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고정비용 0-1 배낭문제에 대한 크바탈-고모리 부등식의 분리문제에 관한 연구*

박경철** · 이경식***†

On the Separation of the Rank-1 Chvatal-Gomory Inequalities for the Fixed-Charge 0-1 Knapsack Problem

Kyungchul Park** · Kyungsik Lee***

■ Abstract ■

We consider the separation problem of the rank-1 Chvatal-Gomory (C-G) inequalities for the 0-1 knapsack problem with the knapsack capacity defined by an additional binary variable, which we call the fixed-charge 0-1 knapsack problem. We analyze the structural properties of the optimal solutions to the separation problem and show that the separation problem can be solved in pseudo-polynomial time. By using the result, we also show that the existence of a pseudo-polynomial time algorithm for the separation problem of the rank-1 C-G inequalities of the ordinary 0-1 knapsack problem.

Keyword : Chvatal-Gomory Inequality, Knapsack Problem, Separation Problem

1. Introduction

Over the past three decades, we have witnessed great advances in computational integer programming, which can be attributed to the various methodologies that have been making it possible to

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**　명지대학교 경영학과
***　한국외국어대학교 산업경영공학과
†　교신저자

strengthen LP relaxations of integer programs. Among them, the Chvatal-Gomory (C-G) procedure is a systematic method to generated valid inequalities to the convex hull of feasible solutions to general integer programs (see Nemhauser and Wolsey [9] for the details).

Though the C-G procedure has potential applicability to general integer programs, it can be very difficult to find sufficiently strong inequalities, which limits its practical applicability. Even though some special cases of the C-G inequalities, which show limited strength, can be separated in polynomial time [1, 2], the separation problem of the C-G inequalities for general integer programs is strongly NP-hard [5]. Fischetti and Lodi [6] studied the separation problem of the rank-1 C-G inequalities for general integer programs. They formulated the separation problem as a mixed integer program and showed that the rank-1 C-G inequalities can significantly tighten the bounds of the LP relaxations of some instances in MIPLIB3.0. However, it is difficult to apply their approach to practical situations since the separation problem itself is a very hard mixed integer program to solve and also there is an issue of round-off errors. Recently, the extension of the C-G inequalities to the case of mixed-integer programs have also been being done [3].

On the other hand, the previous studies also show that by exploiting the structural properties of the C-G inequalities for a specific problem, it can be successfully applied to solve practical problems. Glover et al. [7] devised a nonlinear programming based heuristic procedure for the separation of a subclass of the rank-1 C-G inequalities for the multiple choice 0-1 knapsack problems. They showed that the approach gives promising computational performance. Lee and Park[8] analyzed the exact separation algorithm for the rank-1 C-G inequalities for a variable capacity 0-1 knapsack problem that includes general integer capacity variables without upper bounds. They showed that the separation problem can be solved in pseudo-polynomial time. They also applied their approach to telecommunication network design problems and showed that the rank-1 C-G inequalities can strengthen the bounds of the LP relaxations significantly.

This paper, following a similar line of research to Lee and Park [8], shows that there exists a pseudo-polynomial time algorithm for the exact separation of the rank-1 C-G inequalities for the fixed-charge 0-1 knapsack problem. The convex hull of the feasible solutions to the fixed-charge 0-1 knapsack problem is defined as follows :

$$P = conv\left\{(x, y) \in B_+^{n+1} \mid \sum_{j \in N} a_j x_j - by \le 0\right\},$$

where $N = \{1, \cdots, n\}$ and a_j, $j \in N$ and b are positive integers. Without loss of generality, we assume that $a_j \le b$, for all $j \in N$. The defining inequality of P can be found frequently as a constraint in many network design problems including various location-allocation problems (see Daskin [4]). Hence, the rank-1 C-G inequalities for P can be applied to those practical problems.

