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Weighted Value Sharing and Uniqueness of Entire Functions

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Abstract. In the paper, we study with weighted sharing method the uniqueness of entire functions concerning nonlinear differential polynomials sharing one value and prove two uniqueness theorems, first one of which generalizes some recent results in [10] and [16]. Our second theorem will supplement a result in [17].

1. Introduction, definitions and results

In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. We adopt the standard notations in the Nevanlinna theory of meromorphic functions as explained in [5], [13] and [14]. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a nonconstant meromorphic function h, we denote by $T(r,h)$ the Nevanlinna characteristic of h and by $S(r,h)$ any quantity satisfying $S(r,h) = o\{T(r,h)\}(r \to \infty, r \notin E)$.

Let f and g be two nonconstant meromorphic functions, and let a be a finite value. We say that f and g share the value a CM, provided that $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share a IM, provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities. In addition, we say that f and g share ∞ CM, if $\frac{1}{f}$ and $\frac{1}{g}$ share 0 CM, and we say that f and g share ∞ IM, if $\frac{1}{f}$ and $\frac{1}{g}$ share 0 IM (see[14]). Throughout this paper, we need the following definition.

$$\Theta(a,f) = 1 - \limsup_{r \longrightarrow \infty} \frac{\overline{N}(r,a;f)}{T(r,f)},$$

for any $a \in C \cup \{\infty\}$.

Corresponding to one famous question of Hayman [4], Fang and Hua [1], Yang and Hua [12] obtained the following theorem.

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Theorem A. *Let f and g be two nonconstant entire functions, $n \geq 6$ be a positive integer. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(c_1c_2)^{n+1}c^2 = -1$ or $f \equiv tg$ for a constant t such that $t^{n+1} = 1$.*

Considering kth derivative instead of 1st derivative Fang [2] proved the following theorems.

Theorem B. *Let f and g be two nonconstant entire functions, and let n, k be two positive integers with $n > 2k + 4$. If $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$ or $f \equiv tg$ for a constant t such that $t^n = 1$.*

Theorem C. *Let f and g be two nonconstant entire functions, and let n, k be two positive integers with $n \geq 2k + 8$. If $[f^n(f-1)]^{(k)}$ and $[g^n(g-1)]^{(k)}$ share 1 CM, then $f \equiv g$.*

Now the following question arises:

Is Theorem B and Theorem C hold for some general differential polynomials like $[f^n(f^m - a)]^{(k)}$ or $[f^n(f-1)^m]^{(k)}$?

X. Y. Zhang and W. C. Lin [17] answered the above question and proved the following theorems.

Theorem D. *Let f and g be two nonconstant entire functions, and let n, m and k be three positive integers with $n \geq 2k + m^* + 4$, and λ, μ be constants such that $|\lambda| + |\mu| \neq 0$. If $[f^n(\mu f^m + \lambda)]^{(k)}$ and $[g^n(\mu g^m + \lambda)]^{(k)}$ share 1 CM, then one of the following holds:*
(i) If $\lambda\mu \neq 0$, then $f \equiv g$.
(ii) If $\lambda\mu = 0$, then either $f \equiv tg$, where t is a constant satisfying $t^{n+m^*} = 1$ or $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying

$$(-1)^k \lambda^2 (c_1c_2)^{n+m^*}[(n+m^*)c]^{2k} = 1$$

or

$$(-1)^k \mu^2 (c_1c_2)^{n+m^*}[(n+m^*)c]^{2k} = 1$$

and m^* is defined by $m^* = \chi_\mu m$, where

$$\chi_\mu = \begin{cases} 0 & \text{if } \mu = 0 \\ 1 & \text{if } \mu \neq 0. \end{cases}$$

Theorem E. *Let f and g be two nonconstant entire functions, and let n, m and k be three positive integers with $n > 2k + m + 4$. If $[f^n(f-1)^m]^{(k)}$ and $[g^n(g-1)^m]^{(k)}$ share 1 CM, then either $f \equiv g$ or f and g satisfy the algebraic equation $R(f, g) = 0$, where $R(w_1, w_2) = w_1^n(w_1 - 1)^m - w_2^n(w_2 - 1)^m$.*

To state the next results we need the following definition known as weighted sharing of values introduced by I. Lahiri [7, 8] which measures how close a shared value is to be shared IM or to be shared CM.

Definition 1. Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and k+1 times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k, then z_0 is an a-point of f with multiplicity $m(\leq k)$ if and only if it is an a-point of g with multiplicity $m(\leq k)$ and z_0 is an a-point of f with multiplicity $m(> k)$ if and only if it is an a-point of g with multiplicity $n(> k)$, where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Using the idea of weighted sharing of values, T. Zhang and W. Lu [16] proved the following theorem for entire functions.

Theorem F. *Let f and g be two nonconstant transcendental entire functions, and let $n(\geq 1)$, $k(\geq 1)$, $l(\geq 0)$ be three integers. Suppose that $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share $(1, l)$, if $l \geq 2$ and $n > 2k + 4$ or if $l = 1$ and $n > 3k + 6$ or $l = 0$ and $n > 5k + 7$, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k (c_1 c_2)^n (nc)^{2k} = 1$ or $f \equiv tg$ for some nth root of unity t such that $t^n = 1$.*

Recently L. Liu [10] proved the following theorem which improve Theorem E.

Theorem G. *Let f and g be two nonconstant entire functions, and let n, m and k be three positive integers such that $n > 5k + 4m + 9$. If $[f^n(f-1)^m]^{(k)}$ and $[g^n(g-1)^m]^{(k)}$ share 1 IM, then either $f \equiv g$ or f and g satisfy the algebraic equation $R(f, g) = 0$, where $R(w_1, w_2) = w_1^n(w_1 - 1)^m - w_2^n(w_2 - 1)^m$.*

Regarding Theorems F and G, it is natural to ask the following questions.

Question 1. What can be said about the relation between two nonconstant entire functions f and g, if $\{f^n(f-1)^m\}^{(k)}$ and $\{g^n(g-1)^m\}^{(k)}$ share $(1, l)$ for some $l(\geq 0)$?

