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On a Result of N. Terglane

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Abstract. We prove a uniqueness theorem for meromorphic functions sharing three weighted values, which improves a result given by N. Terglane in 1989 and a result given by X. M. Li and H. X. Yi in 2003. Some examples are provided to show that the result of the paper is best possible.

1. Introduction, definitions and results

In the paper, by meromorphic functions we shall always mean meromorphic functions in the open complex plane $\mathbb{C}$. We adopt the standard notations and definitions of Nevanlinna theory of meromorphic functions as explained in [3, 12]. It is convenient to let E denote any set of positive real numbers of a finite linear measure not necessarily the same at each occurrence. For a non-constant meromorphic function h, we denote by $T(r,h)$ the Nevanlinna characteristic of h and by $S(r,h)$ any quantity satisfying $S(r,h) = o\{T(r,h)\}$ as $r \to \infty$, $r \notin E$.

Let f and g be two non-constant meromorphic functions and let $a \in \mathbb{C} \cup \{\infty\}$. We say that f and g share the value a CM, provided that f and g have the same a-points with the same multiplicities. We say that f and g share the value a IM, provided that f and g have the same a-points ignoring multiplicities (see [12]). Throughout the paper, we need the following definitions.

Definition 1.1. A meromorphic function $a = a(z)$ is called a small function of f if $T(r,a) = S(r,f)$.

Definition 1.2. Let a be a small function of f and g. We denote by $\overline{N}(r,a;f,g)$

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the reduced counting function of the common zeros of $f-a$ and $g-a$.

Definition 1.3. We denote by $N(r,a;f,g\mid\leq 1)$ the counting function of the common simple zeros of $f-a$ and $g-a$ for some small function a of f and g.

Definition 1.4. Let f and g share 0, 1, ∞ IM. We denote by $N(r)$ the counting function of those zeros of $f-g$ which are not the zeros of $f(f-1)$ and $\frac{1}{f}$.

Definition 1.5. We denote by $\overline{N}_0(r,a;f,g)$ the reduced counting function of the common zeros of $f-a$ and $g-a$, where the respective multiplicities are unequal and $a=a(z)$ is a small function of f and g.

Definition 1.6. Let k be a positive integer and a be a small function of f. We denote by $N(r,a;f\mid\leq k)$ $(N(r,a;f\mid\geq k))$ the counting function of those zeros of $f-a$ whose multiplicities are not greater (less) than k.

Also by $\overline{N}(r,a;f\mid\leq k)$ and $\overline{N}(r,a;f\mid\geq k)$ we denote the corresponding reduced counting functions.

In 1989, N. Terglane [11] proved the following result.

Theorem A. *Let f and g be non-constant meromorphic functions and let a, b, c, d be four distinct complex numbers such that the cross-ratio $(a,b,c,d)\in\{-1,2,\frac{1}{2}\}$. If f and g share b, c, d CM and $\overline{N}(r,a;f,g)\neq S(r,f)$ then f is a bilinear transformation of g.*

Q. C. Zhang [13] in 1998 worked in the line of Theorem A and proved the following result.

Theorem B([13]). *Let f and g be non-constant meromorphic functions sharing* 0 , 1 *and* ∞ *CM. Suppose additionally that there exists a complex number $a(\neq 0,1,\infty)$ such that $E_{1)}(a;f)=E_{1)}(a;g)\neq\emptyset$, where $E_{1)}(a;f)$ denotes the set of simple a-points of f and $\emptyset$ denotes the empty set. Then f and g satisfy one of the following relations:* (i) $f\equiv g$, (ii) $f+g\equiv 1$ *with* $a=\frac{1}{2}$, (iii) $(f-1)(g-1)\equiv 1$ *with* $a=2$, (iv) $fg\equiv 1$ *with* $a=-1$.