On the other hand, the fixed-charge 0-1 knapsack problem is closely related to the (ordinary) 0-1 knapsack problem. If we let KP be the convex hull of feasible solutions to the 0-1 knapsack problem, that is,

$$KP = conv\left\{x \in B_+^n \mid \sum_{j \in N} a_j x_j \le b\right\},$$

then KP is the projection of a face of P, $P \cap \{y = 1\}$, onto the x space. It also can be readily seen that for any valid inequality $\pi x - \alpha y \le \beta$ to P, $\pi x \le \alpha + \beta$

is a valid inequality to KP. Moreover, it can be easily shown that if $\pi x - \alpha y \leq \beta$ defines a facet of P, then the inequality $\pi x \leq \alpha + \beta$ defines a facet of KP. The converse is not necessarily true. Hence, the rank-1 C-G inequalities for P can also be applied to KP.

In the next section, we define the separation problem, SEP, of the rank-1 C-G inequalities for P and characterize the optimal solutions to SEP. Specifically, we show that SEP, which is a continuous non-differentiable optimization problem, can be decomposed into $O(nb)$ subproblems. Then, in section 3, we show that each subproblem can be transformed into a combinatorial optimization problem that is NP-hard but can be solved in $O(n^2b^2)$ time, which implies the existence a pseudo-polynomial time algorithm for SEP. We also show that the result also implies that the separation problem of the rank-1 C-G inequalities for KP can be solved in pseudo-polynomial time. Finally, concluding remarks are given in section 4.

2. Analysis of the Separation Problem

The rank-1 C-G inequality for P is define as follows :

$$\sum_{j \in N} \lfloor u_0 a_j + u_j \rfloor x_j + \lfloor -u_0 b + v \rfloor y \leq \lfloor \sum_{j \in J} u_j + v \rfloor, \quad (1)$$

where $u = (u_0, u_1, \cdots, u_n) \in R_+^{n+1}$, $v \in R$, and for a given real number a, $\lfloor a \rfloor$ is the greatest integer less than or equal to a.

For a given $(\bar{x}, \bar{y}) \in R_+^{n+1}$ such that $\bar{x} \leq 1$ and $\bar{y} \leq 1$, and it satisfies the defining inequality of P, let $J = \{ j \in N | \bar{x}_j > 0 \}$. Then, the separation problem, SEP, that determine whether or not there exists an inequality of type (1) which cut off $(\bar{x}, \bar{y})$ can be defined as follows :

$$\text{(SEP)} \ \max \sum_{j \in J} \lfloor u_0 a_j + u_j \rfloor \bar{x}_j + \lfloor -u_0 b + v \rfloor \bar{y} - \lfloor \sum_{j \in J} u_j + v \rfloor$$
$$s.\ t.\ \ u \in R_+^{n+1},\ v \in R_+.$$

Before going into the analysis of SEP, we first derive a useful information on the properties of the *non-dominated* inequalities of type (1). A valid inequality is called *dominated* if it can be obtained by adding other valid inequalities. If an inequality is not dominated, then it is called non-dominated.

Lemma 1 : *For the inequality (1) to defined a non-dominated valid inequality,* $v \leq 1$ *and* $u_j \leq 1$, *for all* $j \in N$.

Proof : Suppose that $v > 1$. Them let us define $v' = (v-1) > 0$. By noting that

$$\lfloor -u_0 b + v \rfloor = \lfloor -u_0 b + v' \rfloor + 1 \text{ and}$$
$$\left\lfloor \sum_{j \in N} u_j + v \right\rfloor = \left\lfloor \sum_{j \in N} u_j + v' \right\rfloor + 1,$$

we can see that the inequality (1) can be obtained by adding another rank-1 C-G inequality of type (1) with v replaced by v' together with the valid inequality $y \leq 1$. By repeatedly applying the same argument, we can conclude that for the inequality (1) to be a non-dominated valid inequality, $v \leq 1$ must hold. Also, by a similar argument, we can see that $u_j \leq 1$, for all $j \in N$ should hold for an inequality (1) to be non-dominated. **QED.**

Lemma 2 : *For the inequality (1) to define a non-dominated valid inequality,* $u_0 \leq 1$.