Question 2. What can be said about the relation between two nonconstant entire functions f and g, if $\{f^n(f^m - a)\}^{(k)}$ and $\{g^n(g^m - a)\}^{(k)}$ share $(1, l)$ for some $l(\geq 0)$?

In this paper, we prove the following two theorems, first one of which will not only provide a supplementary result of Theorem E, also improve and generalize Theorems F and G. Our second theorem will provide a supplementary result of Theorem D. Moreover, Theorem 1 and Theorem 2 deal with Question 1 and Question 2 respectively.

Theorem 1. *Let f and g be two nonconstant entire functions, and let $n(\geq 1)$, $m(\geq 0)$, $k(\geq 1)$ and $l(\geq 0)$ be four integers. Let $[f^n(f-1)^m]^{(k)}$ and $[g^n(g-1)^m]^{(k)}$ share $(1,l)$. Then*
(i) if $m=0$, then conclusion of Theorem F holds provided one of $l \geq 2$, $n > 2k+4$ or $l=1$, $n > 3k+5$ or $l=0$, $n > 5k+7$ holds;
(ii) if $m \geq 1$, then conclusion of Theorem G holds provided one of $l \geq 2$, $n > 2k+m+6$ or $l=1$, $n > 3k+2m+7$ or $l=0$, $n > 5k+4m+9$ holds.

Theorem 2. *Let f and g be two nonconstant entire functions, and let n, m, k and $l(\geq 0)$ be four positive integers. Let $[f^n(\mu f^m+\lambda)]^{(k)}$ and $[g^n(\mu g^m+\lambda)]^{(k)}$ share $(1,l)$ where λ, μ are constants such that $|\lambda|+|\mu| \neq 0$. Then conclusions (i) and (ii) of Theorem D hold respectively for*
(i) $l \geq 2$, $n > 2k+3m+4$ or $l=1$, $n > 3k+4m+5$ or $l=0$, $n > 5k+6m+7$; and
(ii) $l \geq 2$, $n > 2k-m^*+4$ or $l=1$, $n > 3k-m^*+5$ or $l=0$, $n > 5k-m^*+7$.

Remark 1. Since Theorems F and G can be obtained as special cases of Theorem 1, Theorem 1 improves Theorems F and G.

Though the standard definitions and notations of the value distribution theory are available in [5], we explain some definitions and notations which are used in the paper.

Definition 2([6]). For $a \in C \cup \{\infty\}$ we denote by $N(r,a;f \mid = 1)$ the counting functions of simple a-points of f. For a positive integer p we denote by $N(r,a;f \mid \leq p)$ ($N(r,a;f \mid \geq p)$) the counting function of those a-points of f whose multiplicities are not greater (less) than p, where each a-point is counted according to its multiplicity.

$\overline{N}(r,a;f \mid \leq p)$ and $\overline{N}(r,a;f \mid \geq p)$ are defined similarly, where in counting the a-points of f we ignore the multiplicities. Also $N(r,a;f \mid < p)$ and $N(r,a;f \mid > p)$ are defined analogously.

Definition 3([9]). Let p be a positive integer or infinity. We denote by $N_p(r,a;f)$ the counting function of a-points of f, where an a-point of multiplicity m is counted m times if $m \leq p$ and p times if $m > p$. Then

$$N_p(r,a;f) = \overline{N}(r,a;f) + \overline{N}(r,a;f \mid \geq 2) + ... + \overline{N}(r,a;f \mid \geq p).$$

Definition 4. Let f and g be two nonconstant meromorphic functions such that f and g share the value 1 IM. We denote by $N_{11}(r,1;f)$ the counting function for common simple 1-points of f and g where multiplicity is not counted.

Definition 5. Let f and g be two nonconstant meromorphic functions such that f and g share the value 1 IM. We denote by $N_{22}(r,1;f)$ the counting function of those same multiplicity 1-points of f and g where the multiplicity is ≥ 2.

Definition 6. Let f and g be two nonconstant meromorphic functions such that f

and g share the value 1 IM. Let z_0 be a 1-point of f with multiplicity p, a 1-point of g with multiplicity q. We denote by $\overline{N}_L(r,1;f)$ the counting function of those 1-points of f and g where $p>q$, with multiplicity being not counted. $\overline{N}_L(r,1;g)$ is defined analogously.

Definition 7. For $a \in C \cup \{\infty\}$ we put

$$\delta_k(a,f) = 1 - \limsup_{r\to\infty} \frac{N_k(r,a;f)}{T(r,f)}.$$

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1([11]). *Let f be a nonconstant meromorphic function and let $a_n(z)(\not\equiv 0)$, $a_{n-1}(z)$, ... , $a_0(z)$ be meromorphic functions such that $T(r,a_i(z)) = S(r,f)$ for $i = 0,1,2,...,n$. Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + ... + a_1 f + a_0) = nT(r,f) + S(r,f).$$

Lemma 2([5, 13])**.** *Let f be a transcendental entire function, and let k be a positive integer. Then for any non-zero finite complex number c*

$$\begin{aligned} T(r,f) &\leq N(r,0;f) + N\left(r,c;f^{(k)}\right) - N\left(r,0;f^{(k+1)}\right) + S(r,f) \\ &\leq N_{k+1}(r,0;f) + \overline{N}\left(r,c;f^{(k)}\right) - N_0\left(r,0;f^{(k+1)}\right) + S(r,f), \end{aligned}$$

where $N_0\left(r,0;f^{(k+1)}\right)$ denotes the counting function which only counts those points such that $f^{(k+1)} = 0$ but $f\left(f^{(k)} - c\right) \neq 0$.