In 2003, X. M. Li and H. X. Yi [10] considered the problem of removing the hypothesis $(a,b,c,d)\in\{-1,2,\frac{1}{2}\}$ from Theorem A. They considered the special case $b=1$, $c=0$ and $d=\infty$, as the general case can be treated by a suitable bilinear transformation. Following is the result of X. M. Li and H. X. Yi.

Theorem B([10]). *Let f and g be non-constant meromorphic functions sharing 0, 1 and ∞ CM. Further suppose that there exists a complex number $a(\neq 0,1,\infty)$ such that $\overline{N}(r,a;f,g)\neq S(r,f)$. Then f is a bilinear transformation of g, apart from the following three exceptional cases:*

(i) $f = \dfrac{e^{s\gamma}-1}{e^{(p+1)\gamma}-1}$, $g = \dfrac{e^{-s\gamma}-1}{e^{-(p+1)\gamma}-1}$ *with* $1 \le s \le p$ *and* $a = \dfrac{s}{p+1}$,

(ii) $f = \dfrac{e^{(p+1)\gamma}-1}{e^{(p+1-s)\gamma}-1}$, $g = \dfrac{e^{-(p+1)\gamma}-1}{e^{-(p+1-s)\gamma}-1}$ *with* $1 \le s \le p$ *and* $a = \dfrac{p+1}{p+1-s}$,

(iii) $f = \dfrac{e^{s\gamma}-1}{e^{-(p+1-s)\gamma}-1}$, $g = \dfrac{e^{-s\gamma}-1}{e^{(p+1-s)\gamma}-1}$ *with* $1 \le s \le p$ *and* $a = \dfrac{s}{s-p-1}$,

where γ *is a non-constant entire function and* $p(\geq 2)$ *and* s *are positive integers such that* s *and* $p+1$ *are relatively prime. Further the following holds:*

$$N(r, a; f, g \mid\leq 1) = N(r) + S(r, f) = \frac{1}{p}T(r, f) + S(r, f).$$

Following examples show that in Theorem C one cannot replace any of the CM shared values by IM shared values.

Example 1.1. Let $f = \dfrac{(1-e^z)^3}{1-3e^z}$ and $g = \dfrac{4(1-e^z)}{1-3e^z}$. Then f and g share 0 IM and 1, ∞ CM. Since $f - 2 = \dfrac{(1+e^z)(4e^z - e^{2z} - 1)}{1-3e^z}$ and $g - 2 = \dfrac{2(1+e^z)}{1-3e^z}$, we see that $\overline{N}(r, 2; f, g) \neq S(r, f)$ but the conclusion of Theorem C does not hold.

Example 1.2. Let $f = \dfrac{e^{2z}(e^z-3)}{1-3e^z}$ and $g = \dfrac{e^z-3}{1-3e^z}$. Then f, g share 0, ∞ CM and 1 IM. Since $f + 1 = \dfrac{(1+e^z)(1+e^{2z}-4e^z)}{1-3e^z}$ and $g + 1 = \dfrac{-2(1+e^z)}{1-3e^z}$, we see that $\overline{N}(r, -1; f, g) \neq S(r, f)$ but the conclusion of Theorem C does not hold.

Example 1.3. Let $f = \dfrac{1-3e^z}{(1-e^z)^3}$ and $g = \dfrac{1-3e^z}{4(1-e^z)}$. Then f, g share 0, 1 CM and ∞ IM. Since $f - \dfrac{1}{2} = \dfrac{(1+e^z)(1-4e^z+e^{2z})}{2(1-e^z)^3}$ and $g - \dfrac{1}{2} = -\dfrac{1+e^z}{4(1-e^z)}$, we see that $\overline{N}(r, \frac{1}{2}; f, g) \neq S(r, f)$ but the conclusion of Theorem C does not hold.

In view of the above examples, we may ask the following question: *Is it possible in any way to relax the nature of sharing values in Theorem C ?*

We use the notion of weighted sharing of values to answer the above question in affirmative. In the paper we also investigate the problem of replacing the value $a(\neq 0, 1, \infty)$ in Theorem C by a small function $a = a(z)(\not\equiv 0, 1, \infty)$ of f and g.

