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On the Geometry of Lightlike Submanifolds

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Abstract. We give some characterizations of non-existence of lightlike hypersurfaces of an indefinite space form. Some geometric objects for the induced Ricci tensor to be symmetric are studied.

1. Introduction

On lightlike submanifolds of a semi-Riemannian manifold the induced linear connection is torsion free but not metric connection. Hence the induced Ricci tensor of the induced connection, in general, is not symmetric. In [2], some equivalent conditions for the induced connection to be a Levi-Civita connection are studied.

In this article we investigate some geometric objects for the induced Ricci tensor to be symmetric. Furthermore we study the integrability of the screen distribution and non-existence of lightlike hypersurfaces of an indefinite space form. The paper is organized as follows : In section 2, the general theory of lightlike hypersurface is given. In section 3, we give some characterizations of non-existence of lightlike hypersurfaces immersed in an indefinite space form. In section 4, some necessary and sufficient conditions for the induced Ricci tensor to be symmetric are given. In the last section, the induced Ricci tensor of lightlike submanifolds is also studied.

2. Preliminaries

In this section, we review some results from the general theory of lightlike hypersurfaces ([2]). Let (M, g) be a lightlike hypersurface of an $(m+2)$-dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$ with constant index $q(1 \leq q \leq m+1)$. Then the so called *radical distribution* $Rad(TM) = TM \cap TM^{\perp}$ is of rank one, and the induced metric g on M is degenerate and has constant rank m, where $TM^{\perp}$ denotes the normal bundle over M. Also, a complementary vector bundle of $Rad(TM)$ in TM is a non-degenerate distribution of rank m(called a *screen distribution*) over M,

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denoted by $S(TM)$. Thus we have the orthogonal direct sum

$$TM = S(TM) \perp TM^{\perp}. \tag{2.1}$$

Let $tr(TM)$ be a complementary (but not orthogonal) vector bundle (called a *transversal vector bundle*) to TM in $T\bar{M} \mid M$. It is known that for any non-zero section $\xi \in \Gamma(TM^{\perp})$ on a coordinate neighborhood $\mathcal{U} \subset M$ there exists a unique null section N of the transversal vector bundle $tr(TM)$ on $\mathcal{U}$ such that

$$\bar{g}(N,\xi) = 1, \bar{g}(N,N) = \bar{g}(N,X) = 0, \quad \forall X \in \Gamma(S(TM) \mid_{\mathcal{U}}). \tag{2.2}$$

Thus we have the decomposition.

$$T\bar{M} = S(TM) \perp (TM^{\perp} \oplus tr(TM)) = TM \oplus tr(TM). \tag{2.3}$$

Throughout the paper $\Gamma(\bullet)$ denotes the module of smooth sections of the vector bundle $\bullet$.

Now let $\bar{\nabla}$ be the Levi-Civita connection of $\bar{M}$ and ∇ the induced linear connection on the lightlike hypersurface (M, g). According to the decomposition (2.3), we write for any $X, Y \in \Gamma(TM)$

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad \forall X, Y \in \Gamma(TM), \tag{2.4}$$

$$\bar{\nabla}_X V = -A_V X + \nabla^t_X V, \quad \forall X \in \Gamma(TM), \quad \forall V \in \Gamma(tr(TM)), \tag{2.5}$$

where $\nabla_X Y$ and $A_V X$ belong to $\Gamma(TM)$, while h is a $\Gamma(tr(TM))$-valued symmetric $C^{\infty}(M)$-bilinear form on $\Gamma(TM)$, A_V is a $C^{\infty}(M)$-linear operator on $\Gamma(TM)$ and ∇^t is a linear connection on $tr(TM)$. We call h and A_V the *second fundamental form* and the *shape operator* of the lightlike hypersurface M of $\bar{M}$, respectively.

The induced linear connection ∇ on M is a torsion free linear connection on M. Define a symmetric $C^{\infty}(M)$-bilinear form B (called the *local second fundamental form* on TM) and 1-form τ on the coordinate neighborhood $\mathcal{U}$ by

$$B(X,Y) = \bar{g}(h(X,Y),\xi), \quad \forall X, Y \in \Gamma(TM \mid_{\mathcal{U}}), \tag{2.6}$$

$$\tau(X) = \bar{g}(\nabla^t_X N, \xi), \quad \forall X \in \Gamma(TM \mid_{\mathcal{U}}). \tag{2.7}$$

Then (2.4) and (2.5) can be respectively written as follows :

$$\bar{\nabla}_X Y = \nabla_X Y + B(X,Y)N, \quad \forall X, Y \in \Gamma(TM \mid_{\mathcal{U}}), \tag{2.8}$$

$$\bar{\nabla}_X N = -A_N X + \tau(X)N, \quad \forall X \in \Gamma(TM \mid_{\mathcal{U}}). \tag{2.9}$$