Proof : Suppose that $u_0 > 1$. Then let us define $u_0' = (u_0 - 1) > 0$. By noting that

$\lfloor u_0 a_j + u_j \rfloor = \lfloor u_0' a_j + u_j \rfloor + a_j, j \in N$ and
$\lfloor -u_0 b + v \rfloor = \lfloor -u_0' b + v \rfloor - b,$

we can see that the inequality (1) can be obtained by adding another rank-1 C-G inequalities of type (1) with u_0 replaced by u_0' together with the inequality,

$$\sum_{j \in N} a_j x_j - by \le 0.$$

By repeatedly applying the same argument, we can see that the result follows. **QED.**

Note that lemma 1 and 2 imply that there exists an optimal solution (u^*, v^*) to SEP such that $u^* \le 1$ and $v^* \le 1$.

To further analyze SEP, let us consider the case where the coefficient of the variable y in the inequality (1) is fixed to a given integer. That is, let us assume $\lfloor -u_0 b + v \rfloor = -p_0$, where p_0 is a positive integer. Then the corresponding separation problem, SEP(p_0), can be written as follows :

$$\begin{aligned} \text{SEP}(p_0)\ \ max\ & \sum_{j \in J} \lfloor u_0 a_j + u_j \rfloor \bar{x}_j - \\ & p_0 \bar{y} - \lfloor \sum_{j \in J} u_j + v \rfloor \\ s.\ t.\ & u \le 1,\ v \le 1, \\ & p_0 - 1 < u_0 b - v \le p_0, \\ & u \in R_+^{n+1}, v \in R_+. \end{aligned}$$

Theorem 1 : *An optimal solution to SEP can be obtained by solving SEP(p_0), for* $1 \le p_0 \le b$.

Proof : By lemma 1 and 2 together with the fact that if $u_0 = 0$ or the coefficient of the variable y is equal to 0, then the resulting inequality (1) is trivial, we can see that for a non-dominated rank-1 C-G inequality of type (1), the coefficient of the variable y should satisfies the following relation :

$$-b \le \lfloor -u_0 b + v \rfloor \le -1.$$

Therefore, the result follows. **QED.**

Theorem 1 states that SEP can be decomposed into $O(b)$ subproblems, SEP(p_0), for all $1 \le p_0 \le b$, where p_0 is a positive integer. Observe that for a given v, there exists an optimal solution to (u, v) to SEP(p_0) that satisfies $u_0 = (p_0 + v)/b$. This observation leads to the following reformulation of SEP(p_0) :

$$\begin{aligned} max\ & \sum_{j \in J} \lfloor (p_0 + v) a_j / b + u_j \rfloor \bar{x}_j - \\ & p_0 \bar{y} - \lfloor \sum_{j \in J} u_j + v \rfloor \\ s.\ t.\ & u \le 1, v \le 1, \\ & u \in R_+^n, v \in R_+. \end{aligned}$$

To further characterize the properties of the optimal solutions to SEP(p_0), let us introduce some additional notations. Let $\lfloor p_0 a_j / b \rfloor = p_j$, for all $j \in N$ so that $p_0 a_j / b = p_j + q_j / b$, for all $j \in N$, where q_j is an integer and $0 \le q_j < b$. In addition, let $\mu_j = b - q_j$, for all $j \in N$. For notational convenience, let us assume that $\mu_1 / a_1 \le \cdots \le \mu_n / a_n$. Note that for all $j \in N$, we have the following relation :

$$(p_0 + \mu_j / a_j) a_j / b = \lfloor p_0 a_j / b \rfloor + 1.$$

Next, let us define $k = \max\{j \in N | \mu_j / a_j < 1\}$ and if such k does not exist, set $k = 0$. Finally, let $\alpha_0 = 0$, $\alpha_j = \mu_j / a_j$, for all $1 \le j \le k$, and $\alpha_{k+1} = 1$. Then, the following theorem presents an important decomposition property of the optimal solutions to SEP(p_0).