Lemma 3([15]). *Let f be a nonconstant meromorphic function and p, k be positive integers, then*

$$N_p\left(r,0;f^{(k)}\right) \leq N_{p+k}(r,0;f) + k\overline{N}(r,\infty;f) + S(r,f).$$

Lemma 4([5, 13])**.** *Let f be a transcendental meromorphic function, and let $a_1(z)$, $a_2(z)$ be two distinct meromorphic functions such that $T(r,a_i(z)) = S(r,f)$, i=1, 2. Then*

$$T(r,f) \leq \overline{N}(r,\infty;f) + \overline{N}(r,a_1;f) + \overline{N}(r,a_2;f) + S(r,f).$$

Lemma 5([15]). *Let f and g be two nonconstant entire functions, and let $k(\geq 1)$, $l(\geq 0)$ be integers. Suppose that $f^{(k)}$ and $g^{(k)}$ share $(1,l)$ and one of*

(i) $l \geq 2$ *and* $\Delta_1 = \Theta(0, f) + \Theta(0, g) + \delta_{k+1}(0, f) + \delta_{k+1}(0, g) > 3$;
(ii) $l = 1$ *and* $\Delta_2 = \Theta(0, f) + \Theta(0, g) + 2\delta_{k+1}(0, f) + \delta_{k+1}(0, g) > 4$;
(iii) $l = 0$ *and* $\Delta_3 = \Theta(0, f) + \Theta(0, g) + 3\delta_{k+1}(0, f) + 2\delta_{k+1}(0, g) > 6$,
holds, then either $f^{(k)}g^{(k)} \equiv 1$ *or* $f^{(k)} = c_1 g^{(k)} + 1 - c_1$*, where* $c_1 (\neq 0)$ *is a constant.*

Proof. Let

$$H(z) = \frac{F''}{F'} - \frac{2F'}{F-1} - \frac{G''}{G'} + \frac{2G'}{G-1}. \tag{2.1}$$

where $F \equiv f^{(k)}$ and $G \equiv g^{(k)}$. It is obvious from (2.1) that if z_0 is a common simple 1-point of F and G, then it is a zero of H. Thus

$$\begin{aligned} N_{11}(r, 1; F) = N_{11}(r, 1; G) &\leq \overline{N}(r, 0; H) \leq T(r, H) + O(1) \\ &\leq N(r, \infty; H) + S(r, f) + S(r, g). \end{aligned} \tag{2.2}$$

By the assumptions $H(z)$ have poles only at zeros of F' and G' and 1-points of F whose multiplicities are not equal to the multiplicities of the corresponding 1-points of G. So

$$\begin{aligned} N(r, \infty; H) \leq \overline{N}(r, 0; f) + \overline{N}(r, 0; g) + \overline{N}_L(r, 1; F) + \overline{N}_L(r, 1; G) \\ + N_0(r, 0; F') + N_0(r, 0; G'), \end{aligned} \tag{2.3}$$

where $N_0(r, 0; F')$ and $N_0(r, 0; G')$ has the same meaning as in Lemma 2. By Lemma 2, we have

$$T(r, f) \leq N_{k+1}(r, 0; f) + \overline{N}(r, 1; F) - N_0(r, 0; F') + S(r, f), \tag{2.4}$$

and

$$T(r, g) \leq N_{k+1}(r, 0; g) + \overline{N}(r, 1; G) - N_0(r, 0; G') + S(r, g). \tag{2.5}$$

Since F and G share 1 IM, we have

$$\begin{aligned} &\overline{N}(r, 1; F) + \overline{N}(r, 1; G) \\ &= 2N_{11}(r, 1; F) + 2\overline{N}_L(r, 1; F) + 2\overline{N}_L(r, 1; G) + 2\overline{N}_{(22}(r, 1; F). \end{aligned} \tag{2.6}$$

By (2.2) and (2.3), we have

$$\begin{aligned} &\overline{N}(r, 1; F) + \overline{N}(r, 1; G) \\ &\leq N_{11}(r, 1; F) + 3\overline{N}_L(r, 1; F) + 3\overline{N}_L(r, 1; G) + 2\overline{N}_{(22}(r, 1; F) \\ &\quad + \overline{N}(r, 0; f) + \overline{N}(r, 0; g) + N_0(r, 0; F') + N_0(r, 0; G') \\ &\quad + S(r, f) + S(r, g). \end{aligned} \tag{2.7}$$

We consider following three cases:

Case 1. Let $l \geq 2$. Since g is an entire function, we have

$$\begin{aligned}(2.8)\qquad & N_{11}(r,1;F)+2\overline{N}_{(22}(r,1;F)+3\overline{N}_L(r,1;F)+3\overline{N}_L(r,1;G)\\ &\leq N(r,1;G)+S(r,f)+S(r,g)\\ &\leq T(r,G)+S(r,f)+S(r,g)\\ &\leq T(r,g)+S(r,f)+S(r,g).\end{aligned}$$

From (2.4), (2.5), (2.7) and (2.8), we obtain

$$\begin{aligned}T(r,f) \quad \leq \quad & \overline{N}(r,0;f)+\overline{N}(r,0;g)+N_{k+1}(r,0;f)+N_{k+1}(r,0;g)\\ &+S(r,f)+S(r,g).\end{aligned}$$

We suppose that there exists a set I of infinite measure such that $T(r,g) \leq T(r,f)$ for $r \in I$. Then for $r \in I$,

$$\begin{aligned}T(r,f) \leq \{[4-\Theta(0,f)-\Theta(0,g)-\delta_{k+1}(0,f)\\ -\delta_{k+1}(0,g)]+\epsilon\}T(r,f)+S(r,f),\end{aligned}$$

$0<\epsilon<\Delta_1-3$. From this we get

$$T(r,f) \leq S(r,f)$$

for $r \in I$, which is a contradiction.

Case 2. Let $l=1$. Then

$$\begin{aligned}(2.9)\qquad & N_{11}(r,1;F)+2\overline{N}_{(22}(r,1;F)+2\overline{N}_L(r,1;F)+3\overline{N}_L(r,1;G)\\ &\leq N(r,1;G)+S(r,f)+S(r,g)\\ &\leq T(r,G)+S(r,f)+S(r,g)\\ &\leq T(r,g)+S(r,f)+S(r,g).\end{aligned}$$

By Lemma 3 and that f is an entire function, we have

$$\begin{aligned}(2.10)\qquad \overline{N}_L(r,1;F) \quad &\leq \quad N(r,1;F)-\overline{N}(r,1;F)\\ &\leq \quad N\left(r,\infty;\frac{F}{F'}\right)\\ &\leq \quad N\left(r,\infty;\frac{F'}{F}\right)+S(r,f)\\ &\leq \quad \overline{N}(r,0;F)+S(r,f)\\ &\leq \quad N_{k+1}(r,0;f).\end{aligned}$$