We now explain the notion of weighted sharing of values which measures how close a shared value is to being shared IM or to being shared CM.

Definition 1.7([4, 5])**.** Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k then z_o is a zero of $f-a$ with multiplicity $m(\leq k)$ if and only if it is a zero of $g-a$ with multiplicity $m(\leq k)$ and z_o is a zero of $f-a$ with multiplicity $m(>k)$ if and only if it is a zero of $g-a$ with multiplicity $n(>k)$ where m is not necessarily equal to n.

We write f, g share (a,k) to mean that f, g share the value a with weight k. Clearly if f, g share (a,k) then f, g share (a,p) for all integers p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a,0)$ or (a,∞) respectively.

We now state the main result of the paper.

Theorem 1.1. *Let f and g be two distinct non-constant meromorphic functions sharing $(0,1)$, $(1,m)$, (∞,k), where $(m-1)(mk-1)>(1+m)^2$. Further let there exist a small function $a=a(z)(\not\equiv 0,1,\infty)$ of f and g such that $\overline{N}(r,a;f,g)\neq S(r,f)$. Then f and g assume one of the following forms:*

(i) $f=\dfrac{e^{s\gamma}-1}{e^{(p+1)\gamma}-1}$, $g=\dfrac{e^{-s\gamma}-1}{e^{-(p+1)\gamma}-1}$ *with* $1\leq s\leq p$ *and* $a\equiv\dfrac{s}{p+1}$,

(ii) $f=\dfrac{e^{(p+1)\gamma}-1}{e^{(p+1-s)\gamma}-1}$, $g=\dfrac{e^{-(p+1)\gamma}-1}{e^{-(p+1-s)\gamma}-1}$ *with* $1\leq s\leq p$ *and* $a\equiv\dfrac{p+1}{p+1-s}$,

(iii) $f=\dfrac{e^{s\gamma}-1}{e^{-(p+1-s)\gamma}-1}$, $g=\dfrac{e^{-s\gamma}-1}{e^{(p+1-s)\gamma}-1}$ *with* $1\leq s\leq p$ *and* $a\equiv\dfrac{s}{s-p-1}$,

where γ is a non-constant entire function and p and s are positive integers such that s and $p+1$ are relatively prime. Further the following holds:

$$N(r,a;f,g\mid\leq 1)=N(r)+S(r,f)=\frac{1}{p}T(r,f)+S(r,f).$$

Remark 1.1. Examples 1.1, 1.2 and 1.3 show that in Theorem 1.1 the weight of value sharing cannot be reduced to zero.

Remark 1.2. Theorem 1.1 is valid for the following pairs of least values of m and k : $m=2,k=6$; $m=3,k=4$; $m=6,k=2$ and $m=4,k=3$.

Following corollary is immediate from Theorem 1.1.

Corollary 1.1. *Let f and g be two distinct non-constant meromorphic functions sharing $(0,1)$, $(1,m)$, (∞,k), where $(m-1)(mk-1)>(1+m)^2$. Further suppose that $a=a(z)(\not\equiv\infty)$ is a small function of f and g, which is not identically equal to a rational number. Then $\overline{N}(r,a;f,g)=S(r,f)$.*

2. Lemmas

In this section, we present some necessary lemmas.

Lemma 2.1([2]). *If f, g share $(0,0)$, $(1,0)$, $(\infty,0)$ then $T(r,f) \leq 3T(r,g)+S(r,f)$ and $T(r,g) \leq 3T(r,f)+S(r,g)$.*

This shows that $S(r,f) = S(r,g)$ and we denote them by $S(r)$.