Theorem 2 : *There exists an optimal solution to SEP(p_0) such that $v=\alpha_i$ for some $0 \le i \le k+1$.*

Proof : Let $J=\{1,\cdots,n\}$ without loss of generality. Let us assume that an optimal solution (u, v) to SEP(p_0) is given. Since $0 \le v \le 1$, there exists i such that $0 \le i \le k$ and $\alpha_i \le v \le \alpha_{i+1}$. If $v=\alpha_i$ or $v=\alpha_{i+1}$, then the result follows. So, suppose $\alpha_i < v < \alpha_{i+1}$. Let $A=\{j \in J \mid u_j > 0\}$ and let A_1 be a subset of A such that $A_1=\{j \in A \mid j \le i\}$. If $A=\varnothing$, then it can be easily shown that another solution with v replaced by α_i is also an optimal solution. Thus let us assume that $A \ne \varnothing$. If we reset u_0 and u_j for all $j \in A$ such that

$$\lfloor -u_0 b+v \rfloor = -u_0 b+v,$$
$$\lfloor u_0 a_j+u_j \rfloor = u_0 a_j+u_j,\ j \in A,$$

then we can see that the new (u, v) is also optimal to SEP(p_0). Therefore, we can assume that the given optimal solution (u, v) satisfies the following property, where $A_2=A \setminus A_1$:

$$u_0=(p_0+v)/b \text{ and}$$
$$u_j=\begin{cases} 1-\dfrac{a_j}{b}\left(v-\dfrac{\mu_j}{a_j}\right), & j \in A_1, \\ \dfrac{\mu_j}{b}-\dfrac{a_j}{b}v, & j \in A_2, \\ 0, & j \in J \setminus A, \end{cases}$$

If we let $J_1=\{j \in J \mid j \le i\}$ and $J_2=J \setminus J_1$, then from the above property we can see that the following relations hold :

$$\lfloor u_0 a_j+u_j \rfloor = \begin{cases} \lfloor p_0 a_j/b \rfloor +2, & j \in A_1, \\ \lfloor p_0 a_j/b \rfloor +1, & j \in A_2 \cup (J_1 \setminus A_1), \\ \lfloor p_0 a_j/b \rfloor, & j \in J_2 \setminus A_2, \end{cases}$$
$$\lfloor -u_0 b+v \rfloor = -p_0, \text{ and}$$
$$\left\lfloor \sum_{j \in J} u_j+v \right\rfloor = \left\lfloor |A_1|+\sum_{j \in A}\frac{\mu_j}{b}+v\left(1-\sum_{j \in A}\frac{a_j}{b}\right) \right\rfloor.$$

Now we will show that we can construct another optimal solution (u', v') with either $v'=\alpha_i$ or $v'=\alpha_{i+1}$. First, suppose that $1-\sum_{j \in A}\frac{a_j}{b} \ge 0$. In this case, consider the following solution (u', v') :

$$v'=\alpha_i=\mu_i/a_i,$$
$$u'_0=(p_0+v')/b, \text{ and}$$
$$u'_j=\begin{cases} 1-\dfrac{a_j}{b}\left(v'-\dfrac{\mu_j}{a_j}\right), & j \in A_1, \\ \dfrac{\mu_j}{b}-\dfrac{a_j}{b}v', & j \in A_2, \\ 0, & j \in J \setminus A. \end{cases}$$

Then, we can verify that the following relations hold :

$$\lfloor u'_0 a_j+u'_j \rfloor = \lfloor u_0 a_j+u_j \rfloor,\ j \in J,$$
$$\lfloor -u'_0 b+v' \rfloor = \lfloor -u_0 b+v \rfloor, \text{ and}$$
$$\lfloor \sum_{j \in J} u'_j+v' \rfloor \le \lfloor \sum_{j \in J} u_j+v \rfloor.$$

Therefore, (u', v') is an optimal solution. Second, suppose that $1-\sum_{j \in A}\frac{a_j}{b} < 0$. Then, similar to the first case, we can show that the following solution is also optimal to SEP(p_0) :

$$v'=\alpha_{i+1}=\mu_{i+1}/a_{i+1},$$
$$u'_0=(p_0+v')/b, \text{ and}$$
$$u'_j=\begin{cases} 1-\dfrac{a_j}{b}\left(v'-\dfrac{\mu_j}{a_j}\right), & j \in A_1, \\ \dfrac{\mu_j}{b}-\dfrac{a_j}{b}v', & j \in A_2, \\ 0, & j \in J \setminus A. \end{cases}$$