From (2.4), (2.5), (2.7), (2.9) and (2.10), we obtain

$$\begin{aligned}T(r,f) \quad \leq \quad & \overline{N}(r,0;f)+\overline{N}(r,0;g)+2N_{k+1}(r,0;f)+N_{k+1}(r,0;g)\\ &+S(r,f)+S(r,g).\end{aligned}$$

We suppose that there exists a set I of infinite measure such that $T(r,g) \leq T(r,f)$, $r \in I$. Then for $r \in I$,

$$\begin{aligned} T(r,f) \leq \{[5 - \Theta(0,f) - \Theta(0,g) - 2\delta_{k+1}(0,f) \\ -\delta_{k+1}(0,g)] + \epsilon\}T(r,f) + S(r,f), \end{aligned}$$

$0 < \epsilon < \Delta_2 - 4$. From this we get

$$T(r,f) \leq S(r,f)$$

for $r \in I$, which is a contradiction.

Case 3. Let $l = 0$. Then

$$\begin{aligned} N_{11}(r,1;F) + 2\overline{N}_{(22}(r,1;F) + \overline{N}_L(r,1;F) + 2\overline{N}_L(r,1;G) \\ \leq N(r,1;G) + S(r,f) + S(r,g) \\ \leq T(r,G) + S(r,f) + S(r,g) \\ \leq T(r,g) + S(r,f) + S(r,g). \end{aligned} \tag{2.11}$$

From (2.4), (2.5), (2.7), (2.10) and (2.11), we obtain

$$\begin{aligned} T(r,f) \leq\ & \overline{N}(r,0;f) + \overline{N}(r,0;g) + 3N_{k+1}(r,0;f) + 2N_{k+1}(r,0;g) \\ & +S(r,f) + S(r,g). \end{aligned}$$

We suppose that there exists a set I of infinite measure such that $T(r,g) \leq T(r,f)$, $r \in I$. Then for $r \in I$,

$$\begin{aligned} T(r,f) \leq \{[7 - \Theta(0,f) - \Theta(0,g) - 3\delta_{k+1}(0,f) \\ -2\delta_{k+1}(0,g)] + \epsilon\}T(r,f) + S(r,f), \end{aligned}$$

$0 < \epsilon < \Delta_3 - 6$. From this we get

$$T(r,f) \leq S(r,f)$$

for $r \in I$, which is a contradiction.
Hence in all the cases $H(z) \equiv 0$, that is

$$\frac{F''}{F'} - \frac{2F'}{F-1} \equiv \frac{G''}{G'} - \frac{2G'}{G-1}.$$

By integrating two sides of the above equality twice we get

$$\frac{1}{F-1} \equiv \frac{BG + A - B}{G-1}, \tag{2.12}$$

where $A(\neq 0)$ and B are constants.
We consider the case when $l \geq 2$. The case $l = 1$ and $l = 0$ are similar. Now we

consider the following three subcases.

Subcase I. Let $B \neq 0$ and $A = B$.
If $B = -1$, we obtain by (2.12) $FG \equiv 1$.
If $B \neq -1$, from (2.12) we get

$$\frac{1}{F} \equiv \frac{BG}{(1+B)G - 1}.$$

So by Lemma 3 we have

$$\overline{N}\left(r, \frac{1}{1+B}; G\right) \leq \overline{N}(r, 0; F) \leq N_{k+1}(r, 0; f) + S(r, f).$$

By Lemma 2 we obtain

$$\begin{aligned} T(r,g) &\leq N_{k+1}(r,0;g) + \overline{N}\left(r, \frac{1}{1+B}; G\right) - N_0(r,0;G') + S(r,g) \\ &\leq N_{k+1}(r,0;g) + N_{k+1}(r,0;f) + S(r,f) + S(r,g) \\ &\leq \overline{N}(r,0;f) + \overline{N}(r,0;g) + N_{k+1}(r,0;f) + N_{k+1}(r,0;g) \\ &\quad + S(r,f) + S(r,g) \\ &\leq (4 - \Delta_1)T(r,g) + S(r,g). \end{aligned}$$

Thus we obtain

$$(\Delta_1 - 3)T(r,g) \leq S(r,g),$$

$r \in I$, which is a contradiction.

Subcase II. Let $B \neq 0$ and $A \neq B$.
If $B = -1$, from (2.12) we obtain

$$F \equiv \frac{A}{-[G - (a+1)]}.$$

If $B \neq -1$, then we get from (2.12) that

$$F - \left(1 + \frac{1}{B}\right) \equiv \frac{-A}{B^2\left(G + \frac{A-B}{B}\right)}.$$

Since f is an entire function, by Lemma 2, Lemma 3 and by using the same argument as in subcase I, we get a contradiction in both cases.

Subcase III. Let $B = 0$ and $A \neq 0$. Then we obtain from (2.12) that

$$f^{(k)} = \frac{1}{A} g^{(k)} + 1 - \frac{1}{A}. \tag{2.13}$$

This proves the lemma. □

Lemma 6([3]). *Let $f(z)$ be a nonconstant entire function, and let $k \geq 2$ be a positive integer. If $f(z)f^{(k)}(z) \neq 0$, then $f(z) = e^{az+b}$, where $a \neq 0$, b are constants.*

3. Proofs of the Theorems

Proof of Theorem 1. We consider $F(z) = f^n(f-1)^m$ and $G(z) = g^n(g-1)^m$. Then by using Lemma 1, we get

$$\begin{aligned}
\Theta(0, F) &= 1 - \limsup_{r \longrightarrow \infty} \frac{\overline{N}(r, 0; F)}{T(r, F)} \\
&= 1 - \limsup_{r \longrightarrow \infty} \frac{\overline{N}(r, 0; f^n(f-1)^m)}{(n+m)T(r, f)} \\
&\geq 1 - \limsup_{r \longrightarrow \infty} \frac{(1+m^{**})T(r, f)}{(n+m)T(r, f)} \\
&\geq \frac{n+m-1-m^{**}}{n+m},
\end{aligned} \tag{3.1}$$

where

$$m^{**} = \begin{cases} 0 & \text{if } m = 0 \\ 1 & \text{if } m \geq 1. \end{cases}$$