Lemma 2.2([9]). *Let f and g be two distinct non-constant meromorphic functions sharing $(0,1)$, $(1,m)$ and (∞,k), where $(m-1)(mk-1) > (1+m)^2$. If $N(r) \neq S(r)$, then one of the following holds:*

(i) *f is a bilinear transformation of g with $N(r) = T(r,f)+S(r) = T(r,g)+S(r)$,*

(ii) *f is not a bilinear transformation of g with $T(r,f) = T(r,g) + S(r)$ and $N(r) \leq \frac{1}{2}T(r,f) + S(r)$.*

Lemma 2.3([9]). *Let f and g be two distinct non-constant meromorphic functions sharing $(0,1)$, $(1,m)$ and (∞,k), where $(m-1)(mk-1) > (1+m)^2$. If $N(r) \geq \lambda T(r,f)+S(r)$ for some $\lambda > \frac{1}{2}$ then f is a bilinear transformation of g and $N(r) = T(r,f) + S(r) = T(r,g) + S(r)$. Further f and g satisfy one of the following : (i) $f + g \equiv 1$, (ii) $(f-1)(g-1) \equiv 1$ and (iii) $fg \equiv 1$.*

Lemma 2.4([9]). *Let f and g be two distinct non-constant meromorphic functions sharing $(0,1)$, $(1,m)$ and (∞,k), where $(m-1)(mk-1) > (1+m)^2$. If $N(r) \leq \lambda T(r,f) + S(r)$ for some $\lambda (0 < \lambda < 1)$ and $N(r) \neq S(r)$ then f is not a bilinear transformation of g and $N(r) = \frac{1}{p}T(r,f) + S(r)$, $T(r,f) = T(r,g) + S(r)$ and f, g satisfy one of the following :*

(i) $f = \dfrac{e^{s\gamma}-1}{e^{(p+1)\gamma}-1}$ *and* $g = \dfrac{e^{-s\gamma}-1}{e^{-(p+1)\gamma}-1}$;

(ii) $f = \dfrac{e^{(p+1)\gamma}-1}{e^{(p+1-s)\gamma}-1}$ *and* $g = \dfrac{e^{-(p+1)\gamma}-1}{e^{-(p+1-s)\gamma}-1}$;

(iii) $f = \dfrac{e^{s\gamma}-1}{e^{-(p+1-s)\gamma}-1}$ *and* $g = \dfrac{e^{-s\gamma}-1}{e^{(p+1-s)\gamma}-1}$;

where s and $p(\geq 2)$ are positive integers with $1 \leq s \leq p$ and s, $p+1$ are relatively prime and γ is a non-constant entire function.

Lemma 2.5([6]). *Let f and g be two distinct non-constant meromorphic functions sharing $(0,1)$, $(1,m)$ and (∞,k), where $(m-1)(mk-1) > (1+m)^2$. Then $\overline{N}(r,b;f \mid\geq 2) = S(r)$ and $\overline{N}(r,b;g \mid\geq 2) = S(r)$ for $b = 0,1,\infty$.*

Lemma 2.6([7, 8])**.** *Let f and g be two distinct non-constant meromorphic functions sharing $(0,1)$, $(1,m)$ and (∞,k), where $(m-1)(mk-1) > (1+m)^2$. If f is not a bilinear transformation of g then each of the following holds:*

(i) $T(r,f)+T(r,g) = N(r,0;f \mid\leq 1)+N(r,1;f \mid\leq 1)+N(r,\infty;f \mid\leq 1)+N_0(r)+S(r)$,

(ii) $T(r,f) = N(r,0;g' \mid\leq 1) + N_0(r) + S(r)$,

(iii) $T(r,g) = N(r,0;f' \mid\leq 1) + N_0(r) + S(r)$,

(iv) $N_1(r) = S(r)$,

(v) $N_0(r,0;f' \mid\geq 2) = S(r)$,

(vi) $N_0(r,0;g' \mid\geq 2) = S(r)$,

where $N_0(r)$ ($N_1(r)$) denotes the counting function of those simple (multiple) zeros of $f-g$ which are not the zeros of $f(f-1)$ and $\frac{1}{f}$, also $N_0(r,0;f' \mid\geq 2)$ ($N_0(r,0;g' \mid\geq 2)$) is the counting function of those multiple zeros of f' (g') which are not the zeros of $f(f-1)$.