Hence in any case, we can find an optimal solution to SEP(p_0) with either $v=\alpha_i$ or $v=\alpha_{i+1}$. Therefore the result follows. **QED.**

Theorem 2 states that for a given integer p_0 such that $1 \le p_0 \le b$, SEP(p_0) can be solved by solving $O(n)$ subproblems, SEP(p_0, α_i) for all $i=0, \cdots, k+1$, each of which obtained by setting $v=\alpha_i$, $0 \le i \le k+1$ as follows :

$$\text{SEP}(p_0, \alpha_i)\ max \sum_{j\in J} \lfloor (p_0+\alpha_i)a_j/b+u_j \rfloor \overline{x_j} - p_0\overline{y} - \lfloor \sum_{j\in J} u_j + \alpha_i \rfloor$$
$$s.\ t.\ u_j \le 1, j\in J.$$

Therefore, by theorem 1 and 2, we can get an optimal solution to SEP by solving $O(nb)$ subproblems.

In the next section, we show that SEP(p_0, α_i) can be solved in pseudo-polynomial time.

3. A Pseudo-polynomial time Algorithm for SEP

The existence of a pseudo-polynomial time algorithm for SEP can be established by showing that SEP(p_0, α_i) for a given p_0 and α_i can be solved in pseudo-polynomial time. To do that, we introduce some additional notations.

For each $j\in N$, let $\lfloor (p_0+\alpha_i)a_j/b \rfloor = r_j$ so that $(p_0+\alpha_i)a_j/b = r_j + s_j/(ba_i)$, where s_j is an integer such that $0 \le s_j < ba_i$. In addition, let $\lfloor \alpha_i \rfloor = \lfloor \mu_i/a_i \rfloor = r_0$ and then for some integer s_0, $\alpha_i = r_0 + s_0/(ba_i)$. Then, SEP($p_0$, α_i) can be restated as follows :

$$\text{SEP}(p_0, \alpha_i)\ max \sum_{j\in J} \lfloor r_j + f_j + u_j \rfloor \overline{x_j} - p_0\overline{y} - \lfloor \sum_{j\in J} u_j + r_0 + f_0 \rfloor$$
$$s.\ t.\ u_j \le 1,\ j\in J,$$

where $f_0 = s_0/(ba_i)$ and $f_j = s_j/(ba_i)$, for all $j\in J$.

Theorem 3 : *For a given p_0 and α_i, SEP(p_0, α_i) can be solved in $O(n^2b^2)$ time.*

Proof : Since $\overline{x_j} \le 1$, for all $j\in J$, it can be easily seen that there exists an optimal solution such that $u_j\in\{0, 1-f_j\}$, for all $j\in J$. Therefore, if we define a binary variable t_j for each $j\in J$ such that $t_j=1$ if and only if $u_j = 1-f_j$, SEP(p_0, α_i) can be reformulated as the following combinatorial optimization problem, SP(p_0, α_i) :

$$max \sum_{j\in J} \overline{x_j} t_j + \lfloor \sum_{j\in J} (1-f_j) + f_0 \rfloor + C$$
$$s.\ t.\ t_j\in\{0, 1\},\ j\in J,$$

where $C = \sum_{j\in J} r_j \overline{x_j} - p_0\overline{y} - r_0$ which is a constant.

Lee and Park [8] studied a special case of SP (p_0, α_i) where $f_0=0$. They showed that the special is NP-hard which implies that SP(p_0, α_i) is generally NP-hard. They also showed that there exists a pseudo-polynomial time algorithm for the special case of SP(p_0, α_i). They provided a dynamic programming algorithm whose time complexity is $O(n\lambda)$, where λ is the common denominator of f_j's. The fact that $f_0>0$ or not does not make any difference in the essential logic of their algorithm (see Lee and Park [8] for the details of their algorithm). Therefore, by using their algorithm with a minor modification in the calculation of the recursive equations to consider a non-zero f_0, we can obtain $O(n^2ba_i)$ algorithm. Since $a_j \le b$, for all $j\in J$, the algorithm has $O(n^2b^2)$ time complexity. **QED.**

Theorem 3 together with theorem 1 and 2 given in the previous section implies the following corollary.

