (x_n) is a bounded monotonic increasing (or monotonic decreasing) sequence, then $\lim x_n = \sup x_n$(or $\inf x_n$). But for a bounded I-monotonic increasing (or I-monotonic decreasing) sequence it may or may not be. For this we consider the following example.

Example 3.1. Let $I = I_\delta$. Consider the sequence (x_n) defined as

$$x_n = \begin{cases} 1, & \text{for } n = k^2, k \in N; \\ n^{-1}, & \text{otherwise.} \end{cases}$$

Then (x_n) is I-monotonic decreasing and $I - \lim x_n = 0$ but $\sup x_n = 1$.

Result 3.4. *Let $(x_n) \in c^I$. Then $(x_{\pi(n)}) \in c^I$, if $(x_{\pi(n)}) = (x_{\pi(S) \cup \pi(S^c)})$, where $S = \{k \in N : z_k \neq 0\} \in I$ and $x_n = y_n + z_n$, for all $n \in N$ such that $(y_n) \in c^I$ and $(z_n) \in c_0^I$.*

Result 3.5. *Let $x_n \leq y_n$, for all $n \in A(\in I)$. If $I - \lim x_n$ and $I - \lim y_n$ exist, then $I - \lim x_n \leq I - \lim y_n$. If $I - \lim y_n < B$, then $I - \lim x_n < B$.*

Result 3.6. *Let $(x_n) > 0$, for all $n \in A(\in I)$ and $x_n \neq 0$, for all $n \in N$. Then $I - \lim x_n = \infty$ if and only if $I - \lim x_n^{-1} = 0$.*

Result 3.7. (i) *If* $I-\lim x_n = L$ *then* $I-\lim |x_n| = L$, *but the converse is not necessarily true.*
(ii)*A sequence* (x_n) *is I-null if and only if* $(|x_n|)$ *is I-null.*

Result 3.8. *A real sequence* (x_n) *is I-bounded if and only if we can write* $(x_n) = (y_n) + (z_n)$, *where* (y_n) *is bounded and* (z_n) *is I-null.*

4. I-Convergent series

The notion of statistical convergence for series was introduced by Tripathy [13]. In this section we introduce the notion of I-convergence for series of real or complex numbers. This generalizes and unifies different notions of convergence of series. We investigate its different properties.

Definition 4.1. A series $\sum\limits_{k=1}^{\infty} x_k$ is said to be *I-convergent* if it's sequence of partial sums (s_n), where $s_n = x_1 + x_2 + ... + x_n$ is I-convergent.

Definition 4.2. A series $\sum\limits_{k=1}^{\infty} x_k$ is said to be *I-bounded* if it's sequence of partial sums (s_n), where $s_n = x_1 + x_2 + ... + x_n$ is I-bounded.

Theorem 4.1. *If a series* $\sum x_n$ *is I-convergent then there exists a subset* $K = \{k_1 < k_2 < ...\}$ *of* N *such that* $K \in I$ *and* $\sum_i x_{k_i}$ *is convergent.*

Proof. Let $\sum x_n$ is I-convergent, then there exists a subset $G = \{n_1 < n_2 < ---\}$ of N with $G \in I$ then s_{n_i} is convergent.

Let us construct a set $P = \{p_1 < p_2 < ...\}$ of N as $n_{2i} = p_i + 1$; $n_{2i-1} = p_i$, where $p_i \in G^c$, if there is repetition that is $n_i = n_{i+1}$, for some i, then we count n_i not n_{i+1}. Then $G \notin I$ implies that $G^c \in I$. If we take $K = G^c$, then $\sum\limits_{k \in K} x_k$ is convergent. $\square$

Remark 4.1. The converse of the above result fails even though $\sum x_n$ is bounded. It is clear from the following example.

Example 4.1. Let $\sum x_n$ be defined by

$$x_n = \begin{cases} (-1)^n, & \text{for } n = k^2,\ k \in N; \\ 0, & \text{otherwise.} \end{cases}$$

Proof of the following results are easy, so omitted.