If P denotes the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$ with respect to the decomposition (2.1), we obtain

$$\nabla_X PY = \nabla^*_X PY + h^*(X, PY), \quad \forall X, Y \in \Gamma(TM), \tag{2.10}$$

$$\nabla_X U = -A^*_U X + \nabla^{*t}_X U, \quad \forall X \in \Gamma(TM), \quad \forall U \in \Gamma(TM^{\perp}), \tag{2.11}$$

where $\nabla^*_X PY$ and $A^*_U X$ belong to $\Gamma(S(TM))$, ∇ and ∇^{*t} are linear connection on $S(TM)$ and $TM^\perp$ respectively, h^* is a $\Gamma(TM^\perp)$-valued $C^\infty(M)$-bilinear form on $\Gamma(TM) \times \Gamma(S(TM))$ and A^*_U is $\Gamma(S(TM))$-valued $C^\infty(M)$-linear operator on $\Gamma(TM)$. We call h^* and A^*_U the *second fundamental form* and the *shape operator* of the screen distribution $S(TM)$, respectively. Now we define locally a $C^\infty(M)$-bilinear form C (called the *local second fundamental form* on $S(TM)$) and a 1-form $\epsilon(X)$ as follows :

$$C(X, PY) = \bar{g}(h^*(X, PY), N), \quad \forall X, Y \in \Gamma(TM\,|_{\mathcal{U}}), \tag{2.12}$$

$$\epsilon(X) = \bar{g}(\nabla^{*t}_X \xi, N), \quad \forall X \in \Gamma(TM\,|_{\mathcal{U}}). \tag{2.13}$$

Note that $\epsilon(X) = -\tau(X)$. Thus, locally (2.10) and (2.11) become respectively

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)\xi, \quad \forall X, Y \in \Gamma(TM\,|_{\mathcal{U}}), \tag{2.14}$$

$$\nabla_X \xi = -A^*_\xi X - \tau(X)\xi, \quad \forall X, Y \in \Gamma(TM\,|_{\mathcal{U}}). \tag{2.15}$$

It is easy to see that the two local second fundamental forms B and C are related to their shape operators by

$$B(X, Y) = g(A^*_\xi X, Y), \quad \bar{g}(A^*_\xi X, N) = 0, \tag{2.16}$$

$$C(X, PY) = g(A_N X, PY), \quad \bar{g}(A_N X, N) = 0. \tag{2.17}$$

Note that in general, A_N is not symmetric with respect to g, the local second fundamental form B is independent of the choice of screen distribution $S(TM)$ and satisfies

$$B(X, \xi) = 0, \quad \forall X \in \Gamma(TM). \tag{2.18}$$

Furthermore, the induced linear connection ∇ is not a metric connection. Indeed we have

$$(\nabla_X g)(Y, Z) = B(X, Y)\eta(Z) + B(X, Z)\eta(Y) \tag{2.19}$$

for any $X, Y \in \Gamma(TM)$, where η is a differential 1-form locally defined on M by

$$\eta(X) = \bar{g}(X, N), \quad \forall X \in \Gamma(TM). \tag{2.20}$$

Finally, we are concerned with the structure equations for a lightlike hypersurface
$(M, g, S(TM))$ of $(\bar{M}, \bar{g})$. Denote by $\bar{R}$ and R the the curvature tensor of $\bar{\nabla}$ and ∇, respectively. By definition of curvature tensor, we obtain from (2.4) and (2.5).

$$\begin{aligned} \bar{R}(X, Y)Z = \; & R(X, Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X \\ & + (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z) \end{aligned} \tag{2.21}$$

for any $X, Y, Z \in \Gamma(TM)$, where we set

$$(\nabla_X h)(Y, Z) = \nabla^t_X(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \tag{2.22}$$

Then we have the Gauss-Codazzi equations of the lightlike hypersurfaces

$$\begin{aligned}\bar g(\bar R(X,Y)Z,PW) = \ & g(R(X,Y)Z,PW) + \bar g(h(X,Z),h^*(Y,PW)) \\ & - \bar g(h(Y,Z),h^*(Y,PW)),\end{aligned} \tag{2.23}$$

$$\bar g(\bar R(X,Y)Z,U) = \bar g((\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z),U), \tag{2.24}$$

$$\bar g(\bar R(X,Y)Z,V) = \bar g(R(X,Y)Z,V) \tag{2.25}$$

for any $X,Y,Z,W \in \Gamma(TM)$, $U \in \Gamma(TM^\perp)$ and $V \in \Gamma(tr(TM))$.

Making use of (2.2), (2.6), (2.7) and (2.12) we have local expressions for (2.23) $\sim$ (2.25):

$$\begin{aligned}\bar g(\bar R(X,Y)Z,PW) = \ & g(R(X,Y)Z,PW) + B(X,Z)C(Y,PW) \\ & - B(Y,Z)C(X,PW),\end{aligned} \tag{2.26}$$

$$\begin{aligned}\bar g(\bar R(X,Y)Z,\xi) = \ & (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) \\ & + B(Y,Z)\tau(X) - B(X,Z)\tau(Y)\end{aligned} \tag{2.27}$$