Similarly

$$\Theta(0, G) \geq \frac{n+m-1-m^{**}}{n+m}. \tag{3.2}$$

$$\begin{aligned}
\delta_{k+1}(0, F) &= 1 - \limsup_{r \longrightarrow \infty} \frac{N_{k+1}(r, 0; F)}{T(r, F)} \\
&= 1 - \limsup_{r \longrightarrow \infty} \frac{N_{k+1}(r, 0; f^n(f-1)^m)}{(n+m)T(r, f)} \\
&\geq 1 - \limsup_{r \longrightarrow \infty} \frac{(k+m+1)T(r, f)}{(n+m)T(r, f)} \\
&\geq \frac{n-k-1}{n+m}.
\end{aligned} \tag{3.3}$$

Similarly

$$\delta_{k+1}(0, G) \geq \frac{n-k-1}{n+m}. \tag{3.4}$$

Since $F^{(k)}$ and $G^{(k)}$ share $(1, l)$, we discuss the following three cases:

Case 1. $l \geq 2$. From (3.1)-(3.4), we obtain $\Delta_1 > 3$ provided $n > 2k+m+2m^{**}+4$. Since

$$2k + m + 2m^{**} + 4 = \begin{cases} 2k + 4 & \text{if } m = 0 \\ 2k + m + 6 & \text{if } m \geq 1, \end{cases}$$

by Lemma 5(i) we have either $F^{(k)}G^{(k)} \equiv 1$ or

$$F^{(k)} = \frac{1}{A}G^{(k)} + 1 - \frac{1}{A}. \tag{3.5}$$

Case 2. $l = 1$. From (3.1)-(3.4), it is obvious that $\Delta_2 > 4$ provided $n > 3k + 2m + 2m^{**} + 5$. Since

$$3k + 2m + 2m^{**} + 5 = \begin{cases} 3k + 5 & \text{if } m = 0 \\ 3k + 2m + 7 & \text{if } m \geq 1, \end{cases}$$

by Lemma 5(ii) we have either $F^{(k)}G^{(k)} \equiv 1$ or (3.5).

Case 3. $l = 0$. Similarly as above, $\Delta_3 > 6$ provided $n > 5k+4m+2m^{**}+7$. Since

$$5k + 4m + 2m^{**} + 7 = \begin{cases} 5k + 7 & \text{if } m = 0 \\ 5k + 4m + 9 & \text{if } m \geq 1, \end{cases}$$

by Lemma 5(iii) we have either $F^{(k)}G^{(k)} \equiv 1$ or (3.5).

Let

$$F^{(k)}G^{(k)} \equiv 1.$$

i.e.,

$$[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)} \equiv 1. \tag{3.6}$$

Then we consider following two subcases.

Subcase(I) Let $m = 0$. Then

$$[f^n]^{(k)}[g^n]^{(k)} \equiv 1. \tag{3.7}$$

By the nature of f and g it is clear from above that $f \neq 0$, $g \neq 0$. And so $[f^n]^{(k)} \neq 0$ and $[g^n]^{(k)} \neq 0$. If $k \geq 2$, then by Lemma 6 we obtain that $f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying

$$(-1)^k(c_1c_2)^n(nc)^{2k} = 1.$$

Suppose that $k = 1$. Let $f(z) = e^{\alpha(z)}$, $g(z) = e^{\beta(z)}$ where $\alpha(z)$ and $\beta(z)$ are two entire functions. So from (3.7) we have

$$n^2\alpha'\beta'e^{n(\alpha+\beta)} \equiv 1. \tag{3.8}$$

Thus α' and β' have no zeros and we may take $\alpha' = e^{\gamma(z)}$ and $\beta' = e^{\delta(z)}$, where γ and δ are two entire functions. So from (3.8) we get

$$n^2 e^{n(\alpha+\beta)+\gamma+\delta} \equiv 1.$$

Differentiating we get

$$ne^{\gamma} + \gamma' \equiv -(ne^{\delta} + \delta'). \tag{3.9}$$

Since γ and δ are entire, we have $T(r, \gamma') = S(r, e^{\gamma})$ and $T(r, \delta') = S(r, e^{\delta})$. From this we have

$$T(r, e^{\gamma}) = T(r, e^{\delta}) + S(r, e^{\gamma}) + S(r, e^{\delta}).$$

This implies that $S(r, e^{\gamma}) = S(r, e^{\delta}) = S(r)$, say.
Let $\rho = -(\gamma' + \delta')$. Then $T(r, \rho) = S(r)$. If $\rho \not\equiv 0$, (3.9) can be written as

$$\frac{e^{\gamma}}{\rho} + \frac{e^{\delta}}{\rho} \equiv \frac{1}{n}.$$

From this and second fundamental theorem of Nevanlinna, we get

$$\begin{aligned} T(r, e^{\gamma}) &\leq T\left(r, \frac{e^{\delta}}{\rho}\right) + S(r) \\ &\leq \overline{N}\left(r, \infty; \frac{e^{\delta}}{\rho}\right) + \overline{N}\left(r, 0; \frac{e^{\delta}}{\rho}\right) + \overline{N}\left(r, \frac{1}{n}; \frac{e^{\delta}}{\rho}\right) + S(r) \\ &\leq S(r), \end{aligned}$$

a contradiction. So by (3.9) we have

$$\alpha' + \beta' = e^{\gamma} + e^{\delta} = -\left(\frac{\gamma'}{n} + \frac{\delta'}{n}\right) \equiv 0$$

i.e., $\gamma = \delta + (2s+1)\pi i$ for some integer s. Again $\gamma' + \delta' \equiv 0$ implies $\gamma + \delta = d$, where d is a constant. Taking $\gamma = d_1$ we get $\delta = d - d_1 = d_2$, where d_1, d_2 are constants. Again $\alpha' + \beta' \equiv 0$ implies $\alpha = cz + logc_1$ and $\beta = -cz + logc_2$. Since $f = e^{\alpha}$ and $g = e^{\beta}$, by (3.8) we obtain that $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying

$$(-1)^k (c_1 c_2)^n (nc)^{2k} = 1.$$

Subcase(II) Let $m \geq 1$. Since f and g are entire functions, we have $f \neq 0$ and $g \neq 0$. Let $f(z) = e^{\alpha(z)}$, where $\alpha(z)$ is a nonconstant entire function. Clearly

$$[f^{n+m}(z)]^{(k)} = s_m(\alpha', \alpha'', ..., \alpha^{(k)}) e^{(n+m)\alpha(z)}. \tag{3.10}$$

