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A CONTINUOUS ONE-TO-ONE FUNCTION WHOSE INVERSE IS NOWHERE CONTINUOUS

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Abstract. Main purpose of this note is to construct an example of a continuous one-to-one function $f : \mathbb{Q}^* \to \mathbb{R}$ whose inverse is nowhere continuous, and to show that the completeness is not necessary for the continuous inverse theorem.

1. Preliminary

It is well-known that there exits a continuous one-to-one function f on an interval whose inverse is not continuous. For this example, it is necessary that the interval *not* be a closed bounded interval, and that the function *not* be strictly real-valued. Indeed, as in [4, p. 27], let us consider the function $f : [0, 2\pi) \to \mathbb{R}^2$ defined by

$$f(t) := (\cos t, \sin t) \qquad \text{for each} \quad t \in [0, 2\pi).$$

Then it is easy to see that f is a continuous one-to-one function whose image is the unit circle $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. And the inverse function f^{-1} of f maps from S^1 into $[0, 2\pi)$, and actually $f^{-1}(P)$ is its radian where $P(x, y) \in S^1$. It is easy to see that f^{-1} is not continuous at $(1, 0)$. And, in this example, the completeness on the domain of f is essential for the continuity of the inverse function f^{-1}.

On the other hand, in many analysis texts (e.g., see [1-3, 5]), it is not easy to find an example of a continuous one-to-one function f which maps from a subset of $\mathbb{R}$ into $\mathbb{R}$ whose inverse is *nowhere continuous*. So it is interesting to introduce such an instructive example in the mathematical analysis.

2. Main Results

Now we will construct an example of a real-valued continuous one-to-one function defined on a subset of $\mathbb{R}$ whose inverse function is nowhere continuous.

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Theorem 1. *There does exist a continuous one-to-one function* $f : \mathbb{Q}^* = \mathbb{Q}\setminus\{0\} \to \mathbb{R}$ *whose inverse function* f^{-1} *is nowhere continuous on* $f(\mathbb{Q}^*)$.

Proof. Let $f : \mathbb{Q}^* \to \mathbb{R}$ be a function defined by for each $x \in \mathbb{Q}^*$,

$$f(x) := x - k\sqrt{2}, \quad \text{if} \quad k\sqrt{2} < x < (k+1)\sqrt{2} \quad \text{for some} \quad k \in \mathbb{Z}.$$

Then it is easy to see that f is continuous and one-to-one in $\mathbb{Q}^*$. Indeed, if $f(x) = f(y),\ x, y \in \mathbb{Q}^*$, then $x - k_1\sqrt{2} = y - k_2\sqrt{2}$ for some $k_1, k_2 \in \mathbb{Z}$. Hence $x - y = (k_1 - k_2)\sqrt{2}$. Since $x, y \in \mathbb{Q}^*$ and $k_1 - k_2 \in \mathbb{Z}$, we have $x = y$ and $k_1 = k_2$ so that f is one-to-one in $\mathbb{Q}^*$. Also, f is continuous on $\Big(k\sqrt{2}, (k+1)\sqrt{2}\Big) \cap \mathbb{Q}^*$ for each $k \in \mathbb{Z}$ so that f is continuous in $\mathbb{Q}^*$.

Note that the image of f is the set $f(\mathbb{Q}^*) = \{x - k\sqrt{2} \mid x \in \mathbb{Q}^*,\ k\sqrt{2} < x < (k+1)\sqrt{2}$ for some $k \in \mathbb{Z}$ $\}$ which is denumerable and dense proper subset of $(0, \sqrt{2})$. Also, we can see that $\frac{\sqrt{3}}{2} \notin f(\mathbb{Q}^*)$. Indeed, if $\frac{\sqrt{3}}{2} = x - k\sqrt{2}$ for some $x \in \mathbb{Q}^*,\ k \in \mathbb{Z}$, then $2x = 2k\sqrt{2} + \sqrt{3}$ which is impossible.