$$\bar g(\bar R(X,Y)Z,N) = \bar g(R(X,Y)Z,N), \tag{2.28}$$

for any $X,Y,Z,W \in \Gamma(TM)$, respectively, where we set

$$(\nabla_X B)(Y,Z) = XB(Y,Z) - B(\nabla_X Y,Z) - B(Y,\nabla_X Z).$$

Also, from the right hand side of (2.25) with (2.14) and (2.15) we get

$$\begin{aligned}\bar g(\bar R(X,Y)PZ,N) = \ & (\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) \\ & + C(X,PZ)\tau(Y) - C(Y,PZ)\tau(X),\end{aligned} \tag{2.29}$$

$$\bar g(\bar R(X,Y)\xi,N) = C(Y,A^*_\xi X) - C(X,A^*_\xi Y) - 2d\tau(X,Y) \tag{2.30}$$

for any $X,Y,Z \in \Gamma(TM)$, where we set

$$(\nabla_X C)(Y,PZ) = XC(Y,PZ) - C(\nabla_X Y,PZ) - C(Y,\nabla^*_X PZ). \tag{2.31}$$

On the other hand, using again the formulas (2.4) and (2.5) of Gauss and Weingarten, we obtain

$$\begin{aligned}\bar R(X,Y)N \ = \ & R^t(X,Y)N - h(X,A_N Y) + h(Y,A_N X) \\ - \ & (\nabla_X A)_N Y + (\nabla_Y A)_N X,\end{aligned} \tag{2.32}$$

where

$$R^t(X,Y)N = \nabla^t_X \nabla^t_Y N - \nabla^t_Y \nabla^t_X N - \nabla^t_{[X,Y]} N, \tag{2.33}$$

is the curvature tensor of the transversal vector bundle $tr(TM)$ with respect to the transversal connection ∇^t, and

$$(\nabla_X A)_N Y = \nabla_X (A_N Y) - A_N(\nabla_X Y) - A_{\nabla^t_X N} Y. \tag{2.34}$$

Similarly, using (2.10) and (2.11), we have

$$\begin{aligned} R(X,Y)\xi &= R^{*t}(X,Y)\xi - h^*(X, A^*_\xi Y) + h^*(Y, A^*_\xi X) \\ &- (\nabla^*_X A^*)_\xi Y + (\nabla^*_Y A^*)_\xi X, \end{aligned} \tag{2.35}$$

where

$$R^{*t}(X,Y)\xi = \nabla^{*t}_X \nabla^{*t}_Y \xi - \nabla^{*t}_Y \nabla^{*t}_X \xi - \nabla^{*t}_{[X,Y]}\xi, \tag{2.36}$$

is the curvature tensor of the radical distribution $Rad(TM)$ with respect to ∇^{*t}, and

$$(\nabla^*_X A^*)_\xi Y = \nabla^*_X (A^*_\xi Y) - A^*_\xi(\nabla_X Y) - A^*_{\nabla^{*t}_X \xi} Y. \tag{2.37}$$

3. Non-existence of lightlike hypersurfaces

Let $(M, S(TM), g)$ be an $(m+1)$-dimensional lightlike hypersurface of an $(m+2)$-dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$. We say that M(resp. $S(TM)$) is *totally umbilical* if, on each coordinate neighborhood $\mathcal{U}$, there exists a smooth function ρ such that for any $X, Y \in \Gamma(TM\mid_{\mathcal{U}})$

$$B(X,Y) = \rho g(X,Y) \quad (\text{resp.} \quad C(X,PY) = \rho g(X,PY)), \tag{3.1}$$

or equivalently, $A^*_\xi(PX) = \rho PX$(resp. $A_N X = \rho PX$). In case $\rho = 0$ on $\mathcal{U}$, we say that M and $S(TM)$ are *totally geodesic*, respectively. In case $\rho \neq 0$ on $\mathcal{U}$, we say that M and $S(TM)$ are *proper totally umbilical*, respectively.

The second fundamental form h of M is said to be *parallel* if $(\nabla_X h)(Y,Z) = 0$, $\forall X, Y, Z \in \Gamma(TM)$, which is equivalent to

$$(\nabla_X B)(Y,Z) = -\tau(X)B(Y,Z). \tag{3.2}$$

The screen second fundamental form h^* is said to be *parallel* if $(\nabla_X h^*)(Y, PZ) = 0$, $\forall X, Y, Z \in \Gamma(TM)$, which is equivalent to

$$(\nabla_X C)(Y, PZ) = \tau(X)C(Y,PZ), \tag{3.3}$$

where $(\nabla_X h^*)(Y,PZ) = \nabla_X^{*t}(h^*(Y,PZ)) - h^*(\nabla_X Y,PZ) - h^*(Y,\nabla_X^* PZ)$.