Lemma 2.7([1]). *Let f and g be two distinct non-constant meromorphic functions sharing $(0,1)$, $(1,m)$ (∞,k), where $(m-1)(mk-1) > (1+m)^2$. If $N(r,a;f \mid\leq 2) \neq T(r,f) + S(r,f)$ for some small function $a(\not\equiv 0,1,\infty)$ of f and g, then $\overline{N}(r,a;f) = S(r,f)$.*

3. Proofs of the main results

In this section we present the proofs of the theorems.

Proof of Theorem 1.1. Let $\phi_1 = \frac{F'}{F} - \frac{G'}{G}$ and $\phi_2 = \frac{f'(f-a)}{f(f-1)} - \frac{g'(g-a)}{g(g-1)}$, where $aF = f$ and $aG = g$.

If $\phi_1 \equiv 0$, then $f \equiv Ag$, where A is a constant. Since $\overline{N}(r,a;f,g) \neq S(r)$, we get $A = 1$ and so $f \equiv g$, which is a contradiction. So $\phi_1 \not\equiv 0$.

We see that $m(r,\phi_1) \leq m\left(r, \frac{F'}{F}\right) + m\left(r, \frac{G'}{G}\right) + O(1) = S(r)$ and by Lemma 2.5

$$\begin{aligned} N(r,\infty;\phi_1) &= \overline{N}(r,\infty;\phi_1) \\ &\leq \overline{N}(r,0;f \mid\geq 2) + \overline{N}(r,\infty;f \mid\geq 2) + N(r,0;a) + N(r,\infty;a) \\ &= S(r). \end{aligned}$$

Hence $T(r,\phi_1) = S(r)$.

Suppose that $\phi_2 \not\equiv 0$. Since $\phi_2 = a\left(\frac{f'}{f} - \frac{g'}{g}\right) + (1-a)\left(\frac{f'}{f-1} - \frac{g'}{g-1}\right)$, we get $m(r,\phi_2) = S(r)$ and by Lemma 2.5 we obtain

$$\begin{aligned} N(r,\infty;\phi_2) &\leq \overline{N}(r,\infty;\phi_2) + N(r,\infty;a) \\ &\leq \overline{N}(r,0;f \mid\geq 2) + \overline{N}(r,\infty;f \mid\geq 2) + \overline{N}(r,1;f \mid\geq 2) + N(r,\infty;a) \\ &= S(r). \end{aligned}$$

Hence $T(r,\phi_2) = S(r)$. Now we see that

$$\begin{aligned}\overline{N}(r,a;f,g) &\leq N(r,0;\phi_2) + N(r,0;a) + N(r,1;a) + N(r,\infty;a)\\ &\leq T(r,\phi_2) + S(r)\\ &= S(r),\end{aligned}$$

which is a contradiction. Hence $\phi_2 \equiv 0$ and so

$$\frac{f'(f-a)}{f(f-1)} \equiv \frac{g'(g-a)}{g(g-1)}. \tag{3.1}$$

Let z_0 be a zero of $f-a$ and $g-a$ with respective multiplicities p and q. If $a(z_0) \neq 0, 1, \infty$ and $a'(z_0) \neq 0$, then from (3.1) we see that $p = q$.

Let $a(z_0) = 0$. Then z_0 is a zero of $f = (f-a)+a$ and a zero of $g = (g-a)+a$. Hence z_0 is a zero of $\dfrac{f'(f-a)}{f(f-1)}$ and $\dfrac{g'(g-a)}{g(g-1)}$ with multiplicities $p-1$ and $q-1$ respectively. So from (3.1) we get $p = q$. Similarly if $a(z_0) = 1$, we get $p = q$.