Result 4.1. *A series* $\sum z_n$ *of complex terms is I-convergent if and only if the series of real part and imaginary part are I-convergent.*

Result 4.2. *A series* $\sum x_n$ *is I-convergent if and only if there exist series* $\sum y_n$ *and* $\sum z_n$ *such that* $x_n = y_n + z_n$, *for all* $n \in N$, *where* $\sum y_n$ *is convergent and* $\sum z_n$ *is I-null.*

Result 4.3. *If* $\sum x_n$ *and* $\sum y_n$ *two I-convergent series, then for complex numbers* α, β; $\sum(\alpha x_n + \beta y_n)$ *is I-convergent to the sum* $\alpha \sum x_n + \beta \sum y_n$.

References

[1] K. Demirci, *I-limit superior and limit inferior*, Math. Commun, **6(2)**(2001), 165-172.

[2] H. Fast, *Surla convergence statistique*, Colloq. Math., **2**(1951), 241-244.

[3] P. Kostyrko, T. Salat and W. Wilczynski, *I-convergence*, Real Analysis Exchange,**26(2)**(2000-2001), 669-686.

[4] D. Rath and B. C. Tripathy, *Matrix maps on sequence spaces and associated with sets of integers*, Indian Jour. Pure Appl. Math., **27(2)**(1996), 197-206.

[5] T. Salat, *On statistically convergent sequences of real numbers*, Math. Slovaca, **30**(1980), 139-150.

[6] T. Salat, B. C. Tripathy and M. Ziman, *On some properties of I-convergence*, Tatra Mt. Math. Publ, **28**(2004), 279-286.

[7] T. Salat, B. C. Tripathy and M. Ziman, *On I-convergence field*, Italian Jour. Pure Appl. Math, **17**(2005), 45-54.

[8] I. J. Schoenberg, *The integrability of certain functions and related summability methods*, Amer. Math. Monthly, **66**(1959), 361-375.

[9] B. C. Tripathy, *Matrix transformation between some classes of sequences*, Jour. Math. Anal. Appl., **206**(1997), 448-450.

[10] B. C. Tripathy, *On statistical convergent sequences*, Bull. Cal. Math. Soc., **90**(1998), 259-262.

[11] B. C. Tripathy, *On statistical convergence*, Proc.Estonion Acad. Sci. Phy. Math., **47(4)**(1998), 299-303.

[12] B. C. Tripathy,*On statistical convergent and statistical bounded sequences*, Bull. Malyasian Math. Soc., **20**(1998), 31-33.

[13] B. C. Tripathy, *On statistical convergent series*, Journal of Math. Pun. Univ., **XXXII**(1999), 1-8.

[14] B. C. Tripathy, *On generalized difference paranormed statistical convergent sequences*, Indian J. Pure Appl. Math., **35(5)**(2004), 655-663.

[15] B. C. Tripathy and A. Baruah, *Lacunary statistically convergent and lacunary strongly convergent generalized difference sequences of fuzzy real numbers*, Kyungpook Mathematical Journal, **50**(2010), 565-574.

[16] B. C. Tripathy and B. Hazarika, *I-convergent sequence spaces associated with multiplier sequences*, Math. Ineq. Appl., **11(3)**(2008), 543-548.

[17] B. C. Tripathy and B. Hazarika, *Paranorm I-convergent sequence spaces*, Math. Slovaca, **59(4)**(2009), 485-494.

[18] B. C. Tripathy and B. Hazarika, *Some I-convergent sequence spaces defined by Orlicz function*, Acta Mathematica Applicatae Sinica, **27(1)**(2011), 149-154.

[19] B. C. Tripathy and B. Sarma, *Statistically convergent difference double sequence spaces*, Acta Mathematica Sinica, **24(5)**(2008), 737-742.

[20] B. C. Tripathy and S. Mahanta, *On I-acceleration convergence of sequences*, Journal of the Franklin Institute, **347**(2010), 591-598.

[21] B. C. Tripathy and M. Sen, *On generalized statistically convergent sequences*, Indian Jour. Pure Appl. Math., **32(11)**(2001), 1689-1694.

[22] B. C. Tripathy and M. Sen, *Characterization of some matrix classes involving paranormed sequence spaces*, Tamkang Jour. Math., **37(2)**(2006), 155-162.