Proposition 3.1. *Let $(M,S(TM),g)$ be a lightlike hypersurface of a semi-Riemannian manifold $\bar{M}$. If M is totally umbilical and the second fundamental form is parallel, then M is totally geodesic.*

Proof. Substituting (3.1) into (3.2) and taking account of (2.19), we obtain

$$-\tau(X)\rho g(Y,Z) = (X\rho)g(Y,Z) + \rho^2\{g(X,Y)\eta(Z) + g(X,Z)\eta(Y)\}$$

for any vector fields X, Y and Z on M. Putting $Z = \xi$ yields $\rho^2 g(X,Y) = 0$ for any $X, Y \in \Gamma(TM)$, which means that $\rho = 0$, i.e., $B = 0$. Hence M is totally geodesic. □

Proposition 3.2. *Let $(M,S(TM),g)$ be a lightlike hypersurface of a semi-Riemannian manifold $\bar{M}$. If $S(TM)$ is proper totally umbilical and the screen second fundamental form is parallel, then M is totally geodesic.*

Proof. Taking account of (3.1), (3.3), (2.31) and (2.19) we obtain

$$\tau(X)\rho g(Y,PZ) = (X\rho)g(Y,PZ) + \rho\{B(X,Y)\eta(PZ) + B(X,PZ)\eta(Y)\}$$

for any vector fields X, Y and Z on M. Putting $Y = \xi$ yields $\rho B(X,PZ) = 0$ for any $X, Z \in \Gamma(TM)$, which means that $B(X,PZ) = 0$, since $\rho \neq 0$. Hence M is totally geodesic. □

Proposition 3.3. *Let $(M,S(TM),g)$ be a lightlike hypersurface of a semi-Riemannian manifold $\bar{M}$. If $\mathcal{L}_X\eta = 0$ for any vector field X tangent to M, then $S(TM)$ is integrable.*

Proof. It follows from (2.8), (2.9) and (2.20) that

$$\begin{aligned} 0 &= (\mathcal{L}_X\eta)(Y) \\ &= X\bar{g}(Y,N) - \eta([X,Y]) \\ &= \bar{g}(\nabla_X Y,N) - \bar{g}(Y,A_N X) + \tau(X)\bar{g}(Y,N) - \eta([X,Y]), \forall X,Y \in \Gamma(TM). \end{aligned}$$

Putting $Y = PY$ in the equation and using (2.14) and (2.17) yield $\eta([X,PY]) = 0$, which means that $S(TM)$ is integrable. □

If a semi-Riemannian manifold $(\bar{M},\bar{g})$ has a constant sectional curvature c, then we say that $\bar{M}$ is an *indefinite space form* and denote it by $\bar{M}(c)$. In this case, the curvature tensor field $\bar{R}$ of $\bar{M}(c)$ is given by

$$\bar{R}(X,Y)Z = c\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y\} \tag{3.4}$$

for any vector fields X, Y and Z on $\bar{M}(c)$.

Proposition 3.4. *There exist no lightlike hypersurfaces of $\bar{M}(c)$ $(c \neq 0)$ with η-parallel.*

Proof. Taking account of (2.8), (2.9) and (2.20), we get

$$-\bar{g}(Y, A_N X) + \tau(X)\bar{g}(Y, N) = 0, \forall X, Y \in \Gamma(TM).$$

Putting $Y = PY$ in this equation yields $C(X, PY) = 0$. Thus the right hand side of (2.29) vanishes identically, and so we obtain

$$0 = \bar{g}(\bar{R}(X, Y)PZ, N) = c\{\bar{g}(Y, PZ)\bar{g}(X, N) - \bar{g}(X, PZ)\bar{g}(Y, N)\} \tag{3.5}$$

$\forall X, Y \in \Gamma(TM)$. Setting $X = \xi$, we have $c\bar{g}(Y, PZ) = 0$, and so $c = 0$. □

Proposition 3.5. *There exist no lightlike hypersurfaces of $\bar{M}(c)$ ($c \neq 0$) with parallel screen second fundamental form.*

Proof. (3.3) and (2.29) imply (3.5). By the same argument $c = 0$. □

Proposition 3.6. *There exist no lightlike hypersurfaces of $\bar{M}(c)$ ($c \neq 0$) satisfying $(\nabla_X A)_N Y = (\nabla_Y A)_N X$.*

Proof. Our assumption and (2.32) imply (3.5). Hence $c = 0$. □

Define the *null sectional curvature* of M at $u \in M$ with respect to ξ_u as a real number

$$K_{\xi_u}(M) = \frac{g(R(X_u, \xi_u)\xi_u, X_u)}{g(X_u, X_u)}, \tag{3.6}$$

where X_u is non-null vector in $T_u(M)$.

Proposition 3.7. *There exist no lightlike hypersurfaces of $\bar{M}(c)$ with non-zero null sectional curvature.*

Proof. Since $B(X, \xi) = 0$, we obtain from (2.26) and (3.4) that
$g(R(X_u, \xi_u)\xi_u, X_u) = 0$, which showes that $K_{\xi_u}(M) = 0$. □

4. Induced Ricci tensor on hypersurfaces

In this section we study the induced Ricci tensor of a lightlike hypersurface $(M, S(TM), g)$ of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. The induced Ricci tensor on M is given by

$$\breve{R}ic(X, Y) = \text{trace}\{Z \to R(Z, X)Y\}, \quad \forall X, Y \in \Gamma(TM). \tag{4.1}$$

We note (cf. [1])that the induced Ricci tensor $\breve{R}ic$ does not depend on the choice of ξ. Since the induced connection ∇ on M is not a Levi-Civita connection, $\breve{R}ic$ is not symmetric, in general. To begin with we give

Proposition 4.1([1]). *Let $(M, S(TM), g)$ be a lightlike hypersurface of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then the induced Ricci tensor $\breve{R}ic$ is given by*

$$\breve{R}ic(X, Y) = \bar{R}ic(X, Y) - B(X, Y)\,TrA_N + g(A^*_\xi Y, A_N X) + \bar{g}(\bar{R}(\xi, Y)X, N), \tag{4.2}$$

where TrA_N *denotes the trace of* A_N.