Let z_0 be a pole of $a = a(z)$ with multiplicity r. Since z_0 is a zero of $f-a$ and $g-a$, it follows that z_0 is a pole of f and g with multiplicity r. Hence z_0 is a zero of $\dfrac{f'(f-a)}{f(f-1)}$ and $\dfrac{g'(g-a)}{g(g-1)}$ with respective multiplicities $p+r-1$ and $q+r-1$. So from (3.1) we get $p = q$. Therefore we obtain

$$\begin{aligned}\overline{N}_0(r,a;f,g) &\leq N(r,0;a') + N(r,0;a) + N(r,1;a) + N(r,\infty;a)\\ &\leq 5T(r,a) = S(r).\end{aligned} \tag{3.2}$$

We now verify that a zero of $f-a$, which is not a zero of $g-a$, cannot be a pole of f and g. Let z_1 be a zero of $f-a$ with multiplicity $p(\geq 1)$ which is not a zero of $g-a$. Let z_1 be a pole of f and g with respective multiplicities q and r. Since z_1 is a zero of $f-a$, it is also a pole of $a = a(z)$ with multiplicity q. Then we see that z_1 is a zero of $\dfrac{f'(f-a)}{f(f-1)}$ with multiplicity $p+q-1$.

Suppose $q < r$. Then z_1 is a pole of $g-a$ with multiplicity r. So z_1 is a simple pole of $\dfrac{g'(g-a)}{g(g-1)}$, which is impossible by (3.1).

Suppose $q > r$. Then z_1 is a pole of $g-a$ with multiplicity q. So z_1 is a pole of $\dfrac{g'(g-a)}{g(g-1)}$ with multiplicity $q+1-r$, which is impossible by (3.1).

Suppose $q = r$. Let z_1 be a pole of $g-a$ with multiplicity $s(\leq r)$. If $s < r$ then z_1 is a zero of $\dfrac{g'(g-a)}{g(g-1)}$ with multiplicity $q-s-1$. If $s = r-1$, then z_1 is a regular point of $\dfrac{g'(g-a)}{g(g-1)}$ but not a zero of it, which is impossible by (3.1). If $s < r-1$, then from (3.1) we see that $q-s-1 = p+q-1$ and so $p+s = 0$, which is impossible. If $s = r$, then z_1 is a simple pole of $\dfrac{g'(g-a)}{g(g-1)}$, which is impossible by

(3.1). Suppose that z_1 is a regular point of $g-a$ such that $g(z_1)-a(z_1)\neq 0$. Then $q=r$ and z_1 is a zero of $\dfrac{g'(g-a)}{g(g-1)}$ with multiplicity $q-1$. So from (3.1) we get $p+q-1=q-1$, which is impossible.

Let $N_1(r,a;f\mid g\neq a)$ and $N_*(r,a;f\mid g\neq a)$ respectively denote the counting functions of the simple and multiple zeros of $f-a$ which are not the zeros of $g-a$, where each zero is counted according to its multiplicity.

Noting that a is a small function of f and g, we get from (3.1) that a common zero of $f-a$ and $f(f-1)$ is a zero of $g-a$. This together with Lemma 2.5 and Lemma 2.6 (vi) gives

$$N_1(r,a;f\mid g\neq a)=N(r,0;g'\mid\leq 1)+S(r) \tag{3.3}$$

and

$$N_*(r,a;f\mid g\neq a)\leq N_0(r,0;g'\mid\geq 2)+S(r)=S(r). \tag{3.4}$$

We now denote by $N_2(r,a;f,g)$ the counting function of common double zeros of $f-a$ and $g-a$, where zeros are counted according to multiplicity. Since $f-a=a(F-1)$ and $g-a=a(G-1)$, we get