.

$$(3.11) \qquad (-1)^{m-i}[{}^{m}C_i f^{n+i}(z)]^{(k)} = s_i(\alpha', \alpha'', ..., \alpha^{(k)})e^{(n+i)\alpha(z)}.$$

.

$$(3.12) \qquad (-1)^{m}[f^{n}(z)]^{(k)} = s_0(\alpha', \alpha'', ..., \alpha^{(k)})e^{n\alpha(z)}.$$

where $s_i(\alpha', \alpha'', ..., \alpha^{(k)})$ $(i = 0, 1, 2, ..., m)$ are differential polynomials. Obviously

$$s_i(\alpha', \alpha'', ..., \alpha^{(k)}) \not\equiv 0$$

for $i = 0, 1, 2, ..., m$, and

$$[f^n(f-1)^m]^{(k)} \neq 0.$$

From (3.10) and (3.12) we have

$$(3.13) \qquad s_m(\alpha', \alpha'', ..., \alpha^{(k)})e^{m\alpha(z)} + ... + s_0(\alpha', \alpha'', ..., \alpha^{(k)}) \neq 0.$$

Since $\alpha(z)$ is an entire function, we obtain $T(r, \alpha') = S(r, f)$ and $T(r, \alpha^{(j)}) = S(r, f)$ for $j = 1, 2, ..., k$. Hence $T(r, s_i) = S(r, f)$ for $i = 0, 1, 2, ..., m$. So from (3.13), Lemmas 1 and 4 we obtain

$$\begin{aligned} mT(r,f) &= T(r, s_m e^{m\alpha} + ... s_1 e^{\alpha}) + S(r,f) \\ &\leq \overline{N}(r, 0; s_m e^{m\alpha} + ... + s_1 e^{\alpha}) + \overline{N}(r, 0; s_m e^{m\alpha} + ... + s_1 e^{\alpha} + s_0) + S(r,f) \\ &\leq \overline{N}(r, 0; s_m e^{(m-1)\alpha} + ... + s_1) + S(r,f) \\ &\leq (m-1)T(r,f) + S(r,f), \end{aligned}$$

which is a contradiction.

We now suppose that (3.5) holds. Then

$$(3.14) \qquad F = \frac{1}{A}G + P(z),$$

where $P(z)$ is a polynomial of degree atmost k. By the assumptions, we know that either both f and g are transcendental entire function or both f and g are polynomials. First we consider the case when f and g are transcendental entire functions. Then it follows from (3.14) and Lemma 1 that

$$(3.15) \qquad T(r,f) = T(r,g) + S(r,f).$$

If $P(z) \not\equiv 0$, then by (3.14) and (3.15) and Lemma 4 we obtain

$$\begin{aligned} (n+m)T(r,f) &= T(r,F) + O(1) \\ &\leq \overline{N}(r,0;F) + \overline{N}(r,0;G) + S(r,f) \\ &\leq 4T(r,f) + S(r,f), \end{aligned}$$

which is a contradiction because $n > 5k+4m+9$. Hence $P(z) \equiv 0$, and so $F = G/A$.

Now we consider the case when f and g are polynomials. We suppose that f and g have γ and δ pairwise distinct zeros respectively. Then f and g are of the form

$$f(z) = c(z-a_1)^{l_1}(z-a_2)^{l_2}...(z-a_\gamma)^{l_\gamma},$$

$$g(z) = d(z-b_1)^{m_1}(z-b_2)^{m_2}...(z-b_\delta)^{m_\delta},$$

so that

$$f^n(z) = c^n(z-a_1)^{nl_1}(z-a_2)^{nl_2}...(z-a_\gamma)^{nl_\gamma}, \tag{3.16}$$

$$g^n(z) = d^n(z-b_1)^{nm_1}(z-b_2)^{nm_2}...(z-b_\delta)^{nm_\delta}, \tag{3.17}$$

where c and d are nonzero constants, $nl_i > 5k+4m+9$, $nm_j > 5k+4m+9$, $i = 1,2,...,\gamma$, and $j = 1,2,...,\delta$. Differentiating (3.5) we obtain

$$[f^n(f-1)^m]^{(k+1)} = \frac{1}{A}\left[g^n(g-1)^m\right]^{(k+1)},$$

i.e.,

(3.18)
$$\begin{aligned}&\left(f^{n+m}\right)^{(k+1)} + ... + (-1)^i\ {}^mC_{m-i}\left(f^{n+m-i}\right)^{(k+1)} + ... + (-1)^m\left(f^n\right)^{(k+1)}\\ &= \frac{1}{A}\left[\left(g^{n+m}\right)^{(k+1)} + ... + (-1)^i\ {}^mC_{m-i}\left(g^{n+m-i}\right)^{(k+1)} + ... + (-1)^m\left(g^n\right)^{(k+1)}\right].\end{aligned}$$

Using (3.16) and (3.17), (3.18) can be written as

$$\begin{aligned}&(z-a_1)^{nl_1-(k+1)}(z-a_2)^{nl_2-(k+1)}...(z-a_\gamma)^{nl_\gamma-(k+1)}p(z)\\ &= (z-b_1)^{nm_1-(k+1)}(z-b_2)^{nm_2-(k+1)}...(z-b_\delta)^{nm_\delta-(k+1)}q(z),\end{aligned} \tag{3.19}$$

where $p(z)$ and $q(z)$ are polynomials such that $deg\ \ p = m\sum_{i=1}^{\gamma} l_i + (\gamma-1)(k+1)$ and $deg\ \ q = m\sum_{j=1}^{\delta} m_j + (\delta-1)(k+1)$, respectively. Now

$$\begin{aligned}\sum_{i=1}^{\gamma}[nl_i-(k+1)] - m\sum_{i=1}^{\gamma} l_i &= \sum_{i=1}^{\gamma}[(n-m)l_i-(k+1)]\\ &> \gamma(4k+3m+8)\\ &> (\gamma-1)(k+1),\end{aligned}$$

i.e.,

$$\sum_{i=1}^{\gamma}[nl_i-(k+1)] > m\sum_{i=1}^{\gamma} l_i + (\gamma-1)(k+1).$$