Corollary 4.2. *Let* $(M, S(TM), g)$ *be a lightlike hypersurface of an indefinite space form* $\bar{M}(c)$. *Then the induced Ricci tensor* $\breve{R}ic$ *is symmetric if and only if* $A^*_\xi \circ A_N$ *is self-adjoint with respect to* g.

Proof. Our assumption (3.4) showes that $\bar{g}(\bar{R}(\xi, Y)X, N) = -cg(X, Y)$. Then it is clear from (2.16) that $\breve{R}ic$ is symmetric if and only if $g(Y, A^*_\xi A_N X) = g(A^*_\xi A_N Y, X)$. □

Theorem 4.3([2]). *The induced Ricci tensor* $\breve{R}ic$ *of the induced connection* ∇ *is symmetric, if and only if, each* 1*-form* τ *induced by* $S(TM)$ *is closed.*

Now we prove

Theorem 4.4. *Let* $(M, S(TM), g)$ *be a lightlike hypersurface of a semi-Riemannian manifold* $(\bar{M}, \bar{g})$. *Then the following assertions are equivalent :*
(i) *The induced Ricci tensor* $\breve{R}ic$ *of the induced connection* ∇ *is symmetric.*
(ii) *The transversal connection is flat, i.e.,* $R^t = 0$.
(iii) *The radical connection is flat, i.e.,* $R^{*t} = 0$.
(iv) *Each* 1*-form* τ *induced by* $S(TM)$ *is closed, i.e.,* $d\tau = 0$, *on any coordinate neighborhood* $\mathcal{U}$ *of* M.

Proof. It follows from (4.2), (2.17) and the first Bianchi identity that

$$\begin{aligned} \breve{R}ic(X, Y) - \breve{R}ic(Y, X) &= g(A^*_\xi Y, A_N X) - g(A^*_\xi X, A_N Y) + \bar{g}(\bar{R}(X, Y)\xi, N) \\ &= C(X, A^*_\xi Y) - C(Y, A^*_\xi X) + \bar{g}(\bar{R}(X, Y)\xi, N). \end{aligned} \tag{4.3}$$

From (2.35) together with (2.16) and (2.17), we get

$$\bar{g}(\bar{R}(X, Y)N, \xi) = \bar{g}(R^t(X, Y)N, \xi) - C(Y, A^*_\xi X) + C(X, A^*_\xi Y). \tag{4.4}$$

Substituting this equation into (4.3), we obtain

$$\breve{R}ic(X, Y) - \breve{R}ic(Y, X) = -\bar{g}(R^t(X, Y)N, \xi). \tag{4.5}$$

Also, from (2.35) together with (2.12) and (2.28) we have

$$\bar{g}(\bar{R}(X, Y)\xi, N) = \bar{g}(R^{*t}(X, Y)\xi, N) + C(Y, A^*_\xi X) - C(X, A^*_\xi Y). \tag{4.6}$$

Then comparing this equation with (2.30), we have

$$2d\tau(X, Y) = -\bar{g}(R^{*t}(X, Y)\xi, N). \tag{4.7}$$

On the other hand, comparing (4.4) with (2.30) yields

$$2d\tau(X, Y) = \bar{g}(R^t(X, Y)N, \xi). \tag{4.8}$$

Hence the proof follows from (4.5), (4.7) and (4.8). □

Proposition 4.5. *Let $(M, S(TM), g)$ be a proper totally umbilical lightlike hypersurface of $\bar{M}(c)$. Then one of the assertions $(i) \sim (iv)$ stated in Theorem 4.3 holds if and only if the screen distribution $S(TM)$ is integrable.*

Proof. It follows from (4.4) that $R^t = 0$ iff $C(X, A^*_\xi Y) - C(Y, A^*_\xi X) = 0$, which is equivalent to the equation

$$\rho\{g(Y, A_N X) - g(X, A_N Y)\} = 0,$$

since

$$C(X, A^*_\xi Y) = g(A_N X, A^*_\xi Y) = B(Y, A_N X) = \rho g(Y, A_N X),$$

where we have used (2.16) and (2.17). Thus A_N is self-adjoint with respect to g, or equivalently, the screen distribution $S(TM)$ is integrable([2]). □

Proposition 4.6. *Let $(M, S(TM), g)$ be a lightlike hypersurface of $\bar{M}(c)$. If $S(TM)$ is totally umbilical, then one of the assertions $(i) \sim (iv)$ stated in Theorem 4.3 holds*

Proof. The proof follows from (4.5). □

5. Induced Ricci tensor on lightlike submanifolds

In this section, we recall briefly some results from the general theory of lightlike submanifolds (cf. [2], [3]).