$$N_2(r,a;f,g)\leq 2N(r,0;\phi_1)+2N(r,0;a)+2N(r,\infty;a)=S(r). \tag{3.5}$$

Now from (3.2), (3.3), (3.4) and (3.5) we get

$$N(r,a;f\mid\leq 2)=N(r,a;f,g\mid\leq 1)+N(r,0;g'\mid\leq 1)+S(r). \tag{3.6}$$

Since $\overline{N}(r,a;f,g)\leq\overline{N}(r,a;f)$ and $\overline{N}(r,a;f,g)\neq S(r,f)$, by Lemma 2.7 we obtain

$$T(r,f)=N(r,a;f\mid\leq 2)+S(r). \tag{3.7}$$

Now from (3.6), (3.7) and Lemma 2.6 (ii) we get

$$N(r,a;f,g\mid\leq 1)=N_0(r)+S(r). \tag{3.8}$$

From (3.8) we see by Lemma 2.6 (iv) that

$$N_0(r)+S(r)=N(r,a;f,g\mid\leq 1)\leq\overline{N}(r,a;f,g)\leq N_0(r)+S(r)$$

and so

$$\overline{N}(r,a;f,g)=N_0(r)+S(r). \tag{3.9}$$

Since by Lemma 2.6 (iv) $N(r)=N_0(r)+N_1(r)=N_0(r)+S(r)$, from (3.8) and (3.9) we get

$$N(r,a;f,g\mid\leq 1)=N(r)+S(r) \tag{3.10}$$

and

$$\overline{N}(r,a;f,g) = N(r) + S(r). \tag{3.11}$$

From (3.11) and the hypotheses of the theorem we see that $N(r) \neq S(r)$. We now consider the following two cases.

Case I. Let $T(r,f) = N(r) + S(r)$. Then by Lemma 2.2 and Lemma 2.3 we see that f and g satisfy one of the following relations : $f+g \equiv 1$, $(f-1)(g-1) \equiv 1$ and $fg \equiv 1$.

Let $f+g \equiv 1$. Then f and g do not assume the values 0 and 1. So there exists a non-constant entire function γ such that $f = \dfrac{1}{1+e^{\gamma}}$ and $g = \dfrac{1}{1+e^{-\gamma}}$, which is the possibility (i) of the theorem for $p = 1$. Also in this case we get from (3.1) that $a \equiv \dfrac{1}{2}$.

Let $(f-1)(g-1) \equiv 1$. Then f and g do not assume the values 1 and ∞ . Hence there exists a non-constant entire function γ such that $f = 1 + e^{\gamma}$ and $g = 1 + e^{-\gamma}$, which is the possibility (ii) of the theorem for $p = 1$. Also in this case we get from (3.1) that $a \equiv 2$.

Let $fg \equiv 1$. Then f and g do not assume the values 0 and ∞. Hence there exists a non-constant entire function γ such that $f = -e^{\gamma}$ and $g = -e^{-\gamma}$, which is the possibility (iii) of the theorem for $p = 1$. Also in this case we get from (3.1) that $a \equiv -1$.

Case II. Let $T(r,f) \neq N(r) + S(r)$. Then by Lemma 2.2, Lemma 2.4 and (3.10) we see that f and g assume one of the forms (i), (ii) and (iii) of the theorem and

$$N(r,a;f,g \mid \leq 1) = N(r) + S(r,f) = \frac{1}{p}T(r,f) + S(r,f).$$

If f and g assume the form (i), then $\dfrac{f'}{f} - \dfrac{g'}{g} \equiv (s-p-1)\gamma'$ and $\dfrac{f'}{f-1} - \dfrac{g'}{g-1} \equiv s\gamma'$. Since $\gamma' \not\equiv 0$, from (3.1) we get $a \equiv \dfrac{s}{p+1}$. Similarly if f and g assume the forms (ii) and (iii), then we respectively obtain $a \equiv \dfrac{p+1}{p+1-s}$ and $a \equiv \dfrac{s}{s-p-1}$. This proves the theorem. □

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