Similarly,

$$\sum_{j=1}^{\delta}[nm_j-(k+1)] > m\sum_{j=1}^{\delta} m_j + (\delta-1)(k+1).$$

Thus from (3.18) we deduce that there is α such that

$$f^n(\alpha)\,(f(\alpha)-1)^m = g^n(\alpha)\,(g(\alpha)-1)^m = 0,$$

where α has multiplicity greater than $5k+4m+9$. This together with (3.14) implies $P(z)=0$. Thus from (3.5) and (3.14) we obtain $A=1$ and so

$$f^n(f-1)^m \equiv g^n(g-1)^m. \tag{3.20}$$

If $m=0$, then by (3.20) we have $f\equiv tg$ for a constant t such that $t^n=1$.
If $m\geq 1$, then from (3.20) we get

$$\begin{aligned} &f^n[f^m+...+(-1)^i\ {}^mC_{m-i}\,f^{m-i}+...+(-1)^m]\\ &= g^n[g^m+...+(-1)^{i\,m}C_{m-i}\,g^{m-i}+...+(-1)^m]. \end{aligned} \tag{3.21}$$

Let $h=\frac{f}{g}$. If h is a constant, by putting $f=gh$ in (3.21) we get

$$g^{n+m}(h^{n+m}-1)+...+(-1)^i\ {}^mC_{m-i}\,g^{n+m-i}(h^{n+m-i}-1)+...+(-1)^m g^n(h^n-1)=0,$$

which implies $h=1$. Thus $f\equiv g$.

If h is not a constant, then from (3.20) we can say that f and g satisfy the algebraic equation $R(f,g)=0$, where $R(x,y)=x^n(x-1)^m-y^n(y-1)^m$. This completes the proof of Theorem 1. □

Proof of Theorem 2. We consider $F(z)=f^n(\mu f^m+\lambda)$ and $G(z)=g^n(\mu g^m+\lambda)$. Then $[F(z)]^{(k)}$ and $[G(z)]^{(k)}$ share $(1,l)$. We consider following three cases.

Case 1. Let $\lambda\mu\neq 0$. By using Lemma 1 we have

$$\begin{aligned} \Theta(0,F) &= 1-\limsup_{r\to\infty}\frac{\overline{N}(r,0;F)}{T(r,F)}\\ &= 1-\limsup_{r\to\infty}\frac{\overline{N}(r,0;f^n(\mu f^m+\lambda))}{(n+m)T(r,f)}\\ &\geq 1-\limsup_{r\to\infty}\frac{(m+1)T(r,f)}{(n+m)T(r,f)}\\ &\geq \frac{n-1}{n+m}. \end{aligned} \tag{3.22}$$

Similarly

$$\Theta(0,G)\geq\frac{n-1}{n+m}. \tag{3.23}$$

$$\begin{aligned}
(3.24) \qquad \delta_{k+1}(0,F) &= 1-\limsup_{r\to\infty}\frac{N_{k+1}(r,0;F)}{T(r,F)}\\
&= 1-\limsup_{r\to\infty}\frac{N_{k+1}(r,0;f^n(\mu f^m+\lambda))}{(n+m)T(r,f)}\\
&\geq 1-\limsup_{r\to\infty}\frac{(k+m+1)T(r,f)}{(n+m)T(r,f)}\\
&\geq \frac{n-k-1}{n+m}.
\end{aligned}$$

Similarly

$$\delta_{k+1}(0,G)\geq \frac{n-k-1}{n+m}. \tag{3.25}$$

Using (3.22)-(3.25) we obtain that if $l\geq 2$, then $\Delta_1>3$ provided $n>2k+3m+4$; if $l=1$, then $\Delta_2>4$ provided $n>3k+4m+5$ and if $l=0$, then $\Delta_3>6$ provided $n>5k+6m+7$. So by Lemma 5 we obtain either $F^{(k)}G^{(k)}\equiv 1$ or (3.5). Let

$$F^{(k)}G^{(k)}\equiv 1.$$

i. e.,

$$[f^n(\mu f^m+\lambda)]^{(k)}[g^n(\mu g^m+\lambda)]^{(k)}\equiv 1. \tag{3.26}$$

Since f and g are entire functions from above it is clear that

$$f\neq 0 \qquad \textit{and} \qquad g\neq 0. \tag{3.27}$$

Let $f(z)=e^{\alpha(z)}$, where $\alpha(z)$ is an entire function. Then we obtain

$$[\mu f^{n+m}]^{(k)}=t_1(\alpha',\alpha'',...,\alpha^{(k)})e^{(n+m)\alpha(z)} \tag{3.28}$$

and

$$[\lambda f^n]^{(k)}=t_2(\alpha',\alpha'',...,\alpha^{(k)})e^{n\alpha(z)}, \tag{3.29}$$

where $t_i(\alpha',\alpha'',...,\alpha^{(k)})\not\equiv 0$ $(i=1,2)$ are differential polynomials. Since g is an entire function, we have from (3.26) that $[f^n(\mu f^m+\lambda)]^{(k)}\neq 0$. So from (3.28) and (3.29) we get

$$t_1(\alpha',\alpha'',...,\alpha^{(k)})e^{m\alpha(z)}+t_2(\alpha',\alpha'',...,\alpha^{(k)})\neq 0. \tag{3.30}$$

Since α is an entire function, we have $T(r,\alpha')=S(r,f)$ and

$$T(r,\alpha^{(j)})\leq T(r,\alpha')+S(r,f)=S(r,f)$$

for $j=1,2,...,k$. Hence we have

$$T(r,t_i)=S(r,f) \tag{3.31}$$

for $i = 1, 2$. So by (3.30), (3.31), Lemmas 1 and 4 we get

$$\begin{aligned} mT(r,f) &\leq T(r, t_1 e^{m\alpha}) + S(r,f) \\ &\leq \overline{N}(r,0;t_1e^{m\alpha}) + \overline{N}(r,0;t_1e^{m\alpha}+t_2) + S(r,f) \\ &\leq T\left(r, \frac{1}{t_1}\right) + S(r,f) \\ &= S(r,f), \end{aligned}$$

which is a contradiction.