Let $(\bar{M}, \bar{g})$ be of an $(m+n)$-dimensional semi-Riemannian manifold with constant index q such that $m, n \geq 1$, $q(1 \leq q \leq m+n-1)$ and (M, g) an m-dimensional submanifold of $(\bar{M}, \bar{g})$. Then (M, g) is called a *lightlike submanifold* if it admits a degenerate metric g whose radical distribution $Rad(TM)$ is of rank $r(1 \leq r \leq m)$.

Let $tr(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\bar{M}\mid_M$ and $Rad(TM)$ in $S(TM^\perp)^\perp$, where $S(TM^\perp)^\perp$ denotes the orthogonal complementary vector subbundle to $S(TM^\perp)$ in $S(TM)^\perp$, i.e., $S(TM)^\perp = S(TM^\perp) \perp S(TM^\perp)^\perp$

Then we have the following decompositions :

$$tr(TM) = ltr(TM) \perp S(TM^\perp), \tag{5.1}$$

$$\begin{aligned} T\bar{M}\mid_M &= S(TM) \perp \{Rad(TM) \oplus ltr(TM)\} \perp S(TM^\perp) \\ &= TM \oplus tr(TM). \end{aligned} \tag{5.2}$$

We say that a lightlike submanifold $(M, g, S(TM), S(TM^\perp))$ of $\bar{M}$ is

case 1 : *r-lightlike* if $r < min\{m, n\}$;
case 2 : *co-isotropic* if $r = n < m$, $S(TM^\perp) = \{0\}$;
case 3 : *isotropic* if $r = m < n$, $S(TM) = \{0\}$;
case 4 : *totally lightlike* if $r = m = n$, $S(TM) = \{0\}$ and $S(TM^\perp) = \{0\}$.

Here and in the sequel, we only consider an r-lightlike submanifold of $\bar{M}$.

According to the decomposition (5.1) we put

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y) = \nabla_X Y + h^l(X,Y) + h^s(X,Y), \quad \forall X, Y \in \Gamma(TM), \tag{5.3}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X,N), \quad N \in \Gamma(ltr(TM)) \tag{5.4}$$

$$\bar{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X,W), \quad W \in \Gamma(S(TM^\perp)), \tag{5.5}$$

where $\nabla_X Y, A_N X, A_W X \in \Gamma(TM)$, $h(X,Y) \in \Gamma(tr(TM))$, $h^l(X,Y)$, $\nabla^l_X N$, $D^l(X,W) \in \Gamma(ltr(TM))$, and $h^s(X,Y)$, $D^s(X,N)$, $\nabla^s_X W \in \Gamma(S(TM^\perp))$.

We note that the lightlike second fundamental form h^l of a lightlike submanifold M do not depend on $S(TM), S(TM^\perp)$ and $ltr(TM)$. Then, by using (5.3) $\sim$ (5.5) and the fact that $\bar{\nabla}$ is a metric connection, we obtain

$$\bar{g}(h^l(X,Y),\xi) + \bar{g}(Y, h^l(X,\xi)) + g(Y, \nabla_X \xi) = 0, \tag{5.6}$$

$$\bar{g}(h^s(X,Y),W) + \bar{g}(Y, D^l(X,W)) = g(A_W X, Y), \tag{5.7}$$

$$\bar{g}(A_N X, N') + \bar{g}(A_{N'} X, N) = 0, \tag{5.8}$$

$$\bar{g}(D^s(X,N),W) = \bar{g}(N, A_W X), \tag{5.9}$$

where $\xi \in \Gamma(Rad(TM))$, $W \in \Gamma(S(TM^\perp))$ and $N, N' \in \Gamma(ltr(TM))$.

From the decomposition $TM = S(TM) \perp Rad(TM)$, we set

$$\nabla_X PY = \nabla^*_X PY + h^*(X, PY), \tag{5.10}$$

$$\nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi \tag{5.11}$$

for $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$, where $\{\nabla^*_X PY, A^*_\xi X\}$ and $\{h^*(X,PY), \nabla^{*t}_X \xi\}$ belong to $\Gamma(S(TM))$ and $\Gamma(Rad(TM))$, respectively. It follows that ∇^* and ∇^{*t} are linear connections on distributions $S(TM)$ and $Rad(TM)$, respectively. From (5.10) and (5.11) we obtain

$$\bar{g}(h^l(X,PY),\xi) = g(A^*_\xi X, PY), \tag{5.12}$$

$$\bar{g}(h^*(X,PY),N) = g(A_N X, PY), \quad \forall X, Y \in \Gamma(TM). \tag{5.13}$$