We now show that the inverse function f^{-1} is not continuous in $f(\mathbb{Q}^*)$. If $y_o \in f(\mathbb{Q}^*)$, then $y_o \in (0, \sqrt{2})$ and $y_o + k\sqrt{2} \in \mathbb{Q}^*$ for some $k \in \mathbb{Z}$. And we can choose a rational sequence $\{r_n\} \subset \mathbb{Q}^*$ such that for each $n \in \mathbb{N}$,

$$n\sqrt{2} < r_n < (n+1)\sqrt{2} \qquad \text{and} \qquad |\ r_n - n\sqrt{2} - y_o\ | < \frac{1}{n}.$$

This can be possible since $\mathbb{Q}^*$ is a dense subset of $\mathbb{R}$ and $f(\mathbb{Q}^*)$ is a dense subset of $(0, \sqrt{2})$. We let $y_n := r_n - n\sqrt{2}$ for each $n \in \mathbb{N}$; then the sequence $\{y_n\}$ is a irrational sequence in $f(\mathbb{Q}^*)$ converging to $y_o \in f(\mathbb{Q}^*)$. Thus, for each $n \in \mathbb{N}$,

$$f^{-1}(y_n) = r_n \quad \text{and} \quad f^{-1}(y_o) = y_o + k\sqrt{2};$$

but $\{f^{-1}(y_n)\} \to \infty$ so that f^{-1} is not continuous at y_o. This completes the proof. $\square$

As is well-known, the continuous inverse theorem, e.g., see [2, p. 326], is as follow: *Let K be a non-empty compact subset of $\mathbb{R}$, and let $f : K \to \mathbb{R}$ be a continuous one-to-one function on K. Then f^{-1} is continuous on $f(K)$.*

Also, the following fact is well-known as we can see in [5]: *Suppose that f is a continuous one-to-one function from an interval $A \subseteq \mathbb{R}$ onto a subset $B \subseteq \mathbb{R}$. Then the function f^{-1} is continuous from B onto A.*

However, we note that it is impossible to find a counterexample for the continuous inverse theorem of a continuous one-to-one function f which maps from a non-complete interval $[a, b)$ into $\mathbb{R}$. Indeed, since f is continuous one-to-one on the interval $[a,b)$, the image $f\big([a,b)\big)$ must be an interval so that its shape should be either $[c,d)$ or $(c,d]$ $\big($possibly, either $[c,\infty)$ or $(-\infty,d]\big)$, and hence f^{-1} must be continuous. Therefore, the completeness is not a necessary condition for the continuous inverse theorem. Hence, finding some necessary conditions or even more finding some characterizations on the existence of the continuous inverse function for a given continuous one-to-one function $f : I \to \mathbb{R}$ is very instructive in the mathematical analysis.

Also, we will give a simple result on the existence of continuous inverse function, which shows that the completeness is not necessary for the continuous inverse theorem.:

Theorem 2. *Let $G \subseteq \mathbb{R}$ be a non-empty open subset of $\mathbb{R}$, and let $f : G \to \mathbb{R}$ be a continuous one-to-one function on G. Then f^{-1} is continuous on $f(G)$.*

Proof. By Theorem 11.1.9 in [2], G is the union of countably many disjoint open intervals in $\mathbb{R}$, say $G = \cup_{n\in\mathbb{N}} I_n$, where $I_n = (a_n, b_n)$. Since f is a continuous one-to-one function on G, for each $n \in \mathbb{N}$, f is continuous one-to-one on I_n so that f is strictly monotone on I_n. Hence $f(G) = \cup_{n\in\mathbb{N}} f(I_n)$ is the union of countably many disjoint open intervals in $\mathbb{R}$. Therefore, f^{-1} is a continuous inverse function on $f(G)$. □

Finally, it might be interesting that the characterization on the existence of a continuous inverse function for a continuous one-to-one function $f : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}$, can be stated as the condition on A. As we already mentioned, there can be many sufficient conditions on A, e.g., A is compact, open, finite union of disjoint intervals, etc..

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