Corollary 1 : *There exists a pseudo-polynomial time algorithm for SEP whose running time is $O(n^3b^3)$.*

We now turn our attention to the separation problem of the rank-1 C-G inequality for KP that is the convex hull of feasible solutions to the 0-1 knapsack problem, which is defined in section 1. We show that the above corollary 1 implies the existence of a pseudo-polynomial time algorithm for the separation problem of the rank-1 C-G inequalities for KP.

The rank-1 C-G inequality for KP is defined as follows :

$$\sum_{j\in N}\lfloor u_0a_j+u_j\rfloor x_j \le \left\lfloor u_0b+\sum_{j\in N}u_j\right\rfloor, \tag{2}$$

where $u=(u_0, u_1, \cdots, u_n)\in R_+^{n+1}$.

Let $\bar{x}\in R_+^n$ such that $\bar{x}\le 1$ and it satisfies the defining knapsack inequality of KP.

Theorem 4 : *If there exists an inequality (2) that cut off* $\bar{x}$*, then there exists an inequality (1) that cut off* $(\bar{x}, 1)$.

Proof : Let $u=(u_0, u_1, \cdots, u_n)\in R_+^{n+1}$ be the vector that defines an inequality (2) which is violated by $\bar{x}$. That is,

$$\sum_{j\in N}\lfloor u_0a_j+u_j\rfloor \bar{x}_j > \left\lfloor u_0b+\sum_{j\in N}u_j\right\rfloor.$$

Let $\lfloor u_0b\rfloor = p_A$ and let $\left\lfloor \sum_{j\in N}u_j\right\rfloor = p_B$, so that $u_0b = p_A+q_A$ and $\sum_{j\in N}u_j = p_B+q_B$. Then, by using the same $u=(u_0, u_1, \cdots, u_n)\in R_+^{n+1}$ with $v=q_A$, we can obtain the following inequality (1) which is a rank-1 C-G inequality for P :

$$\sum_{j\in N}\lfloor u_0a_j+u_j\rfloor x_j + \lfloor -u_0b+q_A\rfloor y \le \lfloor \sum_{j\in J}u_j+q_A\rfloor. \tag{3}$$

Observe that

$$\begin{aligned}&\left\lfloor \sum_{j\in N}u_j+q_A\right\rfloor - \lfloor -u_0b+q_A\rfloor\\ &= \lfloor p_B+q_B+q_A\rfloor + p_A\\ &= \left\lfloor u_0b+\sum_{j\in N}u_j\right\rfloor,\end{aligned}$$

which means (3) is violated by $(\bar{x}, 1)$. **QED.**

Theorem 4 states that the separation problem of the inequality (2) for KP can be solved by finding a rank-1 C-G inequality (1) for P. This in turn implies that the separation problem of the rank-1 C-G inequality for KP can be solved in pseudo-polynomial time.

Corollary 2 : *There exists a pseudo-polynomial time algorithm for the separation problem of the rank-1 C-G inequality for KP whose running time is $O(n^3b^3)$.*

4. Concluding Remarks

This paper shows that the separation of the rank-1 C-G inequalities for the fixed-charge 0-1 knapsack problem can be done in pseudo-polynomial time, which also implies the existence of a pseudo-polynomial time algorithm for the separation of the rank-1 C-G inequalities for the (ordinary) 0-1 knapsack problem. To our best knowledge, this is the first proof of the pseudo-polynomial solvability of the exact separation of the rank-1 C-G inequalities for the 0-1 knapsack problem.

As presented in section 2, SEP can be decomposed into $O(nb)$ combinatorial optimization problems, which will be helpful in devising an efficient heuristic procedure to generate strong rank-1 C-G inequalities. An efficient heuristic procedure, which is worthwhile to study, can also be used to develop an effective cut generation method for general 0-1 integer programs, in which every constraint can be transformed into a 0-1 knapsack constraint.

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