In general, the induced connection ∇ on M is not metric connection, since

$$(\nabla_X g)(Y,Z) = \bar{g}(h^l(X,Y),Z) + \bar{g}(h^l(X,Z),Y). \tag{5.14}$$

Let $(M, g, S(TM), S(TM^\perp))$ be an m-dimensional r-lightlike submanifold of an $(m+n)$-dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$. Denote by $\bar{R}, R, R^l$ and R^{*t} the curvature tensors of $\bar{\nabla}, \nabla, \nabla^l$ and ∇^{*t}, respectively. We need following structure equations :

$$\begin{aligned} \bar{R}(X,Y)Z &= R(X,Y)Z + A_{h^l(X,Z)}Y \\ &- A_{h^l(Y,Z)}X + A_{h^s(X,Z)}Y - A_{h^s(Y,Z)}X \\ &- (\nabla_X h^l)(Y,Z) - (\nabla_Y h^l)(X,Z) \\ &+ D^l(X, h^s(Y,Z)) - D^l(Y, h^s(X,Z)) \\ &- (\nabla_X h^s)(Y,Z) - (\nabla_Y h^s)(X,Z) \\ &+ D^s(X, h^l(Y,Z)) - D^s(Y, h^l(X,Z)), \end{aligned} \tag{5.15}$$

$$\begin{aligned}
(5.16)\quad \bar{g}(\bar{R}(X,Y)N,\xi) &= \bar{g}(R^l(X,Y)N,\xi)\\
&+ \bar{g}(h^l(Y,A_NX),\xi) - \bar{g}(h^l(X,A_NY,\xi)\\
&+ \bar{g}(D^s(X,N),h^s(Y,\xi)) - \bar{g}(D^s(Y,N),h^s(X,\xi))\\
(5.17)\quad &= -\bar{g}(R(X,Y)\xi,N)\\
&+ \bar{g}(A_NY,h^l(X,\xi)) - \bar{g}(A_NX,h^l(Y,\xi))\\
&+ \bar{g}(D^s(X,N),h^s(Y,\xi)) - \bar{g}(D^s(Y,N),h^s(X,\xi)),
\end{aligned}$$

$$(5.18)\qquad \bar{g}(R(X,Y)\xi,N)) = \bar{g}(R^{*t}(X,Y)\xi,N) + g(A_NY,A^*_\xi X) - g(A_NX,A^*_\xi Y)$$

for any vector fields X,Y and Z tangent to M.

Consider the following local quasi-orthonormal field of frames of $\bar{M}$ along M([2]):

$$(5.19)\qquad \{\xi_1,\cdots,\xi_r,N_1,\cdots,N_r,X_{r+1},\cdots X_m,W_{r+1},\cdots,W_n\},$$

where $\{\xi_1,\cdots,\xi_r\}$ and $\{N_1,\cdots,N_r\}$ are lightlike bases of $\Gamma(Rad(TM))$ and $\Gamma(ltr(TM))$, respectively satisfying

$$\bar{g}(N_i,\xi_j) = \delta_{ij},\quad \bar{g}(N_i,N_j) = 0, \forall i,j \in \{1,\cdots,r\},$$

$\{X_{r+1},\cdots,X_m\}$ and $\{W_{r+1},\cdots,W_n\}$ are orthonormal bases with causal characters ϵ^*_a, ϵ_α of $\Gamma(S(TM))$ and $\Gamma(S(TM^\perp))$, respectively. In the sequel, we adopt the following range of indices :

$$i \in \{1,\cdots,r\};\quad a \in \{r+1,\cdots,m\};\quad \alpha \in \{r+1,\cdots,n\}.$$

Proposition 5.1. *Let $(M,S(TM),S(TM^\perp),g)$ be an r-lightlike submanifold of a semi-Riemannian manifold $(\bar{M},\bar{g})$. Then the induced Ricci tensor $\breve{R}ic$ is given by*

$$\begin{aligned}
(5.20)\quad \breve{R}ic(X,Y) = \bar{R}ic(X,Y) &- TrA_{h(X,Y)}\\
&+ \sum_i \{g(A_{N_i}X,A^*_{\xi_i}Y) - \bar{g}(A_{N_i}X,h^l(\xi_i,Y))\}\\
&+ \sum_a \epsilon^*_a \bar{g}(h^s(X_a,Y),h^s(X,X_a) + \sum_i \bar{g}(h^s(\xi_i,Y),D^s(X,N_i))\\
&- \sum_\alpha \epsilon_\alpha \bar{g}(\bar{R}(X,W_\alpha)Y,W_\alpha) - \sum_i \bar{g}(\bar{R}(X,N_i)Y,\xi_i).
\end{aligned}$$

Proof. From (4.1) we obtain

$$(5.21)\qquad \breve{R}ic(X,Y) = \sum_a \epsilon^*_a g(R(X,X_a)Y,X_a) + \sum_i \bar{g}(R(X,\xi_i)Y,N_i).$$

It follows from (5.15) that

$$\begin{aligned}(5.22)\quad \breve{R}ic(X,Y) = \sum_a \epsilon_a^* \{ & g(\bar{R}(X,X_a)Y,X_a) + g(A_{h^l(X_a,Y)}X,X_a) \\ & - g(A_{h^l(X,Y)}X_a,X_a) + g(A_{h^s(X_a,Y)}X,X_a) - g(A_{h^s(X,Y)}X_a,X_a)\} \\ & + \sum_i \{\bar{g}(\bar{R}(X,\xi_i)Y,N_i) + \bar{g}(A_{h^l(\xi_i,Y)}X,N_i) - \bar{g}(A_{h^l(X,Y)}\xi_i,N_i) \\ & + \bar{g}(A_{h^s(\xi_i,Y)}X,N_i) - \bar{g}(A_{h^s(X,Y)}\xi_i,N_i)\}.\end{aligned}$$