We now suppose that (3.5) holds. Then we obtain (3.14). Suppose that f and g are transcendental entire functions. Then from (3.14) and Lemma 1 we obtain (3.15). If $P(z) \not\equiv 0$, then by (3.14) and (3.15) and Lemma 4 we obtain

$$\begin{aligned} (n+m)T(r,f) &= T(r,F) + O(1) \\ &\leq \overline{N}(r,0;F) + \overline{N}(r,0;G) + S(r,f) \\ &\leq 2(m+1)T(r,f) + S(r,f), \end{aligned}$$

which is a contradiction because $n > 5k+6m+7$. Hence $P(z) \equiv 0$, and so $F = G/A$.

Next we suppose that f and g are polynomials. Then proceeding similarly as in the proof of Theorem 1 we obtain $P(z) = 0$, and so $F = G/A$. From (3.5) and (3.14) we obtain $A = 1$ and so

$$f^n(\mu f^m + \lambda) \equiv g^n(\mu g^m + \lambda). \tag{3.32}$$

Let $h = \frac{f}{g}$. If $h \not\equiv 1$, from (3.32) we obtain

$$g^m = -\frac{\lambda}{\mu}\frac{1-h^n}{1-h^{n+m}}.$$

Since g is an entire function, every zero of $h^{n+m}-1$ is a zero of $h^n - 1$ and hence of $h^m - 1$. Thus h is a constant, which is a contradiction as f and g are nonconstant. Therefore $h \equiv 1$, that is $f \equiv g$.

Case 2. Let $\lambda = 0$ and $\mu \neq 0$. In this case $F = \mu f^{n+m}$ and $G = \mu g^{n+m}$. Proceeding in the same way as Case 1 of the theorem, we obtain

$$\Theta(0,F) \geq \frac{n+m-1}{n+m}. \tag{3.33}$$

$$\Theta(0,G) \geq \frac{n+m-1}{n+m}. \tag{3.34}$$

$$\delta_{k+1}(0,F) \geq \frac{n+m-k-1}{n+m}. \tag{3.35}$$

$$\delta_{k+1}(0,G) \geq \frac{n+m-k-1}{n+m}. \tag{3.36}$$

Using (3.33)-(3.36) we obtain that if $l \geq 2$, then $\Delta_1 > 3$ provided $n > 2k - m + 4$; if $l = 1$, then $\Delta_2 > 4$ provided $n > 3k - m + 5$ and if $l = 0$, then $\Delta_3 > 6$ provided $n > 5k - m + 7$. So by Lemma 5 we obtain either $F^{(k)}G^{(k)} \equiv 1$ or (3.5).

Let

$$F^{(k)}G^{(k)} \equiv 1.$$

i.e.,

$$[\mu f^{n+m}]^{(k)}[\mu g^{n+m}]^{(k)} \equiv 1. \tag{3.37}$$

By this and proceeding in the same way as Subcase (I) of Theorem 1 we get $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying

$$(-1)^k \mu^2 (c_1 c_2)^{n+m}[(n+m)c]^{2k} = 1.$$

We now suppose that (3.5) holds. Then we obtain (3.14). Suppose that f and g are transcendental entire functions. Then from (3.14) and Lemma 1 we get (3.15). If $P(z) \not\equiv 0$, then by (3.14) and (3.15) and Lemma 4 we obtain

$$\begin{aligned}(n+m)T(r,f) &= T(r,F) + O(1)\\ &\leq \overline{N}(r,0;F) + \overline{N}(r,0;G) + S(r,f)\\ &\leq 2T(r,f) + S(r,f),\end{aligned}$$

which is a contradiction because $n > 5k - m + 7$. Hence $P(z) \equiv 0$, and so $F = G/A$.

Next we suppose that f and g are polynomials such that f and g have γ and δ pairwise distinct zeros respectively. Then we obtain (3.16) and (3.17) where c and d are nonzero constants, $nl_i > 5k - m + 7$, $nm_j > 5k - m + 7$, $i = 1, 2, ..., \gamma$, and $j = 1, 2, ..., \delta$. Differentiating (3.5) we obtain

$$\mu(f^{n+m})^{(k+1)} = \frac{1}{A}[\mu(g^{n+m})^{(k+1)}]. \tag{3.38}$$

Using (3.16) and (3.17), (3.38) can be written as

$$\begin{aligned}&(z-a_1)^{(n+m)l_1-(k+1)}(z-a_2)^{(n+m)l_2-(k+1)}...(z-a_\gamma)^{(n+m)l_\gamma-(k+1)}p(z)\\ &=(z-b_1)^{(n+m)m_1-(k+1)}(z-b_2)^{(n+m)m_2-(k+1)}...(z-b_\delta)^{(n+m)m_\delta-(k+1)}q(z),\end{aligned} \tag{3.39}$$

where $p(z)$ and $q(z)$ are polynomials such that $deg \quad p = (\gamma - 1)(k+1)$ and $deg \quad q = (\delta - 1)(k+1)$, respectively. Now

$$\sum_{i=1}^{\gamma}[(n+m)l_i - (k+1)] > \gamma(4k+6) > (\gamma - 1)(k+1).$$

Similarly,

$$\sum_{j=1}^{\delta}[(n+m)_j-(k+1)]>(\delta-1)(k+1).$$

Thus from (3.39) we deduce that there is α such that

$$f^{n+m}(\alpha)=g^{n+m}(\alpha)=0,$$

where α has multiplicity greater than $5k-m+7$. This together with (3.14) implies $P(z)=0$, and so $F=G/A$. From (3.5) and (3.14) we obtain $A=1$ and so $f^{n+m}\equiv g^{n+m}$. Hence $f\equiv tg$, where t is a constant satisfying $t^{n+m}=1$.
Case 3. Let $\lambda\neq 0$ and $\mu=0$. Then $F=\lambda f^n$ and $G=\lambda g^n$. This case can be proved by using same argument as Case 2. This completes the proof of Theorem 2. □

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