On the other hand the Ricci tensor $\bar{R}ic$ of $\bar{M}$ is given by

$$\begin{aligned}\bar{R}ic(X,Y) &= \sum_a \epsilon_a^* g(\bar{R}(X,X_a)Y,X_a) + \sum_i \bar{g}(\bar{R}(X,\xi_i)Y,N_i) \\ &+ \sum_\alpha \epsilon_\alpha \bar{g}(\bar{R}(X,W_\alpha)Y,W_\alpha) + \sum_i \bar{g}(\bar{R}(X,N_i)Y,\xi_i).\end{aligned}$$

Substituting this equation into (5.22), we have

$$\begin{aligned}(5.23)\quad \breve{R}ic(X,Y) =& \bar{R}ic(X,Y) + \sum_a \epsilon_a^*\{g(A_{h^l(X_a,Y)}X,X_a) - g(A_{h^l(X,Y)}X_a,X_a) \\ & + g(A_{h^s(X_a,Y)}X,X_a) - g(A_{h^s(X,Y)}X_a,X_a)\} \\ & + \sum_i\{\bar{g}(A_{h^l(\xi_i,Y)}X,N_i) - \bar{g}(A_{h^l(X,Y)}\xi_i,N_i) \\ & + \bar{g}(A_{h^s(\xi_i,Y)}X,N_i) - \bar{g}(A_{h^s(X,Y)}\xi_i,N_i)\} \\ & - \sum_\alpha \epsilon_\alpha \bar{g}(\bar{R}(X,W_\alpha)Y,W_\alpha) - \sum_i \bar{g}(\bar{R}(X,N_i)Y,\xi_i).\end{aligned}$$

Making use of (5.6) $\sim$ (5.9), (5.12) and (5.13), (5.23) is transformed into (5.20), where

$$\begin{aligned}TrA_{h(X,Y)} &= \sum_a \epsilon_a^*\{g(A_{h^s(X,Y)}X_a,X_a) + g(A_{h^l(X,Y)}X_a,X_a)\} \\ &+ \sum_i\{\bar{g}(A_{h^s(X,Y)}\xi_i,N_i) + \bar{g}(A_{h^l(X,Y)}\xi_i,N_i)\}.\end{aligned}$$

□

Corollary 5.2. *Let $(M, S(TM), S(TM^\perp), g)$ be an r-lightlike submanifold of an indefinite space form $\bar{M}(c)$. Then the induced Ricci tensor $\breve{R}ic$ is given by*

$$\begin{aligned}(5.24)\quad \breve{R}ic(X,Y) =& c(1-m-n-r)g(X,Y) - TrA_{h(X,Y)} \\ & + \sum_i\{g(A_{N_i}X, A^*_{\xi_i}Y) - \bar{g}(A_{N_i}X, h^l(\xi_i,Y))\} \\ & + \sum_a \epsilon_a^* \bar{g}(h^s(X_a,Y),h^s(X,X_a)) + \sum_i \bar{g}(h^s(\xi_i,Y),D^s(X,N_i)).\end{aligned}$$

Finally we prove

Theorem 5.3. *Let $(M, S(TM), S(TM^{\perp}), g)$ be an r-lightlike submanifold $(\bar{M}, \bar{g})$. Then the following assertions are equivalent:*
(i) The induced Ricci tensor $\breve{R}ic$ of the induced connection ∇ is symmetric.
(ii) The lightlike transversal connection is flat, i.e., $R^l = 0$.
(iii) The radical connection is flat, i.e., $R^{*t} = 0$.

Proof. From (5.20) we have

$$\begin{aligned} & \breve{R}ic(X,Y) - \breve{R}ic(Y,X) \\ = & \sum_i g(A_{N_i}X, A^*_{\xi_i}Y) - \sum_i g(A_{N_i}Y, A^*_{\xi_i}X) \\ - & \sum_i \bar{g}(A_{N_i}X, h^l(\xi_i, Y)) + \sum_i \bar{g}(A_{N_i}Y, h^l(\xi_i, X)) \\ + & \sum_i \bar{g}(h^s(\xi_i, Y), D^s(X, N_i)) - \sum_i \bar{g}(h^s(\xi_i, X), D^s(Y, N_i)) \\ - & \sum_i \bar{g}(\bar{R}(X,Y)N_i, \xi_i), \end{aligned} \tag{5.25}$$

where the last term is due to the first Bianchi identity. Replacing the last term of (5.25) by (5.16) and making use of (5.6), (5.25) is reduced to the form.

$$\breve{R}ic(X,Y) - \breve{R}ic(Y,X) = -\sum_i \bar{g}(R^l(X,Y)N_i, \xi_i). \tag{5.26}$$

On the other hand, replacing the last term of (5.25) by (5.17) and again by (5.18) we also have

$$\breve{R}ic(X,Y) - \breve{R}ic(Y,X) = \sum_i \bar{g}(R^{*t}(X,Y)\xi_i, N_i). \tag{5.27}$$

The proof follows from (5.26) and (5.27). □

Remark 5.4. Theorem 5.3 also holds for a co-isotropic submanifold M.

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